Introduction to the representation theory of algebras

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October 20, 2011

to be announced

Preface

The aim of this notes is to give a brief and elementary introduction to the representation theory of finite-dimensional algebras. The notes had its origin in a undergraduate course I gave in two occasions at Universidad Nacional Autonóma de México.

The plan of the course was to try to cope with two competing demands: to expect as little as possible and to reach as much as possible: to expect only linear algebra as background and yet to make way to substantial and central ideas and results during its progress.

Therefore some crucial decisions were necessary. We opted for the model case rather than the most general situation, to the most illustrating example rather than the most extravagant one. We sought a guideline through this vast field which conducts to as many important notions, techniques and questions as possible in the limited space of a one-semester course. So, it is a book written from a specific point of view and the title should really be *Introduction to the theory of algebras, which are finite dimensional over some algebraically closed field.*

The book starts with the most difficult chapter in front: matrix problems. Conceptually there is little to understand in that chapter but it requires a considerable effort from the reader to follow the argumentation within. However, this chapter is central: it prepares all the main examples which later will guide through the rest. In the following two chapters we consider the main languages of representation theory. Since there are several competing languages in representation theory, a considerable amount of our effort is directed towards mastering and combining all of them. As you will see each of these languages has its own advantages and it therefore enables the reader not only to consult the majority of all research articles in the field, it also enriches the way we may thick about the notions themselves.

The rest of the book is devoted to gain structural insight into the categories of modules of a given algebra. In the chapter about module categories some older results are proved, whereas in the next four chapters some newer insights are transmitted. The last chapter is more of an open minded collection concerning what can be said about the building bricks of the module categories, which are called indecomposables with respect to certain invariants, called dimension vectors, under the action of isomorphisms.

Contents

Chapter 1

Matrix problems

"Matrix problems" provide easy examples for the phenomena we are interested in to study. At the same time they give us a powerful machinery to explicitly calculate these examples and study several main aspects of importance in the theory of representations of algebras. They however do not provide a nice and deep theory as we will see later.

We will therefore not pursue the theory of "**matrix problems**" formally but approach it intuitively by looking at a series of examples. As you will see all these examples have certain common features and all will be of importance throughout the whole book.

1.1 Introduction

Roughly speaking, we look at a set $\mathscr{M}$ and an equivalence relation on $\mathscr{M}$ and ask for a **normal form**, that is, an explicit list of representatives of the equivalence classes. The following example is not only illustrative but essential for all others.

Example 1.1. Let $\mathscr{M}$ be the set of all matrices with entries in a fixed field K and define $M \sim N$ if there exist invertible matrices (of suitable size) U and V such that $N = UMV^{-1}$.

We know from linear algebra that any matrix M can be transformed by elementary row and column transformations into

$$\begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{1.1}$$

where $\mathbf{1}_r$ stands for an **identity matrix** of size r and $\mathbf{0}$ for the **zero matrix** of some size. This transformations correspond to the multiplication with U and V respectively. Hence in each equivalence class there exists at least one matrix of the form (1.1). If we interpret M as the matrix corresponding to a linear map f between two fixed vector spaces, then the equivalence class of M corresponds to all matrices which represent f in different basis of these vector spaces. The number r (the size of the identity matrix in the upper left corner) is just the rank of M. Since the rank and also the size of M itself are invariant under change of basis, we see that only one matrix of the form (1.1) belongs to each equivalence class. The form (1.1) is determined by three integers r and m, n, where $m \times n$ is the size of the matrix. $\diamondsuit$

Comments 1.2 (a) The set $\mathscr{M}$ of Example 1.1 is a disjoint union $\mathscr{M} = \bigcup_{(m,n)\in\mathbb{N}^2} \mathscr{M}_{m,n}$ indexed over pairs of natural numbers, where $\mathscr{M}_{m,n} = K^{m\times n}$ is the set of all matrices of size $m \times n$. Each equivalence class is contained in a specific set $\mathscr{M}_{m,n}$.

We will always consider 0 to be an element of $\mathbb{N}$. The matrix of size $m \times 0$ (resp. $0 \times n$) are those matrices corresponding to the linear maps when one of the two vector spaces is the zero space. Such matrices have no entries but they do have a size. As we will see it will be very convenient to consider such strange matrices.

(b) The equivalence classes can be viewed similarly as the orbits under a group action, with the important difference that the action is given by a disjoint union of groups $G = \bigcup_{m,n\in\mathbb{N}} G_{m,n}$, where $G_{m,n} = \mathrm{GL}_m(K) \times \mathrm{GL}_n(K)$. That is, we have various actions $G_{m,n} \times \mathscr{M}_{m,n} \to \mathscr{M}_{m,n}$, given by $(U, V) \cdot M = UMV^{-1}$. The importance of looking at all sizes at the same time will become clearer later.

In the following we will look at some concrete problems of gradually increasing difficulty. We will be able to solve the first two problems but fail in the last one. The solutions encountered are of outstanding interest for representation theory and that is the reason why we start with them.

Exercises

1.1.1 Let $\mathscr{M}$ be the set of all square matrices with entries in the algebraically closed field K and define the equivalence relation by conjugacy, that is, $M \sim N$

holds if and only if there exists an invertible matrix U such that $N = UMU^{-1}$. What are then the normal forms known from linear algebra? How would you divide $\mathscr{M}$ into disjoint subsets? Describe the group actions in these subsets.

1.1.2 Let $\mathscr{M}$ be the set of all matrices with entries in the field K. Define the equivalence relation by one-sided transformation, that is, $M \sim N$ if and only if there exists an invertible matrix U such that $N = UM$. The normal form is known as *reduced row-echelon form* in linear algebra. How many normal forms of size 3×3 exist if K is the field with two elements?

1.2 The two subspace problem

The **two subspace problem** is given by the set $\mathscr{M}$ of all pairs of matrices with the same number of rows under the equivalence relation: $(A, B) \sim (A', B')$ if there exists invertible matrices U, S and T such that $A' = UAS^{-1}$ and $B' = UBT^{-1}$.

We write $[A|B]$ instead of (A, B) for such a pair and think of it as one big matrix divided into two vertical stripes. The **allowed transformations** for this problem consist of row and column transformations which do not change the given equivalence class, but are powerful enough to ensure transitivity within - in our given problem arbitrary row transformations of $[A|B]$ and arbitrary column transformations within each stripe separately.

Notice that the set $\mathscr{M}$ is naturally divided as $\mathscr{M} = \bigcup_{m,n',n'' \in \mathbb{N}} \mathscr{M}_{m,n',n''}$ where $\mathscr{M}_{m,n',n''} = K^{m \times n'} \times K^{m \times n''}$.

Example 1.3. Before we enter the general case, we consider a concrete example. Let

$$M = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right].$$

We first give the sequence of transformations and comment afterwards.

$$\begin{array}{c} -4 \\ \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] \sim \begin{array}{c} \\ -7 \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{array}\right] \sim \begin{array}{c} \\ (-3)\cdot \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array}\right] \sim \begin{array}{c} \\ \\ 6 \\ \end{array}\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{array}\right] \sim$$

$$\begin{array}{c} -2 \\ \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \sim \begin{array}{ccc} & & (-1)\cdot \\ & \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] & \end{array} \sim \begin{array}{c} \\ 2 \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} -2 \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The arrows on the left of the matrix indicate the elementary row transformation of adding a multiple of a row to another row, the factor is indicated as a small number. Similarly an arrow above the matrix indicates an elementary column transformation. There are only two more elementary transformations, namely the multiplication of the second row with -3 in the third step and the multiplication of the last column by -1 in the fifth step. The last matrix is the normal form we sought. $\Diamond$

Applying (1.1) to the block A in $[A|B]$ (by this, we mean that we use row and column transformation in the left block of the matrix $[A|B]$), we see that $[A|B]$ is equivalent to a matrix of the form

$$\left[\begin{array}{cc|c} \mathbf{1}_r & \mathbf{0} & B_1 \\ \mathbf{0} & \mathbf{0} & B_2 \end{array}\right].$$

We apply (1.1) again to B_2 and see that $[A|B]$ is equivalent to

$$\left[\begin{array}{cc|cc} \mathbf{1}_r & \mathbf{0} & B_{1,1} & B_{1,2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right].$$

By adding multiples of rows of the second row-stripe to the first row-stripe we certainly do not change anything in the first two column-stripes and achieve that $B_{1,1} = \mathbf{0}$. Finally, we apply (1.1) to $B_{1,2}$. Observe that the row transformations will disturb the identity block $\mathbf{1}_r$, but this can by corrected using column transformations in the first column stripe. Hence we obtain a normal form

$$\left[\begin{array}{ccc|ccc} \mathbf{1}_a & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_c & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \tag{1.2}$$

where $a + b = r$. But the normal form depends on more than the three non-negative parameters a, b c, namely also on the number of zero rows at the bottom and the number of zero columns in the third and sixth vertical stripe. We hence have proved the following result.

Proposition 1.4. *The two subspace problem has the normal form given in* (1.2).

Exercises

1.2.1 Determine which elements of the two subspace problem in the following list are equivalent.

$$M_1 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad M_2 = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right], \quad M_3 = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right],$$
$$M_4 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right], \quad M_5 = \left[\begin{array}{cc|cc} 5 & 1 & 1 & 3 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad M_6 = \left[\begin{array}{cc|cc} 5 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 \end{array}\right].$$

1.3 Decomposition into indecomposables

As the problems grow larger the complexity in the division of the matrices will increase. We therefore seek a procedure to simplify the notation and enlarge our understanding.

Given two matrices $A \in K^{m\times n}$ and $C \in K^{r\times s}$ we define a new matrix $A \oplus C \in K^{(m+r)\times(n+s)}$ by

$$A \oplus C = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}.$$

If two elements (A, B) and (C, D) of the two subspace problem are given, we define a new one by:

$$(A, B) \oplus (C, D) = (A \oplus C, B \oplus D)$$

and call it the **direct sum** of (A, B) and (C, D).

We can view $\oplus$ as a binary operation

$$\mathscr{M} \times \mathscr{M} \to \mathscr{M}, (M, N) \mapsto M \oplus N,$$

which satisfies $\mathscr{M}_{m,n',n''} \oplus \mathscr{M}_{p,q',q''} \subseteq \mathscr{M}_{m+p,n'+q',n''+q''}$. We can rewrite this as $\mathscr{M}_d \oplus \mathscr{M}_e \subseteq \mathscr{M}_{d+e}$ for each $d, e \in \mathbb{N}^3$. We shall thus write $\mathscr{M}_0$ instead of $\mathscr{M}_{0,0,0}$.

We call a pair (A, B) **indecomposable** if it does not lie in the equivalence class of two strictly smaller pairs, that is, if (A, B) is equivalent to $(A', B') \oplus (A'', B'')$ then either $(A', B') \in \mathscr{M}_0$ or $(A'', B'') \in \mathscr{M}_0$ but not both.

Notice that the unique element of $\mathscr{M}_0$ is, by definition, not indecomposable. Notice also that indecomposbility is a property which is invariant in each equivalence class.

We can now decompose (1.2) into indecomposables.

Proposition 1.5. *There are only* 6 *indecomposable normal forms for the two subspace problem, namely the following:*

$$[|], {\scriptstyle[|-]}, {\scriptstyle[-|]}, [1|], [|1], [1|1], \tag{1.3}$$

where $[|]$, ${\scriptstyle[|-]}$ *and* ${\scriptstyle[-|]}$ *stand for the unique element in* $\mathscr{M}_{1,0,0}$, $\mathscr{M}_{0,1,0}$ *and* $\mathscr{M}_{0,0,1}$ *respectively.*

Proof. We start rearranging the right block of (1.2) to get an equivalent matrix and then decompose as follows:

$$\left[\begin{array}{ccc|ccc} \mathbf{1}_a & \mathbf{0} & \mathbf{0} & \mathbf{1}_a & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] = [\mathbf{1}_a|\mathbf{1}_a] \oplus \left[\begin{array}{cc|cc} \mathbf{1}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$= [\mathbf{1}_a|\mathbf{1}_a] \oplus [\mathbf{1}_b|] \oplus \left[\begin{array}{c|cc} \mathbf{0} & \mathbf{1}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$= [\mathbf{1}_a|\mathbf{1}_a] \oplus [\mathbf{1}_b|] \oplus [|\mathbf{1}_c] \oplus [\mathbf{0}|\mathbf{0}].$$

But $[\mathbf{1}_a|\mathbf{1}_a] = [1|1] \oplus [1|1] \oplus \ldots \oplus [1|1]$ with a summands on the right hand side. We abbreviate this by $[1|1]^a$. Similarly $[\mathbf{1}_b|] = [1|]^b$ and $[|\mathbf{1}_c] = [|1]^c$. Finally the zero matrix $[\mathbf{0}|\mathbf{0}]$ decomposes as

$$[\mathbf{0}|\mathbf{0}] = [|]^r \oplus {\scriptstyle[|-]}^{s'} \oplus {\scriptstyle[-|]}^{s''},$$

where r is the number of rows of $[\mathbf{0}|\mathbf{0}]$ and s' (resp. s'') is the number of columns of the left (resp. right) stripe. Hence we get altogether the following decomposition

$$M = [1|1]^a \oplus [1|]^b \oplus [|1]^c \oplus [|]^r \oplus {\scriptstyle[|-]}^{s'} \oplus {\scriptstyle[-|]}^{s''}.$$

$\square$

Exercises

1.3.1 Decompose each of the following matrices into indecomposables.

$$M_1 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad M_2 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad M_3 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right].$$

1.3.2 Calculate all indecomposables for the **dual problem** of the two subspace problem: consider pairs of matrices (A, B) with the same number of columns under the equivalence relation: $(A, B) \sim (A', B')$ if there exist invertible U, V, T such that $A' = UAT^{-1}$ and $B' = VBT^{-1}$.

1.3.3 Consider the **three subspace problem** consisting of triples of matrices (A, B, C) with the same number of rows under the equivalence relation: $(A, B, C) \sim (A', B, C')$ if there exist invertible matrices R, S, T, U such that $A' = RAU^{-1}$, $B' = SBU^{-1}$ and $C' = TCU^{-1}$. Define indecomposability and then calculate a complete list of indecomposables (up to equivalence). You should get a list of 12 indecomposables.

1.4 The Kronecker problem

The **Kronecker problem** is the matrix problem of pairs of matrices (A, B) of the same size under the equivalence relation: $(A, B) \sim (A', B')$ if there exist invertible matrices U, T such that $A' = UAT^{-1}$ and $B' = UBT^{-1}$. The direct sum is defined in analogy to the two subspace problem: $(A, B) \oplus (C, D) = (A \oplus C, B \oplus D)$ and induces the notion of indecomposablity.

This problem is substantially more complicated. We will have to manipulate carefully until we get the final list of indecomposables, see Proposition 1.6. However, we shall not start from the solution but show how it can be found.

For this, we think the pair A, B again as one matrix with a vertical separation in the middle and write $[A|B]$ instead of (A, B). But we should not forget that the columns of A are **coupled** with the columns of B. Given such a pair $[A|B]$, we start again by reducing A to normal form (1.1) but introduce a slight modification by putting the identity matrix to the upper right corner. But then the matrix B should be divided into 4 blocks of the corresponding size just as in the left vertical stripe:

$$\left[\begin{array}{cc|c:c} \mathbf{0} & \mathbf{1} & C & D \\ \hdashline \mathbf{0} & \mathbf{0} & E & F \end{array}\right],$$

in particular D is a square matrix. If we just focus on the third and fourth vertical stripe, what are the allowed transformations? Well, we are allowed to do arbitrary row transformations separately in each horizontal stripe, but by doing so in the upper one, we spoil the identity matrix in the second vertical stripe. This can be fixed by applying the inverse transformation to the second vertical stripe (and consequently to the forth, remember the coupling). Therefore D is transformed by conjugation. And clearly we are also allowed to apply arbitrary column transformations to the third vertical stripe (and the first, which is zero and thus doesn't bother us). But we are allowed to do even more. We also can add multiples of the lower stripe to the upper stripe and multiples of the left stripes (first and third) to the right stripes (second and forth).

This is a new challenging problem! We shall call it momentarily **the coupled 4-block**, just to have a name for it, since it will appear again. Consider such a coupled 4-block and apply (1.1) to the lower left corner, i.e. to E, again slightly different. The division in the left vertical and lower horizontal stripe is then prolonged throughout the matrix and we obtain:

$$\left[\begin{array}{cc:c} C' & C'' & D'' \\ \hdashline \mathbf{0} & \mathbf{0} & F' \\ \mathbf{1} & \mathbf{0} & F'' \end{array}\right].$$

By adding multiples of the lower to the upper stripe we can force $C' = \mathbf{0}$ and similarly by adding multiples from the left to the right, we can force $F'' = \mathbf{0}$. Hence, we obtain

$$\left[\begin{array}{c:c} C & D \\ \hdashline E & F \end{array}\right] \sim \left[\begin{array}{cc:c} \mathbf{0} & C'' & D' \\ \hdashline \mathbf{0} & \mathbf{0} & F' \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c:c} C'' & D' \\ \hdashline \mathbf{0} & F' \end{array}\right] \oplus \left[\begin{array}{c:} \hdashline \mathbf{1} \end{array}\right].$$

That is, we could get rid of the lower right block by splitting off a direct summand, composed itself of indecomposables $\left[\begin{array}{c:} \hdashline 1 \end{array}\right]$, the one standing in the lower left corner. Now, apply (1.1) to F', but remember that column transformations in the right stripe are coupled with row transformations in the upper stripe, and therefore the vertical division coming from F' takes over to a horizontal division of the upper stripe:

$$\left[\begin{array}{c:c} C'' & D' \\ \hdashline \mathbf{0} & F' \end{array}\right] \sim \left[\begin{array}{c:cc} \tilde{C}_1 & \tilde{D}_1 & \tilde{D}_2 \\ \tilde{C}_2 & \tilde{D}_3 & \tilde{D}_4 \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \sim \left[\begin{array}{c:cc} \tilde{C}_1 & \tilde{D}_1 & \mathbf{0} \\ \tilde{C}_2 & \tilde{D}_3 & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c:cc} \tilde{C}_1 & \tilde{D}_1 & \mathbf{0} \\ \tilde{C}_2 & \tilde{D}_3 & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right] \oplus \left[\begin{array}{c:c} \hdashline & \end{array}\right],$$

where $\left[\begin{array}{c:c} \hdashline & \end{array}\right]$, stands for a lower zero row with no columns. Observe that $\tilde{D}_1$ is a square matrix again and that we can not split off a direct summand containing the $\mathbf{1}$ in the third vertical stripe because it is coupled with the second horizontal stripe. What are the allowed transformations in the four blocks $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{D}_1$ and $\tilde{D}_3$? To see this we recall how the four blocks are embedded in the original problem:

$$\left[\begin{array}{c:cc|c:cc} \mathbf{0} & \mathbf{1} & \mathbf{0} & \tilde{C}_1 & \tilde{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \tilde{C}_2 & \tilde{D}_3 & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right]. \tag{1.4}$$

Recall that in the original 4-block the row transformations of the block D were coupled with the column transformations of the same block. Hence the row transformations of $\tilde{D}_1$ are also coupled with column transformations of the same block. Furthermore we can still do any column transformation in the left stripe and add multiples of those columns to the right.

How about row transformations in the second horizontal stripe? They are coupled with column transformations in the third vertical stripe by the entry **1** in position $(2, 3)$ in (1.4), see the next figure, where the numbers [1] and [2] refer to those two coupled transformations.

$$\begin{array}{cc}
\begin{array}{cccccc} \phantom{\mathbf{0}} & \multicolumn{2}{c}{[2]} & & \multicolumn{2}{c}{[2]} \end{array} & \\
\left[\begin{array}{c:cc|c:cc} \mathbf{0} & \mathbf{1} & \mathbf{0} & \tilde{C}_1 & \tilde{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \tilde{C}_2 & \tilde{D}_3 & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right] & \begin{array}{l} \\ [1] \\ [3] \end{array}
\end{array}$$

Column transformations in the third vertical stripe, [2], would spoil the achieved **1** in position $(3, 6)$ – but we can repair that damage easily by row transformations in the third horizontal stripe [3]. This shows that, indeed we can do arbitrary row transformations in the second horizontal stripe.

Similarly, if we want to add multiples from the second horizontal stripe to the first horizontal stripe, [1] in the next figure, then this action is coupled with the the column transformation of adding multiples of the second vertical stripe to the third, [2].

$$\begin{array}{ccc}
\begin{array}{cccccc} \phantom{\mathbf{0}} & \multicolumn{2}{c}{[2]} & & \multicolumn{2}{c}{[2]} \end{array} & & \\
\left[\begin{array}{c:cc|c:cc} \mathbf{0} & \mathbf{1} & \mathbf{0} & \tilde{C}_1 & \tilde{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \tilde{C}_2 & \tilde{D}_3 & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right] & [1] & [3]
\end{array}$$

Now transformation [2] would spoil the achieved blocks $\mathbf{0}$ in positions $(1, 6)$ and $(2, 6)$ but that damage can be repaired by adding multiples of the third horizontal stripe to the upper ones, [3]. This shows that indeed we can add multiples of the second to the first row.

So, after all we see that the four matrices $C_1 = \tilde{C}_1$, $D_1 = \tilde{D}_1$, $E_1 = \tilde{C}_2$ and $F_1 = \tilde{D}_3$ constitute again a coupled 4-block.

As it seems, we just can proceed iterating easily. But notice the shift in the division: the original horizontal division between blocks has gone to the lower border, and therefore in the next step the indecomposables split off are of the form

$$\left[\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{:c} 0 \\ \hdashline 1 \end{array}\right].$$

In the subsequent steps the indecomposables which might be split off are:

$$\left[\begin{array}{c:cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right], \left[\begin{array}{:cc} & 0 & 0 \\ & 1 & 0 \\ \hdashline & 0 & 1 \end{array}\right], \dots, \left[\begin{array}{c:cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \hdashline 0 & 0 & \cdots & 1 \end{array}\right], \left[\begin{array}{:ccc} & 0 & \cdots & 0 \\ & 1 & \cdots & 0 \\ & \vdots & & \vdots \\ \hdashline & 0 & \cdots & 1 \end{array}\right].$$

The corresponding indecomposables of the Kronecker problem are of the form

$$[J(m,0) \mid \mathbf{1}_m], \qquad \text{for } m \geq 0, \tag{1.5}$$

where $J(m,0)$ is the matrix of size $m \times m$ having as only non-zero entries $J(m,0)_{ab} = 1$ if $a = b-1$ and

$$\left[\begin{array}{c|c} \mathbf{1}_n & z^\top \\ z^\top & \mathbf{1}_n \end{array}\right], \qquad \text{for } n \geq 1, \tag{1.6}$$

where z denotes a zero column of appropriate size and $M^\top$ the transpose of a matrix M. In each step the remaining matrix is of the form

$$\left[\begin{array}{c:cccccc} C_i & D_i & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ E_i & F_i & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{array}\right].$$

The reduction continues until we reach the case where there are no rows left in E_i and F_i. But then there are no columns in the third vertical stripe and consequently no rows in the forth horizontal stripe and this goes on all down to the bottom right and we see that in fact we are left with a matrix of the form

$$\left[\begin{array}{c:c} C_i & D_i \\ \hdashline \end{array}\right] = \left[\begin{array}{c:c} U_1 & V_1 \\ \hdashline \end{array}\right], \tag{1.7}$$

since there can be no other surviving rows or columns. We proceed similarly:

$$\left[\begin{array}{c:c} U_1 & V_1 \\ \hdashline \end{array}\right] \sim \left[\begin{array}{cc:cc} \mathbf{0} & \mathbf{1} & U_2' & V_2' \\ \mathbf{0} & \mathbf{0} & U_2 & V_2 \\ \hdashline \end{array}\right] \sim \left[\begin{array}{cc:cc} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & U_2 & V_2 \\ \hdashline \end{array}\right] = \left[\begin{array}{:} \hdashline \end{array}\right] \oplus \left[\begin{array}{c:cc} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & U_2 & V_2 \\ \hdashline \end{array}\right].$$

The allowed transformations for $\begin{bmatrix} U_2 & V_2 \end{bmatrix}$ are the same than in (1.7) and therefore we can again proceed iteratively, splitting off indecomposables

$$\left[\begin{array}{:} \hdashline \end{array}\right], \left[\begin{array}{c:c} 1 & 0 \\ \hdashline \end{array}\right], \left[\begin{array}{c:cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline \end{array}\right], \dots, \left[\begin{array}{c:cccc} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \hdashline \end{array}\right].$$

The corresponding indecomposables of the Kronecker problem are therefore of the form

$$[z\ \mathbf{1}_n \mid \mathbf{1}_n\ z], \qquad \text{for } n \geq 1. \tag{1.8}$$

In each step the remaining matrix is of the form

$$\left[\begin{array}{c:ccccc} \mathbf{1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathbf{1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & U_j & V_j \\ \hdashline \end{array}\right]$$

and we can proceed unless U_j has no column, which implies that the stripe above has no rows, which in turn implies that the vertical stripe to the left has now rows and this argument runs through to the top and shows that in fact we are left with the matrix

$$\left[\begin{array}{:c} V_j \\ \hdashline \end{array}\right]$$

with coupled row and column transformation. We know from Exercise 1.1.1, that *if the ground field K is algebraically closed*, then the normal form is given by a Jordan matrix, which, in our language, decomposes as a direct sum of **Jordan blocks** $J(n, \lambda) = \lambda\mathbf{1}_n + J(n, 0)$, where $\lambda \in K$ and $J(n, 0)$ is the matrix having as only non-zero entries $J(n, 0)_{ab} = 1$ if $a = b - 1$. The corresponding indecomposables of the Kronecker problem are of the form

$$[\mathbf{1}_n \mid J(n, \lambda)], \qquad \text{for } n \geq 1. \tag{1.9}$$

If K is not algebraically closed, normal forms still exist but are more complicated.

By collecting all indecomposables split off in the process, see (1.5), (1.6), (1.8) and (1.9), we get the solution of the Kronecker problem in case K is algebraically closed:

Proposition 1.6. *Suppose that the field K is algebraically closed. Then the following is a complete list of indecomposables for the Kronecker problem*

$$\left[\begin{array}{c|c} \mathbf{1}_n & z^\top \\ z^\top & \mathbf{1}_n \end{array}\right], \quad [\mathbf{1}_m \mid J(m, \lambda)], \quad [J(m, 0) \mid \mathbf{1}_m], \quad [z\ \mathbf{1}_n \mid \mathbf{1}_n\ z], \tag{1.10}$$

where $n \geq 0$, $m \geq 1$ are natural numbers, $\lambda \in K$, and $z \in K^n$ denotes the zero column.

Comment 1.7. Note that if K is not algebraically closed then we also get a complete list if we substitute $J(n, \lambda)$ by the appropriate indecomposables of the conjugation problem (which may be very difficult to get, as for example if $K = \mathbb{Q}$). In the forthcoming we always will assume the field K to be algebraically closed unless it is stated otherwise.

We have found an infinite but complete list of indecomposables, indexed by discrete values, namely m and n, and families indexed over the ground field, so-called **one-parameter families**.

Exercises

1.4.1 Let $[A|B]$ be an element of the Kronecker problem with $A, B \in K^{1\times 5}$. Notice that $[A|B]$ must decompose into some number $t > 1$ of smaller indecomposable summands. What possible t might occur? Give for each possibility one example.

1.4.2 Consider those indecomposable elements $[A|B]$ of the Kronecker problem which have square matrices A and B. How do those elements look like where not both matrices A and B are invertible?

1.4.3 Proof that when two invertible matrices T and U satisfy

$$T \left[\begin{array}{c|c} \mathbf{1}_n & z^\top \\ z^\top & \mathbf{1}_n \end{array}\right] U^{-1} = \left[\begin{array}{c|c} \mathbf{1}_n & z^\top \\ z^\top & \mathbf{1}_n \end{array}\right],$$

where z is as in Proposition 1.6, then $T = \alpha \mathbf{1}_{n+1}$ and $U = \alpha \mathbf{1}_n$ for some non-zero scalar $\alpha \in K$.

1.5 The three Kronecker problem

We consider finally three coupled matrices of the same size $[A|B|C]$, that is we consider the following matrix problem.

The **3-Kronecker problem** is given by triples of matrices (A_1, A_2, A_3) having the same size under the equivalence relation given by $(A_1, A_2, A_3) \sim (A_1', A_2', A_3')$ if there exist invertible matrices T, U such that $A_i' = TA_iU^{-1}$ for $i = 1, 2, 3$.

The straightforward generalization of n-tuples $(A_1, \ldots, A_n)$ is called the **n-Kronecker problem**. The direct sum is defined in analogy to the case of the two subspace problem:

$$(A_1, \ldots, A_n) \oplus (B_1, \ldots, B_n) = (A_1 \oplus B_1, \ldots, A_n \oplus B_n).$$

Again we write $[A_1|A_2|A_3]$ instead of (A_1, A_2, A_3) and think of it as a big matrix with coupled columns. In this case our approach is very different. We first show the following result.

Proposition 1.8. *For each $\lambda, \mu \in K$ define*

$$M_{\lambda,\mu} = \left[\begin{array}{c|c|c} 1 & 0 & \lambda \\ 0 & 1 & \mu \end{array}\right].$$

Then the matrices $M_{\lambda,\mu}$ are pairwise non-equivalent indecomposables.

Proof. Suppose that $M_{\lambda,\mu} \sim M_{\rho,\sigma}$, that is there are invertible matrices U and T such that

$$T\begin{bmatrix}1\\0\end{bmatrix}U = \begin{bmatrix}1\\0\end{bmatrix}, \quad T\begin{bmatrix}0\\1\end{bmatrix}U = \begin{bmatrix}0\\1\end{bmatrix}, \quad \text{and } T\begin{bmatrix}\lambda\\\mu\end{bmatrix}U = \begin{bmatrix}\rho\\\sigma\end{bmatrix}. \tag{1.11}$$

Since the multiplication with $U = [u]$ correspond to scalar multiplication with u, it follows from the first two equations that $T\mathbf{1}_2 u = \mathbf{1}_2$, and hence $T = u^{-1}\mathbf{1}_2$. But then $\rho = \lambda$ and $\sigma = \mu$. In other words, the matrices $M_{\lambda,\mu}$ are pairwise non-equivalent for different pairs (λ, μ).

They are also indecomposable: suppose that $M_{\lambda,\mu} \sim [A|B|C] \oplus [D|E|F]$. But then, $[D|E|F]$ (or $[A|B|C]$, the case is treated similarly) does not have columns and can only have rows. Since the first two columns of $M_{\lambda,\mu}$ are linearly independent, the same holds for $[A|B|C]$ and therefore $[A|B|C]$ must contain two rows. Hence $[D|E|F]$ is the matrix of size 0×0. Hence $M_{\lambda,\mu}$ is indecomposable. □

Thus we found a **two parameter family** of pairwise non-equivalent indecomposables. For a representation theorist, this is a point where it is clear that it is hopeless to try to write down a complete list of all indecomposables, since by such a list one would parametrize all modules over any finite-dimensional algebra simultaneously, see the end of Chapter 9 for a brief outline. This is why such a problem is called **wild**. However, if the size of the matrix is fixed then it is possible to write down a complete list of representatives.

Exercises

1.5.1 For each $\lambda = (a, b, c, d, e, f) \in K^6$ define the element M_λ of the 3-Kronecker problem as follows:

$$M_\lambda = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & a & b \\ 0 & 1 & 1 & 0 & c & d \\ 0 & 0 & 0 & 1 & e & f \end{array}\right].$$

Determine when two such elements M_λ and M_μ are equivalent and when they are not.

1.5.2 Use Exercise 1.4.3 to generalize the former exercise to get families of indecomposable and pairwise non-equivalent elements of the 3-Kronecker problem, which are indexed with $n(n+1)$ parameters.

1.6 Comments

We have encountered three different situations:

- A *finite* list of indecomposables as in the two subspace problem. This is the case of **finite representation type**.
- An *infinite but complete* list of indecomposables containing one (or more) *one-parameter family* of pairwise non-equivalent indecomposables. This was the case in the Kronecker problem.
- A *wild* (hopeless) situation as in the 3-Kronecker problem. This is the case of **wild representation type**.

These are the three **representation types** which occur in representation theory studied here. The second and third case are those of **infinite representation type**. The first and second case together are called of **tame representation type**.

Classifying is one of the important problems in representation theory (and that is why we started with it), but clearly there are many others. We used a technique called "matrix problems", which is very general and powerful and has done the job quite satisfactorily. However you should have noticed if you tried to solve the three subspace problem, see Exercise 1.3.3, that the method requires concentration and errors tend to be overseen and might slip through unnoticed.

Furthermore, we gained no structural insight and got just a plain list of indecomposables. In the following chapters we will see that there are better methods, which are somehow "self-correcting", which means that mistakes tend to surface quickly. However, in order to be able to understand those "better" methods, we have to prepare first some theory. We start by reinterpreting the considered problems from a new point of view.

As you may have noticed, we made no effort to formalize the term "matrix problem". This is due to the fact that we were mainly interested in some key examples and the list produced by the techniques. Matrix problem is a very general concept allowing much broader context. For example, the classification of square matrices under transposition ($A \sim A'$ if there exists an invertible T with $A' = TAT^{\top}$) is a nice and important matrix problem, but it will not allow the translation into the language of quivers and representations as studied in the next chapter. Therefore we did not stress the formal definition but stuck to the examples.

Chapter 2

Representations of quivers

All the examples of matrix problems we studied so far can be reformulated in a diagrammatic way which yields a nice general point of view for looking at matrix problems. This approach leads naturally to a more sophisticated language known as categories and functors and it will take large part of this chapter to develop these notions properly.

2.1 Quivers

Look again at the examples of Chapter 1. In the two subspace problem, we considered pairs of matrices (A, B) with the same number of rows under a certain equivalence relation. Identifying matrices with linear maps, we have thus considered diagrams

$$\begin{tikzcd} K^{n'} \arrow[dr, "A"'] & & K^{n''} \arrow[dl, "B"] \\ & K^m & \end{tikzcd} \tag{2.1}$$

and were trying to find "good bases" for the involved vector spaces. The **dual problem** (see Exercise 1.3.2) corresponds to diagrams of the form

$$\begin{tikzcd} K^{n'} & & K^{n''} \\ & K^m \arrow[ul, "A"] \arrow[ur, "B"'] & \end{tikzcd} .$$

In the Kronecker problem, we considered two matrices of the same size and in the three Kronecker problem three matrices of the same size under the equivalence relation of simultaneous row transformations and simultaneous column transformations. Thus $[A|B]$ and $[A|B|C]$ corresponds to two and three "parallel" linear maps, respectively:

$$\begin{tikzcd} K^l \arrow[r, shift left, "A"] \arrow[r, shift right, "B"'] & K^m \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} K^l \arrow[r, shift left=2, "A"] \arrow[r, "B" near end] \arrow[r, shift right=2, "C"'] & K^m. \end{tikzcd}$$

Again, the equivalence relation is given by arbitrary change of basis in the two vector spaces. Thus, these four matrix problems, are encoded by a simple diagram

$$\begin{tikzcd}[column sep=small] \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}, \quad \begin{tikzcd}[column sep=small] \bullet & & \bullet \\ & \bullet \arrow[ul] \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \bullet \end{tikzcd}. \tag{2.2}$$

The matrix problem may be recovered, by replacing the vertices by vector spaces with basis, the arrows by linear maps (between the corresponding vector spaces) and considering two corresponding tuples of matrices as equivalent if one can be obtained from the other by change of basis. The diagrams we consider are therefore oriented graphs, where loops and multiple arrows are explicitly allowed. We formalize this in the following.

A **quiver** is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of **vertices**, Q_1 is the set of **arrows** and s, t are two maps $Q_1 \to Q_0$, assigning the **starting vertex** and the **terminating vertex** for each arrow. We also say that α **starts in** $s(\alpha)$ and **ends in** $t(\alpha)$. A quiver Q is **finite** if Q_0 and Q_1 are finite sets.

Examples 2.1 (a) The quiver on the left in (2.2) is called **two subspace quiver**.

(b) The third quiver from the left in (2.2) is called **Kronecker quiver**.

(c) The quiver on the right in (2.2) is called **three Kronecker quiver**. More generally, the n**-Kronecker quiver** consists of two vertices $1, 2$ and $n > 1$ arrows, which have 1 as starting vertex and 2 as terminating vertex. $\Diamond$

Usually, in examples, we will have $Q_0 = \{1, \ldots, n\}$. For arrows α with $s(\alpha) = i$ and $t(\alpha) = j$, we usually write. $\alpha \colon i \to j$. In general, we denote by $\mathbb{N}^{Q_0}$ the set of all functions $Q_0 \to \mathbb{N}$, thus, if $Q_0 = \{1, \ldots, n\}$ then $\mathbb{N}^{Q_0} = \mathbb{N}^n$.

To each quiver we can associate a matrix problem (the contrary is false, see end of Section 2.2). Let Q be a finite quiver. Then the **matrix problem associated to** Q is the pair $(\mathscr{M}_Q, \sim_Q)$ where

$$\mathscr{M}_Q = \bigcup_{d \in \mathbb{N}^{Q_0}} \mathscr{M}_{Q,d}, \quad \mathscr{M}_{Q,d} = \{(M_\alpha)_{\alpha \in Q_1} \mid M_\alpha \in K^{d_{t(\alpha)} \times d_{s(\alpha)}}\}$$

and $(M_\alpha)_\alpha \sim_Q (N_\alpha)_\alpha$ if and only if there exists a family $(U_i)_{i \in Q_0}$ of invertible matrices such that

$$N_\alpha = U_{t(\alpha)} M_\alpha U_{s(\alpha)}^{-1} \tag{2.3}$$

for each arrow $\alpha \in Q_1$.

For each $M \in \mathscr{M}_{Q,d}$ the vector $d \in \mathbb{N}^{Q_0}$ is called the **dimension vector** of M and shall be denoted by $d = \underline{\dim}\, M$.

Exercises

2.1.1 Draw the diagram of the quiver Q given by $Q_0 = \{1, 2, 3, 4, 5\}$, $Q_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $s(\alpha_i) = i$ and $t(\alpha_i) = i + 1$ for $i = 1, \ldots, 4$.

2.1.2 Draw the quiver corresponding to the three subspace problem, described in Exercise 1.3.3. This quiver is called the **three subspace quiver**.

2.1.3 Let Q be the quiver $1 \xrightarrow{\alpha} 2$. Solve the corresponding matrix problem $(\mathscr{M}_Q, \sim_Q)$, that is, determine the indecomposables.

2.1.4 Determine the dimension vectors of the indecomposables in the Kronecker problem.

2.2 Representations

We will fix the ground field K and omit the dependence on K in our notation if no confusion can arise.

A **representation** of a quiver Q is a pair

$$V = ((V_i)_{i \in Q_0}, (V_\alpha)_{\alpha \in Q_1})$$

of two families: the first, indexed over the vertices of Q, is a family of finite-dimensional vector spaces and the second, indexed over the arrows of Q, consists of linear maps $V_\alpha \colon V_{s(\alpha)} \to V_{t(\alpha)}$.

The **zero representation**, denoted by 0, is the unique family with $V_i = 0$ (the zero vector space) for each $i \in Q_0$.

It is common to write a representation "graphically" by replacing each vertex i by the vectorspace V_i and each arrow $\alpha\colon i \to j$ by the linear map $V_\alpha\colon V_i \to V_j$. The **dimension vector** of a representation V is the vector $(\dim V_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$.

Example 2.2. The indecomposable elements of the Kronecker problem with square matrices are $[\mathbf{1}_m|J(m,\lambda)]$ for $\lambda \in K$ and $[J(m,0)|\mathbf{1}_m]$. They correspond to the following indecomposable representations of the Kronecker quiver

$$
\begin{tikzcd} V_m \arrow[r, shift left, "1"] \arrow[r, shift right, "X+\lambda"'] & V_m \end{tikzcd}
\qquad \text{resp.} \qquad
\begin{tikzcd} V_m \arrow[r, shift left, "X"] \arrow[r, shift right, "1"'] & V_m \end{tikzcd} ,
$$

where $V_m = K[X]/(X^m)$. ◊

Let V and W be two representations of a finite quiver Q. A **morphism** from V to W is a family of linear maps $f = (f_i\colon V_i \to W_i)_{i\in Q_0}$ such that for each arrow $\alpha\colon i \to j$ we have

$$f_j V_\alpha = W_\alpha f_i. \tag{2.4}$$

We denote a morphism just like a function, that is, we write $f\colon V \to W$ to indicate that f is a morphism from V to W. Observe that equation (2.4) states that the following diagram commutes:

$$
\begin{tikzcd}
V_i \arrow[r, "f_i"] \arrow[d, "V_\alpha"'] & W_i \arrow[d, "W_\alpha"] \\
V_j \arrow[r, "f_j"'] & W_j
\end{tikzcd} .
$$

A morphism is an **isomorphism** if each f_i is invertible and we say that V and W are **isomorphic** representations if there exists an isomorphism from V to W.

A **basis** of a representation V is a family $(B_i)_{i\in Q_0}$ where B_i is a basis of the space V_i for each vertex $i \in Q_0$. Each such basis yields a family of matrices $V^B = (V^B_\alpha)_{\alpha\in Q_1}$ where V^B_α represents the linear map V_α in the bases $B_{s(\alpha)}$ and $B_{t(\alpha)}$.

Proposition 2.3. *Let V and W be two representations of a finite quiver Q. Then V is isomorphic to W if and only if for some (any) basis B of V and some (any) basis C of W we have that V^B and W^C are equivalent elements of the matrix problem associated to Q.*

Proof. Notice that by chosing bases, we translate the linear invertible map f_i into an invertible matrix U_i. Condition (2.4) corresponds then to (2.3). □

Let V and W be two representations of a quiver Q. The direct sum $V \oplus W$ is then defined as the representation given by the spaces $(V \oplus W)_i = V_i \oplus W_i$ and the linear maps $(V \oplus W)_\alpha = V_\alpha \oplus W_\alpha$, which are defined componentwise. We denote $V \oplus V$ as a power by V^2 and inductively V^i for lager exponents i.

A representation V is **indecomposable** if and only if $V \neq 0$ and it is impossible to find an isomorphism $V \xrightarrow{\sim} V' \oplus V''$ for any non-zero representations V' and V''.

Proposition 2.4. *Let V be a representation of a finite quiver Q. Then V is indecomposable if and only if V^B is indecomposable for some (any) basis B of V.*

Proof. This an immediate consequence of the definitions and Proposition 2.3. □

We thus achieved a perfect translation. Solving one of the matrix problems above corresponds to *classifying the indecomposable representations up to isomorphism.* For instance, we get the following result.

Proposition 2.5. *Each indecomposable representation of the quiver corresponding to the two subspace problem is isomorphic to precisely one representation of the following list.*

$$
\begin{tikzcd}[column sep=small] 0 \arrow[dr] & & 0 \arrow[dl] \\ & K & \end{tikzcd},
\begin{tikzcd}[column sep=small] K \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd},
\begin{tikzcd}[column sep=small] 0 \arrow[dr] & & K \arrow[dl] \\ & 0 & \end{tikzcd},
\begin{tikzcd}[column sep=small] K \arrow[dr, "{[1]}"'] & & 0 \arrow[dl] \\ & K & \end{tikzcd},
\begin{tikzcd}[column sep=small] 0 \arrow[dr] & & K \arrow[dl, "{[1]}"] \\ & K & \end{tikzcd},
\begin{tikzcd}[column sep=small] K \arrow[dr, "{[1]}"'] & & K \arrow[dl, "{[1]}"] \\ & K & \end{tikzcd}.
$$

Comments 2.6 (a) Observe that the strange matrices of the two subspace problem occurring in (1.3) correspond to natural representations.

(b) Notice that not every matrix problem we considered admits such a straightforward translation. For instance, in the coupled 4-block problem of Section 1.4 we looked at quadruples of matrices

$$\begin{bmatrix} C & D \\ E & F \end{bmatrix},$$

where the row and column transformations for D are coupled by conjugation. The corresponding quiver would look as follows (where we indicated the places of the matrices):

$$\begin{tikzcd}
\bullet \arrow[dr, "C"] \arrow[dd, "E"'] & \\
& \bullet \arrow[dl, "F"] \arrow[loop right, "D"] \\
\bullet &
\end{tikzcd} .$$

This quiver defines a wild case and does not correspond to our original matrix problem, since we have not expressed in our new language that we can add rows from the lower stripe to the upper stripe nor that we can add columns from the left to the right stripe.

Exercises

2.2.1 Write the matrices given in Proposition 1.8 as representations of the corresponding quiver.

2.2.2 Use the spaces V_n and V_{n+1} of Example 2.2 to write those indecomposable representations of the Kronecker quiver which were not already given in the Example.

2.2.3 Decompose the following representation into indecomposables

$$\begin{tikzcd}
K^2 \arrow[dr, "\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}"'] & & K^2 \arrow[dl, "\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}"] \\
& K^2 &
\end{tikzcd}$$

2.3 Categories and functors

We shall briefly explain the language of categories and functors, since it provides a general language for the different concepts we shall encounter. If you are already familiar with categories and functors you can skip all of this section except the examples and read on in Section 2.4.

A **category** $\mathscr{C}$ is a class of **objects** (which we usually denote by the same letter as the whole category) together with a familiy of sets $\mathscr{C}(x, y)$ whose elements are called **morphisms** (one set for each pair of objects $x, y \in \mathscr{C}$) together with a family of **composition maps** $\mathscr{C}(y, z) \times \mathscr{C}(x, y) \to \mathscr{C}(x, z), (g, f) \mapsto g \circ f$ (one for each triple of objects $x, y, z \in \mathscr{C}$) such that for each object $x \in \mathscr{C}$ there exists an **identity morphism** $1_x \in \mathscr{C}(x, x)$, that is an element which satisfies $1_x \circ f = f$ and $g \circ 1_x = g$ for any $f \in \mathscr{C}(w, x)$,

$g \in \mathscr{C}(x,y)$, any $w, y \in \mathscr{C}$ and such that the composition is associative, that is $(h \circ g) \circ f = h \circ (g \circ f)$ for any $f \in \mathscr{C}(w,x)$, $g \in \mathscr{C}(x,y)$, $h \in \mathscr{C}(y,z)$, any $w, x, y, z \in \mathscr{C}$.

This is a long definition! Intuitively, a category is something similar to what you obtain when you throw all sets and all maps between all these sets into one big bag called Set, the category of sets.

To summarize: There are objects, which form a class; between any two objects there is a set of morphisms (possibly the empty set); morphisms may be composed and the composition is associative; and there are identity morphisms. We usually write $f\colon x \to y$ for a morphism $f \in \mathscr{C}(x,y)$ to remind us of the similarity with maps. We also often omit the composition symbol and write gf instead of $g \circ f$.

Examples 2.7 (a) The category Set has as objects the class of all sets and as morphisms just all maps. The composition of morphisms in the category is just the composition of maps and the identity morphisms are the identity maps.

(b) The category Vec has as objects K-vector spaces with the linear maps as morphisms. The composition of morphisms and the identity morphism are again the obvious ones. The category vec has as objects the class of finite-dimensional K-vector spaces, again with the linear maps as morphisms.

(c) The category Top has the topological spaces as objects and continuous functions as morphisms. $\diamondsuit$

As you see you can take for the objects all representatives of a fixed algebraic structure like topological spaces, rings, groups, abelian groups, finitely generated abelian groups and so on and so on. For the morphisms you take the structure preserving maps between them, and for the composition just the composition of maps. You always get a category. Let us look now at some more and stranger categories.

Examples 2.8 (a) Let Q be a finite quiver. We will see that $\mathscr{M}_Q$ can be viewed as a category. The class of objects is by definition just the set $\mathscr{M}_Q$ itself. If $M, N \in \mathscr{M}_Q$ then let

$$\mathscr{M}_Q(M,N) = \{(U_i)_{i \in Q_0} \mid \forall \alpha \in Q_1, N_\alpha U_{s(\alpha)} = U_{t(\alpha)} M_\alpha\}.$$

Observe that the condition $N_\alpha U_{s(\alpha)} = U_{t(\alpha)} M_\alpha$ is the same as (2.3) except that we do not require the matrices U_i to be invertible.

It is easy to verify that $\mathscr{M}_Q$ is indeed a category if the composition is given by componentwise matrix multiplication and the identity morphisms are the tuples of identity matrices. However, morphisms are clearly not functions between two sets in this example.

(b) The category $\operatorname{rep} Q$ has as objects the representations of Q and as morphisms just the morphisms of representations. The composition is given by componentwise composition of linear functions and the identity morphisms are given by tuples of identity functions. Note, that as in the example before, morphisms are not given by a single function; in this case they consist of a family of functions satisfying some compatibility property. If V and W are representations of the quiver Q, we denote the morphism set $\operatorname{rep} Q(V,W)$ also by $\operatorname{Hom}_Q(V,W)$. $\lozenge$

In a category $\mathscr{C}$ a morphism $f\colon x \to y$ is called an **isomorphism** if there exists a morphism $g\colon y \to x$ such that $f \circ g = 1_y$ and $g \circ f = 1_x$. Two objects are said to be **isomorphic in** $\mathscr{C}$ if there exists an isomorphism between them.

Example 2.9. In the category $\mathscr{M}_Q$ (see Example 2.8 (a)) two objects are isomorphic precisely when they are equivalent. So, the categorical concept of isomorphism has just the right meaning we are interested in. $\lozenge$

The following concept will be used to relate different categories among each other.

If $\mathscr{C}$ and $\mathscr{D}$ are categories then a **covariant functor** $F\colon \mathscr{C} \to \mathscr{D}$ associates to each object $x \in \mathscr{C}$ an object of $Fx \in \mathscr{D}$ and to each morphism $g \in \mathscr{C}(x,y)$ a morphism $Fg \in \mathscr{D}(Fx, Fy)$ such that $F1_x = 1_{Fx}$ for each object $x \in \mathscr{C}$ and such that the composition is preserved, that is $F(h \circ g) = Fh \circ Fg$ for any $g \in \mathscr{C}(x,y)$, $h \in \mathscr{C}(y,z)$, any $x,y,z \in \mathscr{C}$. A **contravariant functor** $F\colon \mathscr{C} \to \mathscr{D}$ is very similar to a covariant functor except that it inverts the direction of the morphisms, that is $Fg \in \mathscr{D}(Fy, Fx)$ for any $g \in \mathscr{C}(x,y)$ and consequently $F(h \circ g) = Fg \circ Fh$ for any g and h.

We shall meet many functors during the course of this book and limit ourselves here to two simple examples of covariant functors.

Examples 2.10 (a) Let Q be a finite quiver. Define a functor $F\colon \operatorname{rep} Q \to \operatorname{vec}$ by $FV = \bigoplus_{i \in Q_0} V_i$ for any representation V of Q and $Fg = \bigoplus_{i \in Q_0} g_i$ for any morphism of representations g.

(b) Let Q be a finite quiver. Then define a functor $G\colon \mathscr{M}_Q \to \operatorname{rep} Q$ as follows: for an object M let $(GM)_i = K^{d_i}$, where $d = \underline{\dim}\, M$ is the dimension vector, and $(GM)_\alpha = M_\alpha$. Further, if $U\colon M \to N$ is a morphism in $\mathscr{M}_Q$, then define $GU = U$, with the abuse of notation that U denotes a matrix as well as the associated linear map in the canonical bases. $\diamondsuit$

If $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{E}$ are functors then we obtain a functor $GF\colon \mathscr{C} \to \mathscr{E}$ in the obvious way: $(GF)x = G(Fx)$ for each object x and $(GF)f = G(Ff)$ for each morphism f. The functor GF is called the **composition** of F with G.

If $\mathscr{C}$ is a category, then the functor $1_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ defined by $1_{\mathscr{C}}x = x$ and $1_{\mathscr{C}}f = f$ for each object x and morphism f is called the **identity functor** of $\mathscr{C}$.

Two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ are called **inverse to each other** if $FG = 1_{\mathscr{D}}$ and $GF = 1_{\mathscr{C}}$. Note that F and G necessarily must have the same variance, that is, they both must be covariant or both contravariant, see Exercise 2.3.4. If F and G are covariant, then they are called **isomorphisms** and the categories $\mathscr{C}$ and $\mathscr{D}$ are called **isomorphic**. If F and G are contravariant then they are called **dualizations** and the categories **dual**.

We will later see that it is rather seldom in practice that two categories are isomorphic, see Section 2.5, where we develop a weaker notion called **equivalence**. The last piece of categorical terminology relates two functors.

If $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{C} \to \mathscr{D}$ are two covariant functors, then a **morphism of functors** (or **natural transformation** or just **morphism**) $\varphi\colon F \to G$ is a family $(\varphi_x)_{x \in \mathscr{C}}$ of morphism $\varphi_x \in \mathscr{D}(Fx, Gx)$ such that $\varphi_y \circ Fh = Gh \circ \varphi_x$ for any morphism $h \in \mathscr{C}(x, y)$.

If $F\colon \mathscr{C} \to \mathscr{D}$ is a given covariant functor, then the morphism $F \to F$ given by the family $(1_{Fx})_{x \in \mathscr{C}}$ is called **identity morphism** and will be denoted by 1_F. A morphism $\varphi\colon F \to G$ of covariant functors is an **isomorphism** if there exists a morphism $\psi\colon G \to F$ such that $\psi\varphi = 1_F$ and $\varphi\psi = 1_G$.

Exercises

2.3.1 Prove that a morphism $f\colon V \to W$ of representations of a quiver is an isomorphism if and only if for each vertex i the linear map f_i is bijective. Show that in that case the family $(f_i^{-1})_{i\in Q_0}$ of inverse maps constitutes an isomorphism $W \to V$ of representations.

2.3.2 Prove a generalization of the previous exercise, namely, that a morphism $\varphi\colon F \to G$ of functors $F, G\colon \mathscr{C} \to \mathscr{D}$ is an isomorphism if and only if for each object $x \in \mathscr{C}$ the morphism $\varphi_x\colon Fx \to Gx$ is an isomorphism. Show that in that case the family $(\varphi_x^{-1})_{x\in\mathscr{C}}$ is an isomorphism of functors $G \to F$.

2.3.3 Verify carefully that all the properties stated in the definition of a category are satisfied in the two Examples 2.8.

2.3.4 Investigate when the functor GF is covariant and when it is contravariant, depending on the variance of F and G.

2.3.5 Let $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{C} \to \mathscr{D}$ be two contravariant functors. What is the apropriate condition for a family $(\varphi_x)_{x\in\mathscr{C}}$ of morphisms $\varphi_x \in \mathscr{D}(Fx, Gx)$ to be a *morphism of covariant functors*?

2.4 The path category

A representation looks very much like a functor $Q \to \mathrm{vec}$ where Q is viewed "as a category" with vertices as objects and arrows as morphisms. But of course this is nonsense, since there are no identity morphisms and no composition of arrows in Q. In the following we will enhance the quiver Q to a proper category. Therefore we will need the concept of paths in Q.

Let Q be a quiver (possibly infinite). A **path of length** l is a $(l+2)$-tuple

$$w = (j|\alpha_l, \alpha_{l-1}, \ldots, \alpha_2, \alpha_1|i) \tag{2.5}$$

where $i, j \in Q_0$ and $\alpha_1, \ldots, \alpha_l \in Q_1$ such that $s(\alpha_1) = i$, $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, l-1$ and $t(\alpha_l) = j$.

We explicitly allow $l = 0$ but require then that $j = i$. The corresponding path $e_i := (i||i)$ is called the **identity path** or **trivial path** in i. The length l of a path w is denoted by $\mathrm{len}(w)$.

We extend the functions s and t in the obvious way: $s(w) = i$ and $t(w) = j$ if w is the path (2.5). A path w of positive length $l > 0$ is called a **cycle** if $s(w) = t(w)$. Cycles are often also called **oriented cycles** in the literature. A cycle of length 1 is called **loop**.

The **composition** of two paths $v = (i|\alpha_l, \dots, \alpha_1|h)$ and $w = (j|\beta_m, \dots, \beta_1|i)$ is defined by

$$wv = (j|\beta_m, \dots, \beta_1, \alpha_l, \dots, \alpha_1|h).$$

Notice that we defined the composition of paths in the same order as functions, which is not at all standard in the literature, but rather up to the taste of the author. A path $(i|\alpha_l, \dots, \alpha_1|h)$ will often be denoted by $\alpha_l \alpha_{l-1} \cdots \alpha_1$.

Let Q be a quiver (possibly infinite). The **path category** KQ of Q is the category whose objects are the vertices of Q and the morphisms from i to j form a vector space which has as basis the paths w with $s(w) = i$ and $t(w) = j$. The composition is extended bilinearly from the composition of paths.

At this point we should pause a little and look at the curious fact that we did not define the category of paths having as morphisms *just* the paths, as one might expect first. Indeed that would form a nice category also, but due to reasons which shall become clear in the next chapter, we "linearize" the paths such that we can take sums and multiples.

A category is a K**-category** if its morphism sets are endowed with a K-vector space structure such that the composition is K-bilinear.

A functor $F\colon \mathscr{C} \to \mathscr{D}$ between K-categories is K**-linear** if $\mathscr{C}(x,y) \to \mathscr{D}(Fx, Fy), h \mapsto Fh$ is K-linear for each pair of objects $x, y \in \mathscr{C}$. If $\mathscr{C}$ is a K-category then $\operatorname{mod}\mathscr{C}$ is the **category of** K**-linear functors** $\mathscr{C} \to \mathrm{vec}$, that is, the objects of $\operatorname{mod}\mathscr{C}$ are those functors and the morphisms are the morphisms of functors with the obvious composition. For two functors $F, G \in \operatorname{mod}\mathscr{C}$ we write $\operatorname{Hom}_{\mathscr{C}}(F,G)$ for the set of morphisms $(\operatorname{mod}\mathscr{C})(F,G)$.

Example 2.11. The path category KQ is a K-category. Moreover, each representation V of Q defines a K-linear (covariant) functor

$$\tilde{V}\colon KQ \to \mathrm{vec}\,.$$

Conversely, any such functor gives rise to a representation of Q. $\Diamond$

In a K-category $\mathscr{C}$ the **direct sum** of two objects x and y is defined as object z together with maps

$$\begin{tikzcd} x \arrow[r, shift right, "\iota_x"'] & z \arrow[l, shift right, "\pi_x"'] \arrow[r, shift left, "\pi_y"] & y \arrow[l, shift left, "\iota_y"] \end{tikzcd}$$

such that $\pi_x \iota_x = \mathrm{id}_x$, $\pi_y \iota_y = \mathrm{id}_y$, $\iota_x \pi_x + \iota_y \pi_y = \mathrm{id}_z$, $\pi_y \iota_x = 0$ and $\pi_x \iota_y = 0$. So, formally a direct sum is a quintuple $(z, \pi_x, \pi_y, \iota_x, \iota_y)$. However, the object z is –up to isomorphism– uniquely determined by x and y, see Exercise 2.4.6. This justifies the common abuse of language to call z itself the direct sum of x and y and denote it as $x \oplus y$.

Example 2.12. If $\mathscr{C} = \operatorname{rep} Q$, the categorical direct sum corresponds to the direct sum defined for representations above. Also in case $\mathscr{C} = \mathscr{M}_Q$ the categorical direct sum corresponds to the direct sum in the language of matrix problems. $\diamondsuit$

Exercises

2.4.1 Verify that the morphisms of representations are precisely the morphisms between covariant K-linear functors $KQ \to \mathrm{vec}$. Show that the category $\mathrm{mod}(KQ)$ is isomorphic to the category $\operatorname{rep} Q$.

2.4.2 If Q denotes the Kronecker quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd} \quad ,$$

then there are four morphism spaces in the category KQ. Determine the dimensions of these spaces. Determine $KQ(2,1)$ as set. How many elements does it have?

2.4.3 Let Q be a finite quiver. Show that different objects of KQ are non-isomorphic.

2.4.4 For a finite quiver Q, prove that all morphism spaces in KQ are finite-dimensional if and only if there is no cycle in the quiver Q.

2.4.5 The **adjacency matrix** A_Q of a quiver Q with vertices $1, \ldots, n$ is the matrix of size $n \times n$ whose entry $(A_Q)_{ij}$ is the number of arrows $\alpha \in Q_1$ with $s(\alpha) = j$ and $t(\alpha) = i$. Prove that A_Q is **nilpotent** (that is, there exists some positive integer t such that $A_Q^t = 0$) if and only if there is no cycle in Q. For this, show first, that for each t, the entry $(A_Q^t)_{ij}$ equals the number of paths w of length t with $s(w) = j$, $t(w) = i$.

Conclude from this that in case Q has no cycle then $A_Q^n = 0$, where n is the number of vertices. Furthermore show that the matrix $B = \mathbf{1}_n + A_Q + A_Q^2 + \ldots + A_Q^{n-1}$ measures the dimension of the morphism spaces in KQ, namely $B_{ij} = \dim_K (KQ(j,i))$.

2.4.6 Let $\mathscr{C}$ be a K-category. Suppose that the quintuples $(z, \pi_x, \pi_y, \iota_x, \iota_y)$ and $(z', \pi'_x, \pi'_y, \iota'_x, \iota'_y)$ are two direct sums of the objects x and y in $\mathscr{C}$. Proof that z is isomorphic to z'.

2.5 Equivalence of categories

As we have seen there is a very close relationship between the category of representations $\operatorname{rep} Q$ and the category of matrix problems $\mathscr{M}_Q$ associated to Q. In the following we would like to clarify this relationship completely.

We recall that two categories $\mathscr{C}$ and $\mathscr{D}$ are called **isomorphic** if there exist two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that $GF = 1_{\mathscr{C}}$ and $FG = 1_{\mathscr{D}}$.

However, this notion is in most concrete cases far too restrictive. It is more convenient to look at some slight generalization: two categories $\mathscr{C}$ and $\mathscr{D}$ are called **equivalent** if there exist two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that $GF \simeq 1_{\mathscr{C}}$ and $FG \simeq 1_{\mathscr{D}}$, that is, if there exists isomorphisms of functors $\varphi\colon GF \to 1_{\mathscr{C}}$ and $\psi\colon FG \to 1_{\mathscr{D}}$. In that case the functors F and G are called **equivalences** or **quasi-inverse to each other**.

Proposition 2.13. *Let Q be a finite quiver. Then the categories $\operatorname{rep} Q$ and $\mathscr{M}_Q$ are equivalent.*

Proof. We already have constructed the functor

$$F\colon \mathscr{M}_Q \longrightarrow \operatorname{rep} Q,$$

as application on the objects, see Example 2.10 (b). The definition on morphisms is straightforward.

To define a quasi-inverse G of F we choose a basis B^V for each representation V. Define $n_{V,i} = \dim V_i$ for each representation V and each vertex i. We recall that for each arrow $\alpha\colon i \to j$ we get a matrix $M_\alpha^V \in K^{n_{V,j} \times n_{V,i}}$ representing the linear map V_α in the bases B_i^V and B_j^V. Moreover, the tuple $G(V) = (M_\alpha^V)_{\alpha \in Q_1}$ defines an object of $\mathscr{M}_Q$. Furthermore, for each morphism $f\colon V \to W$ of representations of Q we define $G(f) = (U_i^f)_{i \in Q_0}$, where U_i^f is the matrix representing the linear map f_i in the bases B_i^V and B_i^W.

To see that

$$G\colon \operatorname{rep} Q \longrightarrow \mathscr{M}_Q$$

is a functor we have to verify that G preserves identity morphisms and the composition. Indeed $G(1_V) = 1_{G(V)}$ holds since the morphism 1_V is the family of identity maps $1_{V_i}\colon V_i \to V_i$ which are expressed as identity matrices, since for both spaces we choose the same basis B_i^V. For the composition, let U, V and W be representations of Q and $f\colon U \to V$ and $g\colon V \to W$ be morphisms of representations. Then, for a vertex i of the quiver, $G(f)_i$ is the matrix representing $f_i\colon U_i \to V_i$ in the basis B_i^U and B_i^V respectively. Similarly, $G(g)_i$ represents g_i in the Basis B_i^V and B_i^W respectively. Therefore $G(g \circ f)_i = G(g)_i G(f)_i$ holds for all i. This shows $G(g \circ f) = G(g)G(f)$ and finishes the proof that G is a functor.

It remains to see that F and G are quasi-inverse to each other. Furthermore, we denote by $\psi_{V,i}\colon K^{n_{V,i}} \to V_i$ the linear map representing the identity matrix in the canonical basis and B_i^V respectively. Note that $K^{n_{V,i}} = FG(V)_i$ holds by definition. To see that the family $\psi_V = (\psi_{V,i})_{i \in Q_0}$ defines a morphism $FG(V) \to V$, we have to verify that for each arrow $\alpha\colon i \to j$ the following diagram commutes:

$$\begin{tikzcd} FG(V)_i \arrow[r, "\psi_{V,i}"] \arrow[d, "FG(V)_\alpha"'] & V_i \arrow[d, "V_\alpha"] \\ FG(V)_j \arrow[r, "\psi_{V,j}"'] & V_j \end{tikzcd} .$$

We show this by looking at these maps in special bases: for V_i and V_j we chose the bases B_i^V and B_j^V respectively and for $FG(V)_i$ and $FG(V)_j$ we choose the canonical bases. The linear map V_α is then represented by the matrix $G(V)_\alpha = M_\alpha^V$, the maps $\psi_{V,i}$ and $\psi_{V,j}$ by identity matrices and $FG(V)_\alpha$ also by $G(V)_\alpha$. This shows that ψ_V is a morphism of representations. Since for each vertex $\psi_{V,i}$ is bijective, it is an isomorphism, see Exercise 2.3.1.

Thus we have now a family $(\psi_V)_{V \in \operatorname{rep} Q}$ of isomorphisms $\psi_V\colon FG(V) \to V$ of representations. To see that this family consitutes a morphism $FG \to 1_{\operatorname{rep} Q}$ of functors $\operatorname{rep} Q \to \operatorname{rep} Q$ it must be shown that for each morphism $f\colon V \to W$ of representations the following diagram on the left hand side commutes.

$$\begin{tikzcd} FG(V) \arrow[r, "\psi_V"] \arrow[d, "FG(f)"'] & V \arrow[d, "f"] \\ FG(W) \arrow[r, "\psi_W"'] & W \end{tikzcd} \qquad \begin{tikzcd} FG(V)_i \arrow[r, "\psi_{V,i}"] \arrow[d, "FG(f)_i"'] & V_i \arrow[d, "f_i"] \\ FG(W)_i \arrow[r, "\psi_{W,i}"'] & W_i \end{tikzcd}$$

By definition, this means that for each vertex i the diagram on the right hand side commutes. Indeed, in the bases B_i^V and B_i^W for V_i and W_i respectively and the canonical bases for $FG(V)_i$ and $FG(W)_i$, the linear maps f_i and $FG(f)_i$ are represented by $G(f)_i$, whereas $\psi_{V,i}$ and $\psi_{W,i}$ are represented by identity matrices.

To see that ψ is an isomorphism of functors we have to give an inverse. For this we use Exercise 2.3.2 to see that for each representation V, the family $\psi_V^{-1} = (\psi_{V,i}^{-1})_{i \in Q_0}$ is an isomorphism of representations which is an inverse of ψ.

To get an isomorphism $GF \to 1_{\mathscr{M}_Q}$ we could proceed very similarly. But we will choose a much simpler way by restricting the choice of the bases for each vector space of the form K^t to be always the canonical basis. It then happens that $GF(M) = M$ for each $M \in \mathscr{M}_Q$. Thus $GF = 1_{\mathscr{M}_Q}$ and φ is the identity morphism. $\square$

We have intentionally written down all the details in the proof to show how each categorical definition, which is involved, can be brought down in our setting to statements about linear maps and matrices representing them.

Exercises

2.5.1 Let Q be the quiver which has a single vertex and no arrows. Show that vec and rep Q are isomorphic categories. In this sense the study of representations of quivers generalizes linear algebra of finite-dimensional vector spaces.

2.5.2 Let Mat be the category whose objects are the natural numbers (including 0) and whose morphism spaces $\mathrm{Mat}(n, m)$ are the sets $K^{m \times n}$ of matrices of size $m \times n$ and entries in the field K. The composition in Mat is given matrix multiplication. Show that vec and Mat are equivalent categories.

2.6 A new example

We will consider a new class of problems starting from a family of quivers, which are called **linearly oriented**, and look as follows:

$$Q: \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \arrow[r, "\alpha_2"] & \underset{3}{\bullet} \arrow[r, dash] & \cdots \arrow[r] & \underset{n-2}{\bullet} \arrow[r, "\alpha_{n-2}"] & \underset{n-1}{\bullet} \arrow[r, "\alpha_{n-1}"] & \underset{n}{\bullet} \end{tikzcd}$$

We shall denote this quiver by $\overrightarrow{\mathbb{A}}_n$. The following result shows, that the classification problem can be solved completely for $\overrightarrow{\mathbb{A}}_n$.

Theorem 2.14. *Each indecomposable representation of $\overrightarrow{\mathbb{A}}_n$ is isomorphic to a representation*

$$[j, i]: \quad 0 \to \ldots \to 0 \to K \xrightarrow{[1]} \ldots \xrightarrow{[1]} K \to 0 \to \ldots \to 0,$$

where the first (that is, leftmost) occurrence of K happens in place j and the last (that is, rightmost) in place i for some $1 \leq j \leq i \leq n$.

In particular, $\overrightarrow{\mathbb{A}}_n$ is of finite representation type and there are $\frac{n(n+1)}{2}$ indecomposables, up to isomorphism.

Proof. Let V be an indecomposable representation of $\overrightarrow{\mathbb{A}}_n$. Let i be the minimal index such that V_{α_i} is not injective and set $i = n$ if no such index exists. Similarly, let j be the maximal index such that $V_{\alpha_{j-1}}$ is not surjective and set $j = 1$ if no such index exists. We shall show that V is isomorphic to $[j, i]$.

If $i < n$ then $V_{\alpha_1}, \ldots, V_{\alpha_{i-1}}$ are all injective, but V_{α_i} is not. Then we let S_i be a complement of $L_i = \operatorname{Ker} V_{\alpha_i}$, and set inductively $S_h = V_{\alpha_h}^{-1}(S_{h+1})$, $L_h = V_{\alpha_h}^{-1}(L_{h+1})$ for $h = i-1, i-2, \ldots, 1$. Note, that $S_h \oplus L_h = V_h$ for $h = 1, \ldots i$. We thus see that V decomposes into

$$(L_1 \to \ldots \to L_i \to 0 \to \ldots \to 0) \oplus (S_1 \to \ldots \to S_i \to V_{i+1} \to \ldots \to V_n)$$

and since V is indecomposable and $L_i \neq 0$ the right summand must be zero.

Thus we have shown so far that $V_h = 0$ for $h > i$ and that all maps V_{α_h} are injective for $h < i$. This implies that $j \leq i$. We observe that if $i = n$ then all these statements are trivially true or void.

If $j > 1$ then $V_{\alpha_j}, V_{\alpha_{j+1}}, \ldots, V_{\alpha_{i-1}}$ are surjective and hence bijective, but $V_{\alpha_{j-1}}$ is not. Let R_j be a complement of $M_j = V_{\alpha_{j-1}}(V_{j-1})$ and set inductively $M_h = V_{\alpha_{h-1}}(M_{h-1})$, $R_h = V_{\alpha_{h-1}}(R_{h-1})$, for $h = j+1, \ldots i$. We therefore conclude that V decomposes into

$$\begin{gathered}(0 \to \ldots \to 0 \to R_j \to \ldots \to R_i \to 0 \to \ldots \to 0) \oplus \\ (V_1 \to \ldots \to V_{j-1} \to M_j \to \ldots M_i \to 0 \to \ldots \to 0).\end{gathered}$$

The indecomposability of V implies now that the latter one is zero, since $R_j \neq 0$.

This shows that $V_h = 0$ for $h < j$ and that V_{α_h} is bijective for $h = j, \ldots, i-1$. We observe that in case $j = 1$ all these statements are trivially true or void. Thus V is isomorphic to

$$0 \to \ldots \to 0 \to K^d \xrightarrow{\mathbf{1}_d} \ldots \xrightarrow{\mathbf{1}_d} K^d \to 0 \to \ldots \to 0,$$

where d denotes the dimension of the spaces $V_j, \ldots, V_i$. But this representation is isomorphic to the direct sum of d copies of $[j, i]$. By the indecomposability of V it follows that $d = 1$ and that V is isomorphic to $[j, i]$. $\square$

Thus, we have determined the objects of the category rep $\overrightarrow{\mathbb{A}}_n$: they are, up to isomorphism, direct sums of the representations $[j, i]$. Now we turn our attention to morphisms between representations of $\overrightarrow{\mathbb{A}}_n$. If V and W are two representations, we first write them as direct sum of indecomposable representations, say $V \simeq \bigoplus_{a=1}^{s} V_a$ and $W \simeq \bigoplus_{b=1}^{t} W_b$, where each V_a and each W_b is of the form $[j, i]$ for some $1 \leq j \leq i \leq n$. A morphism $\varphi\colon V \to W$ is then given by a matrix of morphisms $\varphi_{ba}\colon V_a \to W_b$. Therefore, we are reduced to determine the morphisms between two indecomposable representations $[j, i]$ and $[j', i']$.

Lemma 2.15. *The morphism space* $\mathrm{Hom}([j, i], [j', i'])$ *is non-zero if and only if* $j' \leq j \leq i' \leq i$. *Moreover, in that case,* $\mathrm{Hom}([j, i], [j', i'])$ *is one-dimensional.*

Proof. Suppose that $\psi\colon [j, i] \to [j', i']$ is a morphism. Clearly, there is always the **zero morphism** with $\psi_h = 0$ for all h. If we suppose that ψ is not identically zero then the two intervals $[j, i]$ and $[j', i']$ must have some intersection, that is, $m = \max(j, j') \leq \min(i, i') = M$. For each h with $m \leq h \leq M$, the map ψ_h is just scalar multiplication with some factor λ_h. But if $m \leq h, h+1 \leq M$, then the commutative square

$$\begin{tikzcd} K \arrow[r, "{[1]}"] \arrow[d, "{[\lambda_h]}"'] & K \arrow[d, "{[\lambda_{h+1}]}"] \\ K \arrow[r, "{[1]}"'] & K \end{tikzcd}$$

shows that $\lambda_h = \lambda_{h+1}$. Hence, if there is a non-zero morphism then it is just a non-zero scalar multiple of the morphism $\iota = \iota_{j,i}^{j',i'} : [j, i] \to [j', i']$, where $\iota_h = [1]$ for each $h = \max(j, j'), \ldots, \min(i, i')$.

It remains to determine the condition, when a non-zero morphism ψ can exist. If $j < j'$ then we have a commuting square

$$\begin{tikzcd} K \arrow[r, "{[1]}"] \arrow[d, "\psi_{j'-1}"'] & K \arrow[d, "\psi_{j'}"] \\ 0 \arrow[r, "{[1]}"'] & K \end{tikzcd}$$

which shows that $\psi_{j'} = 0$ and consequently $\psi = 0$. Similarly, the case $i < i'$ is excluded. Since $m \leq M$ we get $j \leq i'$ and therefore $j' \leq j \leq i' \leq i$ is a necessary condition for $\operatorname{Hom}([j,i],[j',i'])$ to be non-zero and in that case this space is one-dimensional. $\square$

Hence we have

$$\operatorname{Hom}([j,i],[j',i']) = \begin{cases} K\,\iota_{j,i}^{j',i'}, & \text{if } j' \leq j \leq i' \text{ and } j \leq i' \leq i \\ 0, & \text{else.} \end{cases}$$

Notice that the maps $\iota_{j,i}^{j',i'}$ behave multiplicatively, that is $\iota_{j',i'}^{j'',i''} \circ \iota_{j,i}^{j',i'} = \iota_{j,i}^{j'',i''}$.

We call a non-zero morphism between two indecomposable representations **irreducible** if it cannot be written as a sum of compositions, where each composition consists of two non-isomorphisms between indecomposables.

Hence from $[j,i]$ there are, up to scalar multiples, at most two irreducible morphisms starting, namely $\iota_{j,i}^{j-1,i}$ (if $1 < j$) and $\iota_{j,i}^{j,i-1}$ (if $j < i$).

Putting everything together, we have the following diagram of irreducible maps between indecomposable representations in case $n = 5$.

$$\begin{tikzcd}[column sep=small]
 & & & & {[1,5]} \arrow[dr] & & & & \\
 & & & {[2,5]} \arrow[ur] \arrow[dr] & & {[1,4]} \arrow[dr] & & & \\
 & & {[3,5]} \arrow[ur] \arrow[dr] & & {[2,4]} \arrow[ur] \arrow[dr] & & {[1,3]} \arrow[dr] & & \\
 & {[4,5]} \arrow[ur] \arrow[dr] & & {[3,4]} \arrow[ur] \arrow[dr] & & {[2,3]} \arrow[ur] \arrow[dr] & & {[1,2]} \arrow[dr] & \\
{[5,5]} \arrow[ur] & & {[4,4]} \arrow[ur] & & {[3,3]} \arrow[ur] & & {[2,2]} \arrow[ur] & & {[1,1]}
\end{tikzcd} \tag{2.6}$$

Notice that the whole diagram is commutative, i.e. each square in it is commututaive. But there is still more structure inside of it which we shall discover in Section 6.3.

Exercises

2.6.1 Decompose the following representation V of $\overrightarrow{\mathbb{A}}_n$ into indecomposables: $V_i = K^2$ for all $i = 1, \ldots, n$ and $V_{\alpha_i} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for all $i = 1, \ldots, n-1$.

2.6.2 Show that if $f\colon V \to W$ is a morphism of representations of a quiver Q then $\operatorname{Im} f = (f_i(V_i))_{i \in Q_0}$ defines a **subrepresentation** of W, that is a family of subspaces $W_i' \subseteq W_i$ such that $W_\alpha(W_i') \subseteq W_j'$ for each arrow $\alpha\colon i \to j$. Explain how W_α defines $(\operatorname{Im} f)_\alpha$.

2.6.3 Use the previous exercise to prove that an irreducible morphism $f\colon V \to W$ between indecomposable representations satisfies that either all f_i are injective or all f_i are surjective.

Chapter 3

Algebras

Representations of quivers is a modern language which now will be connected to the classic one of modules over an algebra. This translation will take up the whole chapter. With this three different languages are developed for the same thing, each of which with a distinctive flavour.

3.1 Definition and generalities about algebras

What do matrix problems and representations of quivers have to do with algebras and modules? We will need the whole chapter to answer the question in a satisfactory way.

To fix notation, we start with the definition of an algebra and gradually see how the previous two chapters link to the new concepts.

All rings will be unitary unless explicitly stated. If A is a ring, its **center** is

$$\mathrm{Z}(A) = \{z \in A \mid az = za \text{ for all } a \in A\},$$

that is, $\mathrm{Z}(A)$ consists of those elements which commute with all elements of the ring. Note that by definition $\mathrm{Z}(A) = A$ holds if and only if the ring A is commutative. Furthermore, it is easy to see that $\mathrm{Z}(A)$ is closed under addition and multiplication, contains 0 and 1 and is closed under additive inverses. Therefore $\mathrm{Z}(A)$ is a subring of A.

Example 3.1. The center of $K^{n\times n}$ is $\{\lambda \mathbf{1}_n \mid \lambda \in K\}$, see Exercise 3.1.1. $\lozenge$

An **algebra** over a commutative ring R, or R-algebra for short, is a ring A

together with an inclusion $\iota\colon R \to Z(A)$, that is, an injective homomorphism of rings.

The inclusion ι induces the external multiplication

$$R \times A \to A, (r, a) \mapsto r \cdot a = \iota(r)a.$$

If R is a field then this implies that A is an R-vector space and thus has a well defined **dimension** over R. In most cases considered in this book, this dimension will be finite, but exceptions will occur also.

A **homomorphism** of R-algebras $(A, \iota_A) \to (B, \iota_B)$ is a ring-homomorphism $f\colon A \to B$ such that $\iota_B = f\iota_A$. It is common to write just $f\colon A \to B$ in that case and to omit the information on the inclusions if no confusion can arise. A homomorphism of R-algebras f is called an **isomorphism** if it is invertible: that is, there exists $g\colon B \to A$ such that $gf = \mathbf{1}_A$ and $fg = \mathbf{1}_B$.

In most cases we will describe an algebra over a field K by giving a K-vector space A together with a multiplication which has a unit 1_A. The inclusion ι is then meant to be $\lambda \mapsto \lambda 1_A$.

Let A be an R-algebra. Since A is in particular a ring, we can consider left and right A-modules and in fact these will be the A-modules, even if A is considered as algebra. If M is a left A-module and we restrict the multiplication $A \times M \to M, (a, m) \mapsto am$ to $K \times M \to M, (\lambda, m) \mapsto \iota(\lambda)m$, we see that M is a K-vector space. Hence the dimension of a module M is well defined and denoted by $\dim M$. In this book, we will study such modules as well as the algebras themselves.

Let A be a finite-dimensional K-algebra and M an A-module. A **submodule** U of M is by definition an abelian subgroup $U \subseteq M$ such that $au \in U$ for all $a \in A$ and all $u \in U$. If we restrict A to K we see that U is a subspace of M. The multiplication in M induces naturally a multiplication in the quotient space M/U by $a(m+U) := am+U$. Hence each submodule $U \subseteq M$ yields a **quotient module** M/U. Each K-algebra admits the zero space as module, which is called the **zero module** and denoted by 0.

Given a K-algebra A and two left (resp. right) A-modules M and N. Then each **homomorphism** $f\colon M \to N$ of left (resp. right) A-modules, that is, each homomorphims of abelian groups $f\colon M \to N$ which satisfies $af(m) = f(am)$ for all $a \in A$ and all $m \in M$ is automatically K-linear. Therefore the set of homomorphisms $\mathrm{Hom}_A(M, N)$ is a K-vector space. If both modules are finite-dimensional then also $\mathrm{Hom}_A(M, N)$ is finite-dimensional. We usually consider left modules. The category of finite-dimensional left A-modules will be denoted by $\operatorname{mod} A$.

If $M = N$, we also denote $\operatorname{End}_A(M) = \operatorname{Hom}_A(M, M)$ and call its elements **endomorphisms** of M. The space $\operatorname{End}_A(M)$ is clearly closed under sums and compositions. It contains the zero map and the identity map id_M as the neutral element of the sum and the composition respectively. Furthermore, these two binary operations satisfy the distribution laws. Also, the multiples $\lambda \mathrm{id}_M$ with $\lambda \in K$, commute with any other endomorphism. Hence $\operatorname{End}_A(M)$ is a K-algebra, called the **endomorphism algebra** of M.

If R and S are rings then a **R-S-bimodule** M is a left R-module which is also a right S-module such that $(rm)s = r(ms)$ for all $r \in R$, $s \in S$ and $m \in M$. If R and S are both K-algebras, then it follows that $\lambda m = (1_R\lambda)m = m(\lambda 1_S) = m\lambda$ for all $\lambda \in K$.

Exercises

3.1.1 Prove that $\mathrm{Z}(K^{n\times n}) = \{\lambda \mathbf{1}_n \mid \lambda \in K\}$.

3.1.2 Show that a homomorphism of R-algebras is an isomorphism if and only if it is bijective.

3.1.3 Show that the ring L of lower triangular matrices in $K^{n\times n}$ is a K-algebra; determine $\mathrm{Z}(L)$, the inclusion $K \to \mathrm{Z}(L)$ and the dimension of L over K.

3.1.4 Show that, if A is an R-algebra, then the set $\{r1_A \mid r \in R\}$ is contained in $\mathrm{Z}(A)$.

3.1.5 Explain why complex conjugation $\mathbb{C} \to \mathbb{C}, a + bi \mapsto a - bi$ is an isomorphism of $\mathbb{R}$-algebras, but not even a homomorphism of $\mathbb{C}$-algebras.

3.1.6 Let A be an R-algebra. Prove that any A-module M is also an R-module. In particular, if R is a field then M is an R-vector space.

3.1.7 Prove that there are, up to isomorphism, precisely two $\mathbb{C}$-algebras of dimension 2 over $\mathbb{C}$, namely $\mathbb{C} \times \mathbb{C}$ and $\mathbb{C}[X]/(X^2)$.

3.1.8 Let R be any ring. Observe that R can be considered as left R-module. The **endomorphism ring** $\operatorname{End}_R(R)$ consists then of all homomorphisms of abelian groups $\varphi\colon R \to R$ which satisfy $\varphi(rx) = r\varphi(x)$ for all $r, x \in R$. Prove that $f\colon \operatorname{End}_R(R) \to R^{\mathrm{op}}$ is an isomorphism of rings, where R^{op} is the **opposite ring** of R, that is $R^{\mathrm{op}} = R$ as abelian groups and the multiplication $*$ in R^{op} is given by $r * s = s \cdot r$, where the latter is the multiplication in R. Investigate what happens if R is considered right R-module. Prove then that $\operatorname{End}_R(R)$ is isomorphic to R.

3.2 The path algebra

We now want to associate an algebra A_Q, called **path algebra**, to a finite quiver Q. Recall that KQ denotes the path category, as explained in Section 2.4. As vector space, the algebra A_Q is defined as

$$A_Q = \bigoplus_{i,j \in KQ} KQ(i,j).$$

The multiplication in A_Q is obtained by extending the composition of paths bilinearly, setting $vw = 0$ whenever $s(v) \neq t(w)$.

Then we have that A_Q is an associative ring with unit $1_{A_Q} = \sum_{i \in KQ} (i||i)$. Moreover, the ground field acts centrally, that is, for $\lambda \in K$ and $a \in A_Q$, we always have $(\lambda 1_{A_Q})a = a(\lambda 1_{A_Q})$. Thus, A_Q is a K-algebra.

It is common to denote the path algebra A_Q as KQ, exactly like the path category. Note that A_Q in general is not commutative. For example $A_{\vec{\mathbb{A}}_2}$ is not commutative.

Examples 3.2 (a) Notice, that in $A_{\vec{\mathbb{A}}_n}$, there is a path from i to j if and only if $i \leq j$. Hence the algebra $A_{\vec{\mathbb{A}}_n}$ is isomorphic to the lower triangular matrices of size $n \times n$, under the mapping induced by

$$(j|\alpha_{j-1}, \ldots \alpha_i|i) \mapsto \mathbf{E}^{(ji)},$$

where $\mathbf{E}^{(ji)}$ is the $n \times n$-matrix, whose unique non-zero entry equals 1 and is located in the j-th row and i-th column.

(b) If Q is the Kronecker quiver, see Example 2.1 (b), then the algebra A_Q is called the **Kronecker algebra**. Note that it has dimension 4 over the field K. ◊

We now can start to relate modules over the path algebra A_Q with the representations of the quiver Q.

Theorem 3.3. *For any finite quiver Q, the category* $\operatorname{mod} A_Q$ *of finite-dimensional left modules over the path algebra A_Q is isomorphic to the category* $\operatorname{rep} Q$ *of representations of Q.*

Proof. Given any representation V of Q, we get a left A_Q-module

$$V' = \bigoplus_{i \in KQ} V_i$$

by defining the multiplication V'_w with a path $w = (j|\alpha_l, \dots \alpha_1|i)$ on a family $(v_h)_{h \in KQ}$ as the family having $V_{\alpha_l} \circ \ldots \circ V_{\alpha_1}(v_i)$ in the j-th coordinate and zero elsewhere. Notice that V' is finite-dimensional, since Q is finite and by definition a representation has finite-dimensional vector spaces attached to each vertex. Conversely, given a finite-dimensional left A_Q-module M, we define $M_i = e_i M = \{e_i m \mid m \in M\}$. We have then $M = \bigoplus_{i \in KQ} M_i$ and can easily define a representation by setting $M_\alpha \colon M_i \to M_j, e_i m \mapsto (j|\alpha|i)m$ for any arrow $\alpha \colon i \to j$ in Q.

If $\varphi \colon V \to W$ is a morphism of representations, then we define $\varphi' = \bigoplus_{i \in KQ} \varphi_i \colon V' \to W'$, which is a homomorphisms of A_Q-modules. Conversely, any homomorphism $\psi \colon M \to N$ of finite-dimensional A_Q-modules gives rise to a morphism of representations $(\psi_i)_{i \in Q_0}$, where $\psi_i \colon M_i \to N_i$, $e_i m \mapsto \psi(e_i m) = e_i \psi(m)$. $\square$

Remark 3.4. Since the category $\operatorname{mod} A_Q$ is isomorphic to $\operatorname{rep} Q$, the categorical concepts of direct sum and indecomposablity translate from $\operatorname{rep} Q$ to $\operatorname{mod} A_Q$. Explicitly, the **direct sum** of two modules M and N is given as $M \oplus N$ as vector space with componentwise multiplication and a non-zero module M is **indecomposable** if $M \simeq N \oplus N'$ implies $N = 0$ or $N' = 0$. As with representation we use the notation M^n to denote the direct sum $M \oplus \ldots \oplus M$ with n summands. $\lozenge$

Example 3.5. The algebra A of lower triangular matrices of size 5×5 is isomorphic to $A_{\vec{\mathbb{A}}_5}$. Hence by Theorem 3.3 and 2.14, there are 15 indecomposable A-modules. $\lozenge$

Exercises

3.2.1 Let Q be the quiver with one vertex and one arrow. Notice that this arrow necessarily must be a loop. Prove that the path algebra A_Q is isomorphic to the polynomial algebra in one variable $K[X]$.

3.2.2 Let A be the **two subspace algebra**, that is, the path algebra of the two subspace quiver, see Example 2.1 (a). Show that the algebra A is isomorphic to the algebra B, which consist of the matrices $M \in K^{3 \times 3}$ such that $M_{12} = M_{13} = M_{31} = M_{32} = 0$ with the usual matrix multiplication as multiplication in B. Give an explicit isomorphism, which maps each of the 5 paths in A to a matrix of the form $\mathbf{E}^{(ji)}$ for some i, j.

3.2.3 Determine the number of isomorphism classes of indecomposable A-modules of a fixed dimension d if A is the Kronecker algebra.

3.3 Quotients by ideals

We will show that to study the module categories of finite-dimensional algebras it is enough to consider quotients of path algebras, see Theorem 3.28. It will be useful to see such quotients right from the beginning in order to have concrete examples to illustrate the forthcoming theory.

A **left** (resp. **right**) **ideal** I of a K-algebra A is a subset of A, which defines a subgroup with respect to the addition such that $ai \in I$ (resp. $ia \in I$) for each $a \in A$ and each $i \in I$. A subset which is both a left and right ideal is called **both-sided ideal**, or just **ideal** for short. It follows straightforward from the definitions that each left (resp. right) ideal is always a K-subspace of A.

If I is an ideal of A, the set A/I is an abelian group as quotient of abelian groups. The multiplication in A induces a well-defined multiplication in A/I, precisely because of the ideal property.

Example 3.6. Let Q be the quiver with one vertex and one arrow α, which then must be a loop, see Exercise 3.2.1. Then let $I \subseteq A_Q$ be the ideal generated by the α^n. Then the algebra A_Q/I is isomorphic to the algebra $K[X]/(X^n)$. $\Diamond$

A **left** (resp. **right**) **ideal** of a category $\mathscr{C}$ is a family $(\mathscr{I}(x,y))_{x,y\in\mathscr{C}}$ of subsets $\mathscr{I}(x,y) \subseteq \mathscr{C}(x,y)$ such that for each $j \in \mathscr{I}(x,y)$, each object $z \in \mathscr{C}$ and each $c \in \mathscr{C}(y,z)$, (resp. each $d \in \mathscr{C}(t,x)$) the composition cj belongs to $\mathscr{I}(x,z)$ (resp. jd belongs to $\mathscr{I}(t,y)$). If the category $\mathscr{C}$ is a K-category, the ideal is assumed to consist of subspaces $\mathscr{I}(x,y) \subseteq \mathscr{C}(x,y)$ and not just subsets. Similary a (**both-sided**) **ideal** of a category $\mathscr{C}$ is a left ideal which is also a right ideal.

Example 3.7. Let $Q = \overrightarrow{\mathbb{A}}_n$ be the linearly oriented quiver with n vertices and $\mathscr{I} = (\mathscr{I}(j,i))_{i,j=1}^n$ be the family given by $\mathscr{I}(j,i) = K\overrightarrow{\mathbb{A}}_n(j,i)$ if $j-i > 2$ and $\mathscr{I}(j,i) = 0$ else. Then $\mathscr{I}$ is an ideal of $K\overrightarrow{\mathbb{A}}_n$. $\Diamond$

Whenever we have an ideal $\mathscr{I}$ of a K-category $\mathscr{C}$ we may consider the **quotient category** $\mathscr{C}/\mathscr{I}$, which has the same objects than $\mathscr{C}$, and morphism spaces $(\mathscr{C}/\mathscr{I})(x,y) = \mathscr{C}(x,y)/\mathscr{I}(x,y)$. That the composition of $\mathscr{C}/\mathscr{I}$, which is induced by the composition of $\mathscr{C}$, is well defined is due to the ideal property, as in the theory of rings, see Exercise 3.3.2.

Exercises

3.3.1 Determine $K\overrightarrow{\mathbb{A}}_n/\mathscr{I}$ if $\mathscr{I}$ is the ideal of $K\overrightarrow{\mathbb{A}}_n$ as defined in Example 3.7.

3.3.2 Let $\mathscr{I}$ be an ideal of a category $\mathscr{C}$. Verify that the composition in the quotient category $\mathscr{C}/\mathscr{I}$ is well defined.

3.3.3 Let A and B be two algebras and $f\colon A \to B$ an algebra homomorphism. Prove that $\operatorname{Ker} f = \{a \in A \mid f(a) = 0\}$ is always an ideal of A.

3.3.4 Let Q be a quiver with one vertex and two arrows α, β, which must be loops. Observe that there is an obvious algebra homomorphism $\pi\colon A_Q \to K[X,Y]$, such that $\pi(\alpha) = X$ and $\pi(\beta) = Y$. Show that π is surjective and determine its kernel $\pi^{-1}(0)$.

3.3.5 Let $\mathscr{C}$ be a K-category with finitely many objects and finite-dimensional morphism spaces and let $A_{\mathscr{C}} = \bigoplus_{x,y\in\mathscr{C}} \mathscr{C}(x,y)$. Show that $A_{\mathscr{C}}$ is an algebra and that $\operatorname{mod} A_{\mathscr{C}}$ and $\operatorname{mod} \mathscr{C}$ are isomorphic categories.

3.4 Idempotents

Not any K-algebra is isomorphic to a path algebra of some quiver. However, we will show that to any algebra A we may associate a quiver Q which will be of central importance. First, we give the general idea of how to work our way back from the algebra to a category, and then in a second step back to a quiver. We first outline this construction in the example of lower triangular matrices.

Example 3.8. Let A be the K-algebra of lower triangular $n \times n$-matrices. We constructed this algebra as path algebra of the linearly oriented quiver $\overrightarrow{\mathbb{A}}_n$ and would like to see how this quiver can be reconstructed from the algebra A. Which elements of A do correspond to the vertices and to the arrows of $\overrightarrow{\mathbb{A}}_n$? We have seen in Example 3.2 (a), that the elements $\mathbf{E}^{(ii)}$ for $i = 1, \dots, n$ are related to the vertices whereas $\mathbf{E}^{(i+1\,i)}$ for $i = 1, \dots, n-1$ are related to the arrows. We have now to describe these elements in terms of the multiplication: the elements $e = \mathbf{E}^{(ii)}$ satisfy $e^2 = e$, whereas the elements $y = \mathbf{E}^{(i+1\,i)}$ are vaguely those which cannot be obtained as products in A "without involving" some element $\mathbf{E}^{(jj)}$. $\diamondsuit$

An element e of an algebra A is called **idempotent** if $e^2 = e$. Note that 0 and 1 are always idempotents. If e is an idempotent then also $f = 1 - e$, since $f^2 = 1 - 2e - e^2 = 1 - 2e + e = f$. Furthermore, these two idempotents satisfy $ef = 0 = fe$ and $e + f = 1$. In general, two idempotents e and e' are said to be **orthogonal** if $ee' = 0 = e'e$. A non-zero idempotent e is **primitive** if for any two orthogonal idempotents e' and e'' with $e' + e'' = e$ we must have $e' = 0$ or $e'' = 0$. A set $\{e_1, \ldots, e_t\}$ of pairwise orthogonal idempotents is called **complete** if $\sum_{i=1}^{t} e_i = 1$.

Proposition 3.9. *In each finite-dimensional K-algebra A there exists a finite, complete set of pairwise orthogonal, primitive idempotents.*

Proof. If 1 is primitive the set is $\{1\}$ and we are done. Otherwise, there exist two non-zero, orthogonal idempotents e' and e'' such that $1 = e' + e''$. We set $L_1 = \{1\}$ and $L_2 = \{e', e''\}$and construct for $m \geq 3$ iteratively sets L_m which consist of m pairwise orthogonal, non-zero idempotents such that $\sum_{x \in L_m} x = 1$. This iteration continues as long as we find an idempotent $x \in L_m$ which is not primitive: we then find two non-zero, orthogonal idempotents y' and y'' such that $x = y' + y''$. Note that, $xy' = y'^2 + y'y'' = y' = y'x$ and therefore y' is orthogonal to any $z \in L_m \setminus \{x\}$ since $y'z = y'xz = 0$ and similarly $zy' = 0$. We then set $L_{m+1} = L_m \setminus \{x\} \cup \{y', y''\}$.

It remains to show that there exists some integer M for which each idempotent of L_M is primitive. For this, we consider the sets $xA = \{xa \mid a \in A\}$ for each $x \in L_M$. Note that xA is a K-vector space, that $xA \neq 0$ since $x \neq 0$ is contained in xA and that $A = \bigoplus_{x \in L_M} xA$ as vector spaces. The latter holds since $a = 1a = \sum_{x \in L_M} xa \in \sum_{x \in L_M} xA$ for each element $a \in A$ and that for each $x \in L_M$ we have $xA \cap \left(\sum_{z \neq x} zA\right) = 0$ since $\sum_{z \neq x} zA = (1 - z)A$ and the idempotents z, $1 - z$ are orthogonal.

Thus for each m, we have that $\dim_K A = \sum_{x \in L_m} \dim_K xA$. Since the spaces xA are non-zero and the dimension of A is finite, we must have $m \leq \dim_K A$. This finishes the proof. $\square$

Remark 3.10. Note that the proof shows that each complete set of pairwise orthogonal, primitive idempotents is finite. $\lozenge$

Once we have found such a complete set S of pairwise orthogonal, primitive idempotents we can define a category A_S: the objects form the set S and the morphism sets are given by $A_S(x, y) = yAx = \{yax \mid a \in A\}$ and the composition is induced by the multiplication of A. The category A_S is a K-category.

Proposition 3.11. *Let A be a finite-dimensional K-algebra and S a complete set of pairwise orthogonal, primitive idempotents. Then the category* $\operatorname{mod} A$ *of finite-dimensional left A-modules is equivalent to the category* $\operatorname{mod} A_S$ *of K-linear functors $A_S \to \mathrm{vec}$.*

Proof. For each finite-dimensional left A-module M, we can define a K-linear functor $\Phi(M)\colon A_S \to \mathrm{vec}$ by $\Phi(M)(x) = xM$ and $\Phi(M)(a)\colon M(x) \to M(y), m \mapsto am$ for each $a \in A_S(x,y)$. If $f\colon M \to N$ is a homomorphism of A-modules, then $f(xM) = xf(M) \subseteq xN$ and therefore f induces the K-linear maps $f_x\colon xM \to xN$. The family $(f_x)_{x\in S}$ satisfies $f_y\Phi(M)(a) = \Phi(N)(a)f_x$ for each $a \in A_S(x,y)$, that is, $(f_x)_{x\in S}$ is a morphism of functors $\Phi(f)\colon \Phi(M) \to \Phi(N)$.

Note that $A = \bigoplus_{x,y\in S} A_S(x,y)$. Therefore an element $a \in A$ can be written in unique way $a = \sum_{x,y\in S} a_{x,y}$ with $a_{x,y} \in A_S(x,y)$ for all $x,y \in S$. Hence, if $F\colon A_S \to \mathrm{vec}$ is a K-linear functor, then $\Psi(F) = \bigoplus_{x\in S} F(x)$ is an A-module via the multiplication $a\cdot(m_x)_{x\in S} = \left(\sum_{x\in S} a_{y,x}m_x\right)_{y\in S}$. A morphism $g\colon F \to G$ gives rise to a homomorphism $\Psi(g) = \bigoplus_{x\in S} g_x\colon \Psi(M) \to \Psi(N)$. It is easy to see that Φ and Ψ are inverse to each other. □

It will be convenient to have a different characterization for the primitiveness of an idempotent. A finite-dimensional algebra A is called **local** if each element $a \in A$ is either invertible or **nilpotent**, that is, there exists a positive natural number ℓ such that $a^\ell = 0$. We first need an auxiliary result concerning endomorphisms of an A-module.

Lemma 3.12. *Let A a K-algebra and M a finite-dimensional A-module. Then for each endomorphism $\varphi\colon M \to M$ there exists a natural number $n > 1$ such that $M = \operatorname{Im}\varphi^n \oplus \operatorname{Ker}\varphi^n$.*

Proof. The spaces $\varphi^m(M)$ form a chain

$$M = \varphi^0(M) \supseteq \varphi^1(M) \supseteq \varphi^2(M) \supseteq \dots$$

with decreasing dimensions. Therefore there exists an index n such that $\varphi^n(M) = \varphi^{n+1}(M)$. Consequently $\varphi^n(M) = \varphi^\ell(M)$ for all $\ell \geq n$, in particular for $\ell = 2n$. Denote $\psi = \varphi^n$. Then for each $m \in M$ there exists an element $m' \in M$ such that $\psi(m) = \psi^2(m')$. Consequently $m - \psi(m') \in \operatorname{Ker}\psi$ and hence $m = \psi(m') + (m - \psi(m')) \in \operatorname{Im}\psi + \operatorname{Ker}\psi$. Thus $M = \operatorname{Im}\psi + \operatorname{Ker}\psi$. If $m \in \operatorname{Im}\psi \cap \operatorname{Ker}\psi$ then $m = \psi(m')$ and therefore $0 = \psi(m) = \psi^2(m')$. But ψ induces a bijection $\psi(M) \to \psi^2(M) = \psi(M)$. Therefore $\psi^2(m') = 0$

implies $m = \psi(m') = 0$. This shows that $M = \operatorname{Im}\psi \oplus \operatorname{Ker}\psi$ as vector spaces. The same decomposition holds as A-modules since both $\operatorname{Im}\psi$ and $\operatorname{Ker}\psi$ are A-submodules. This concludes the proof of the assertion. □

Proposition 3.13. *Let A be a K-algebra. A finite-dimensional left (resp. right) A-module M is indecomposable if and only if $\operatorname{End}_A(M)$ is local.*

Proof. Assume M is an indecomposable left A-module and $\varphi \in \operatorname{End}_A(M)$. Then, by Lemma 3.12, there exists an integer $\ell \geq 0$ such that $M = \operatorname{Im}\varphi^\ell \oplus \operatorname{Ker}\varphi^\ell$. Since M is indecomposable either $\operatorname{Ker}\varphi^\ell = M$, that is, φ is nilpotent, or $\operatorname{Im}\varphi^\ell = M$, that is, φ^ℓ is an isomorphism. Since $\operatorname{End}_A(M)$ is a vector space of finite-dimension, the latter can only happen if φ itself is an isomorphism, hence invertible. This shows that $\operatorname{End}_A(M)$ is local.

Assume now that M is not indecomposable. Then there exist $M_1, M_2 \neq 0$ such that $M = M_1 \oplus M_2$. Denote by $\pi_1 \colon M \to M_1$ the canonical projection and by $\iota_1 \colon M_1 \to M$ the canonical inclusion. Then $\varphi = \iota_1\pi_1 \in \operatorname{End}_A(M)$ is idempotent, neither invertible nor nilpotent. Hence $\operatorname{End}_A(M)$ is not local.

The proof for right A-modules is completely similar. □

Proposition 3.14. *A finite-dimensional algebra A is local if and only if 1_A is primitive if and only if A is indecomposable as left (resp. right) A-module.*

Proof. If we consider A as right A-module, we know from Proposition 3.13, that A is indecomposable if and only if $\operatorname{End}_A(A)$ is local. Using Exercise 3.1.8, we see that $\operatorname{End}_A(A)$ is isomorphic to A. If we consider A as right A-module, then we see that A is indecomposable if and only if A^{op} is local. It follows from the definition of being local that A is local if and only if A^{op} is local. It remains to see that these conditions are equivalent to 1_A being primitive.

First suppose that $1 = 1_A$ is primitive. Assume that we have a decomposition of A as left A-module, say $A = M_1 \oplus M_2$, then we denote for $i = 1, 2$ by $\pi_i \colon A \to M_i$ the canonical projections, and by $\iota_i \colon M_i \to A$ the canonical inclusions. Then $1 = f_1 + f_2$, where $f_i = \iota_i\pi_i(1)$ for $i = 1, 2$. Note that f_i are idempotents $f_i^2 = \iota_i\pi_i(1)\iota_i\pi_i(1) = \iota_i\pi_i(\iota_i\pi_i(1)1) = f_i$, where the second equation follows from the fact that π_i and ι_i are homomorphisms of left A-modules and the last equation follows from $\pi_i\iota_i = \mathrm{id}_{M_i}$. Now, since 1 is primitive either $f_1 = 0$ or $f_2 = 0$ and consequently $M_1 = 0$ or $M_2 = 0$. This shows that A is indecomposable as left A-module.

Now suppose that $1 = f_1 + f_2$ where f_1 and f_2 are orthogonal idempotents. Then $A = A_1 \oplus A_2$, where $A_i = Af_i$ are left A-modules, which are non-zero since $f_i \in A_i$. Hence A is not indecomposable as left A-module. $\square$

Corollary 3.15. *An idempotent e of a finite-dimensional algebra A is primitive if and only if the algebra $eAe = \{eae \mid a \in A\}$ is local and this happens if and only if eAe is indecomposable as left (or right) eAe-module.*

Proof. This follows immediately from Proposition 3.14 since $e = 1_B$ for $B = eAe$. $\square$

Exercises

3.4.1 Let $B = K^{2\times 2}$ and $A = B^{2\times 2}$. Determine a complete set of pairwise orthogonal, primitive idempotents for the K-algebra A.

3.4.2 Show that a finite-dimensional algebra is local if and only if the nilpotent elements form an ideal. This is the general definition for a ring R. It is equivalent to each of the following conditions: (i) there exists a unique maximal left ideal; (ii) there exists a unique maximal right ideal; (iii) $0 \neq 1$ and for each $x \in R$ either x or $1 - x$ is invertible.

3.4.3 Show that for any positive integer n the algebra $K[X]/(X^n)$ is local.

3.4.4 Show that for $n > 1$ the algebra $K^{n\times n}$ is not local.

3.4.5 Let $A = K^{n\times n}$ and let $S = \{e_1, \dots, e_n\}$ where $e_i = \mathbf{E}^{(ii)}$. Show that any two idempotents e_i and e_j are isomorphic in the category A_S.

3.5 Morita equivalence

If A is a finite-dimensional algebra, then the elements of a complete set S of pairwise orthogonal, primitive idempotents of A are good candidates for the vertices of the quiver of A. Note that for each quiver Q, different vertices are non-isomorphic as objects of the category KQ, see Exercise 2.4.3. In the category A_S, this must not be the case, see Exercise 3.4.5.

An algebra A is called **basic** if there exists a complete set S of pairwise orthogonal, primitive idempotents, which are pairwise not isomorphic as objects of A_S.

It seems that the idea of finding such a complete set S of pairwise orthogonal, primitive idempotents was not successful after all. We have no chance of assigning a quiver Q to A such that the vertices of Q are S. Fortunately, there is an elegant way around that problem if we are primarily interested in the module categories. This motivates the following definition: two finite-dimensional algebras A and B are said to be **Morita equivalent**, if $\operatorname{mod} A$ and $\operatorname{mod} B$ are equivalent.

Proposition 3.16. *For any finite-dimensional algebra A there exists a basic finite-dimensional algebra B which is Morita equivalent to A.*

Proof. By Proposition 3.11, there exists a finite, complete set S of pairwise orthogonal, primitive idempotents. We will show by induction on the number of elements of S that there exists a basic algebra B which is Morita equivalent to A.

If the elements of S are pairwise non-isomorphic, then A is already basic and we are done setting $B = A$. Otherwise, there exist $e, f \in S$ which are isomorphic in A_S, that is, there exist $u \in A_S(e, f)$ and $v \in A_S(f, e)$ such that $uv = \mathbf{1}_f = f$ and $vu = \mathbf{1}_e = e$. We then define $A' = (1_A - f)A(1_A - f)$. Note that $S' = S \setminus \{f\}$ is a complete set of pairwise orthogonal, primitive idempotents for A'. Thus by the induction hypothesis A' is Morita equivalent to a basic algebra B.

All what remains to do is to show that A and A' are Morita equivalent, that is, we have to show that $\operatorname{mod} A$ and $\operatorname{mod} A'$ are equivalent. By Proposition 3.11 it is enough to show that the categories $\operatorname{mod} A_S$ and $\operatorname{mod} A'_{S'}$ are equivalent.

We define the functor $\Phi\colon \operatorname{mod} A_S \to \operatorname{mod} A'_{S'}$ just as the obvious restriction. To define $\Psi\colon \operatorname{mod} A'_{S'} \to \operatorname{mod} A_S$ we first observe that for any $M \in \operatorname{mod} A_S$, the map $M(u)\colon M(e) \to M(f)$ is bijective since it has an inverse, namely $M(v)$. Therefore we may define $\Psi(M)\colon A_S \to \mathrm{vec}$ on the objects by setting $\Psi(M)(f) = M(e)$ and $\Psi(M)(x) = M(x)$ for all $x \neq f$. On homomorphisms, $\Psi(M)$ is defined as follows:

$$\Psi(M)(\beta) = \begin{cases} M(\beta), & \text{if } s(\beta) \neq f \neq t(\beta), \\ M(\beta v), & \text{if } s(\beta) = f \neq t(\beta), \\ M(u\beta), & \text{if } s(\beta) \neq f = t(\beta), \\ M(u\beta v), & \text{if } s(\beta) = f = t(\beta). \end{cases}$$

Then for any $N \in \operatorname{mod} A_S$ there exists a natural isomorphism $\varphi_N\colon N \to \Psi\Phi(N)$ which is given by $\varphi_f = v$ and $\varphi_x = \mathbf{1}_{N(x)}$ for all $x \neq f$. These isomorphisms φ_N form a family which gives rise to an isomorphism of functors $\mathbf{1}_{\operatorname{mod} A_S} \to \Psi\Phi$. It is easy to see that $\Phi\Psi = \mathbf{1}_{\operatorname{mod} A'_{S'}}$. This finishes the proof. $\square$

Exercises

3.5.1 Prove that $K^{n\times n}$ is Morita equivalent to K for all $n \geq 1$.

3.6 The radical

We recall that if A is a finite-dimensional, local algebra then the nilpotent elements form an ideal $\operatorname{rad} A$, which is maximal as left and right ideal, see Exercise 3.4.2. The quotient $A/\operatorname{rad} A$ is then a skew field extension of K. The ideal is called **Jacobson radical** of A or just **radical** of A.

The radical plays a fundamental role in the forthcoming and therefore we carefully outline these concepts here. First we generalize this notion to arbitrary algebras. The **Jacobson radical** $\operatorname{rad} A$ of an algebra A is the intersection of all maximal left ideals.

Lemma 3.17. *Let A be a finite-dimensional algebra. Then an element x of A belongs to* $\operatorname{rad} A$ *if and only if* $1 - ax$ *is invertible for each $a \in A$.*

Proof. Assume first that x belongs to $\operatorname{rad} A$. Let $a \in A$ be any element. If $1 - ax$ is not invertible, then the left ideal generated by $1 - ax$ is properly contained in A. Since A is finite-dimensional, there exists a maximal left ideal $I \subset A$ such that $1 - ax \in I$. By definition $x \in I$ and hence $ax \in I$. Consequently $1 = (1 - ax) + ax \in I$, which implies $I = A$, a contradiction.

Assume now, that $1 - ax$ is invertible for each $a \in A$. Let $I \subset A$ be some maximal left ideal of A and $\pi \colon A \to A/I$ the canonical projection. Since I is maximal A/I is a field. If $x \notin I$, then $\pi(x) \neq 0$ and therefore there exists an inverse $\pi(a)$, that is, $\pi(ax) = \pi(1)$ and hence $1 - ax \in I$, which again is a contradiction. $\square$

Proposition 3.18. *The radical* $\operatorname{rad} A$ *of a finite-dimensional algebra A is a right ideal and equals the intersection of all maximal right ideals of A. Also $x \in \operatorname{rad} A$ if and only if* $1 - xa$ *is invertible for all $a \in A$.*

Proof. We denote by R' the intersection of all maximal right ideals of A. With a similar argument as given in the proof of Lemma 3.17, we see that $x \in R'$ if and only if $1 - xa$ is invertible for all $a \in A$.

Let $x \in \operatorname{rad} A$. We have to show that $xa \in \operatorname{rad} A$ for each $a \in A$. Indeed, if $xa \notin \operatorname{rad} A$ then by Lemma 3.17 there exists an element $b \in A$ such that $1 - bxa$ is not invertible. Consequently, there exists a maximal left ideal $I \subset A$ such that $1 - bxa \in I$. Since $z = bx \in \operatorname{rad} A$, $1 - az$ is invertible and hence $(1 - az)a = a(1 - za) \in I$ implies that $a \in I$ and then $1 = (1 - za) + za \in I$, a contradiction.

Therefore each $x \in \operatorname{rad} A$ satisfies that $1 - xa$ is invertible for all $a \in A$. This shows $\operatorname{rad} A \subseteq R'$. By a similar argument, R' is a left ideal and $R' \subseteq \operatorname{rad} A$. □

Let A be a finite-dimensional algebra and M a left A-module. Then the **radical** of M is defined as $\operatorname{rad} M = (\operatorname{rad} A)M$, that is, $\operatorname{rad} M$ consists of all elements of M which can be written as finite sums $\sum_{i=1}^{t} r_i m_i$ with $r_i \in \operatorname{rad} A$ and $m_i \in M$.

Remark 3.19. It follows from the definition that $\operatorname{rad}(M \oplus N) = (\operatorname{rad} M) \oplus (\operatorname{rad} N)$ for two A-modules M and N and that $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$ for each homomorphism $f\colon M \to N$. ◊

Proposition 3.20 (Nakayama's Lemma)**.** *If A is a finite-dimensional algebra and M a finite-dimensional left A-module, then* $\operatorname{rad} M$ *is a proper submodule of M.*

Proof. Let S be a subset of M. We say that S **generates** M or that S is a **generating set** for M, if the smallest submodule of M which contains S is M itself. This is equivalent to the fact that each M can be written as sum $\sum_{i=1}^{t} a_i s_i$ with $a_i \in A$ and $s_i \in S$.

Since M is finite-dimensional, any K-basis is a generating set for M. So, let now S be a generating set with minimal cardinality. Notice that S is finite, $S = \{m_1, \ldots, m_t\}$.

Now, we will prove that $\operatorname{rad} M$ is a proper subset of M by contradiction: assume that $\operatorname{rad} M = M$. Then $m_1 \in \operatorname{rad} M$. By the definition of $\operatorname{rad} M$, we have $m_1 = \sum_{j=1}^{\ell} r_j n_j$ for some $r_j \in \operatorname{rad} A$ and some $n_j \in M$. If we now express each n_j in the generating set S, we get $n_j = \sum_{i=1}^{t} a_{ij} m_i$ for some

$a_{ij} \in A$ and altogether

$$m_1 = \sum_{j=1}^{\ell} r_j \sum_{i=1}^{t} a_{ij} m_i = \sum_{i=1}^{t} \left(\sum_{j=1}^{\ell} r_j a_{ij} \right) m_i = \sum_{i=1}^{t} r_i' m_i. \tag{3.1}$$

Observe that the coefficient $r_i' = \sum_{j=1}^{\ell} r_j a_{ij}$ belongs to $\operatorname{rad} A$ since $\operatorname{rad} A$ is a right ideal by Proposition 3.18. Now, if $t = 1$, then we get from equation (3.1) that $(1 - r_1')m_1 = 0$. This implies $m_1 = 0$ since $1 - r_1'$ is invertible, a contradiction to the minimality of S. If $t > 1$, then equation (3.1) can be written as $(1 - r_1')m_1 = \sum_{i=2}^{t} r_i' m_i$, which again contradicts the minimality of S, since $1 - r_1'$ is invertible. Thus we have shown $m_1 \notin \operatorname{rad} M$ □

Let I be an ideal of A. Then the **powers** I^m are defined inductively $I^m = II^{m-1} = \{\sum_{i=1}^{t} a_i b_i \mid a_i \in I, b_i \in I^{m-1}\}$. Note that I^m is an ideal of A. The ideal will be called **nilpotent** if there exists an integer $m \geq 1$ such that $I^m = 0$.

Corollary 3.21. *The radical* $\operatorname{rad} A$ *of a finite-dimensional algebra* A *is a nilpotent ideal.*

Proof. Note that we have tow possible interpretation of $\operatorname{rad} A$, namely the Jacobson radical and the radical of A as left A-module. However, it follows directly from the definition that they coincide.

Let $I = \operatorname{rad} A$. By definition we have $I^m = \operatorname{rad} I^{m-1}$, that is, I^m equals the radical of I^{m-1}, viewed as left A-module. Hence by Nakayama's Lemma 3.20, I^m is a proper submodule of I^{m-1} and therefore a proper subspace. Since A is finite-dimensional, we must have $I^m = 0$ for $m \geq \dim_K A$. □

We now translate these concepts into the language of categories. We call a category $\mathscr{C}$ a **spectroid** if it satisfies the following three properties: (i) the morphism spaces $\mathscr{C}(x, y)$ are finite-dimensional, (ii) the endomorphism algebra $\mathscr{C}(x, x)$ is local for each object x and (iii) objects are pairwise non-isomorphic. A spectroid is **finite** if it has only finitely many objects.

Example 3.22. If A is a finite-dimensional algebra and S a complete set of pairwise orthogonal, primitive idempotents, then the category A_S is a finite spectroid. ◊

A morphism $f \in \mathscr{C}(x,y)$ of a K-category $\mathscr{C}$ is called **radical** if for each $g \in \mathscr{C}(y,x)$ the element $1_x - gf$ is invertible in the algebra $\mathscr{C}(x,x)$. The radical morphisms form an ideal which is called the **radical** of $\mathscr{C}$ and will be denoted by $\mathrm{rad}_{\mathscr{C}}$.

Example 3.23. If $\mathscr{C}$ is a spectroid then the radical $\mathrm{rad}\,\mathscr{C}$ is given by: the subspace $(\mathrm{rad}\,\mathscr{C})(x,x)$ is the Jacobson radical $\mathrm{rad}(\mathscr{C}(x,x))$ of the finite-dimensional algebra $\mathscr{C}(x,x)$ for each object x and $(\mathrm{rad}\,\mathscr{C})(x,y) = \mathscr{C}(x,y)$ for $x \neq y$ since the objects are pairwise non-isomorphic. It therefore makes sense to omit the parenthesis as both ways of interpretation of $\mathrm{rad}\,\mathscr{C}(x,y)$ amounts to the same thing. ◊

If $\mathscr{I}$ is an ideal of a K-category $\mathscr{C}$ its **powers** are defined by

$$\mathscr{I}^m(x,y) = \sum_{z \in \mathscr{C}} \mathscr{I}(z,y)\mathscr{I}^{m-1}(x,z).$$

Note that $\mathscr{I}^m$ is again an ideal of $\mathscr{C}$. We say that $\mathscr{I}$ is **nilpotent** if there exists some integer $m \geq 1$ such that $\mathscr{I}^m$ is the **zero ideal**, that is, $\mathscr{I}^m(x,y) = 0$ for all $x, y \in \mathscr{C}$. The following result will be important in the forthcoming.

Corollary 3.24. *If $\mathscr{C}$ is a finite spectroid, then $\mathrm{rad}\,\mathscr{C}$ is a nilpotent ideal.*

Proof. Let $x_1, \dots, x_t$ be the objects of $\mathscr{C}$. Then for each $i = 1, \dots, t$ the algebra $\mathscr{C}(x_i, x_i)$ is finite-dimensional and local. Therefore, by Corollary 3.21 there exists some $m_i \geq i$ such that $\mathrm{rad}^{m_i}\,\mathscr{C}(x_i,x_i) = 0$. Let $M = \max\{m_1, \dots, m_t\}$. We now argue that $\mathrm{rad}\,\mathscr{C}^\ell = 0$ for $\ell \geq (M+1)t$. Indeed, if $f = f_\ell \cdots f_1$ is a composition of ℓ radical morphisms with $f_j \in \mathscr{C}(x_{h_{j-1}}, x_{h_j})$ and $\ell > (M+1)t$, then at least one index i is repeated at least $M+1$ times in the sequence $h_0, h_1, \dots, h_\ell$. Hence f can be written in the form $f = v g_M g_{M-1} \cdots g_1 u$, where $u \in \mathscr{C}(x_{h_0}, x_i)$, $v \in \mathscr{C}(x_i, x_{h_\ell})$ and g_i are endomorphisms $g_i \in \mathscr{C}(x_i,x_i)$. Therefore $g_M \cdots g_1 = 0$ and consequently $f = 0$. □

Exercises

3.6.1 Show that the element $\mathbf{E}^{(ij)}$ for $i \neq j$ is nilpotent, but is not radical in the algebra $K^{n \times n}$. Determine $\mathrm{rad}(K^{n\times n})$.

3.6.2 Calculate the radical of the algebra of lower triangular matrices.

3.6.3 Show that for a quiver Q, the radical of the category KQ is the ideal generated by the arrows, that is, $(\operatorname{rad} KQ)(x, y)$ is the set of linear combinations of paths from x to y of positive length.

3.6.4 Let $\mathscr{C}$ be a K-category with finitely many objects and finite-dimensional morphism spaces. Denote by $A_{\mathscr{C}}$ the algebra $A_{\mathscr{C}} = \bigoplus_{x,y\in\mathscr{C}} \mathscr{C}(x, y)$, see Exercise 3.3.5. Show that if $\mathscr{I}$ is some ideal of $\mathscr{C}$ then $I = \bigoplus_{x,y\in\mathscr{C}} \mathscr{I}(x, y)$ is an ideal of $A_{\mathscr{C}}$. Prove that $I = \operatorname{rad} A_{\mathscr{C}}$ if $\mathscr{I} = \operatorname{rad}\mathscr{C}$ and more generally $I = \operatorname{rad}^m A_{\mathscr{C}}$ if $\mathscr{I} = \operatorname{rad}^m \mathscr{C}$. Also prove that $A_{\mathscr{C}/\mathscr{I}} = A_{\mathscr{C}}/I$ holds in general.

3.7 The quiver of an algebra

We now set out to define the quiver of an algebra A. For this we first construct a basic algebra B which is Morita equivalent to A. Let S be a complete set of pairwise orthogonal, primitive idempotents of B and B_S the category constructed in Section 3.4. The category B_S is a finite spectroid, see Example 3.22. In the following it is explained how to associate a quiver $Q_{\mathscr{C}}$ to any spectroid $\mathscr{C}$. The **quiver** Q_A of the algebra A is by definition the quiver Q_{B_S}. We will have to wait until Remark 4.27 to understand why Q_A does not depend on the choice of S.

Let $\mathscr{C}$ be a spectroid. For each pair of objects $x, y \in \mathscr{C}$, we choose morphisms

$$\alpha_1^{(x,y)}, \ldots, \alpha_{n_{xy}}^{(x,y)} \in \operatorname{rad}\mathscr{C}(x, y) \tag{3.2}$$

which become a K-basis in the vector space $\operatorname{rad}\mathscr{C}(x, y)/\operatorname{rad}^2\mathscr{C}(x, y)$ under the canonical projection. Note that by definition $\mathscr{C}(x, y)$ is finite-dimensional and therefore each space $\operatorname{rad}\mathscr{C}(x, y)/\operatorname{rad}^2\mathscr{C}(x, y)$ is also finite-dimensional. A morphism $f \in \mathscr{C}(x, y)$ is called **irreducible** if it is radical but does not belong to the radical square.

The **quiver** $Q_{\mathscr{C}}$ associated to the spectroid $\mathscr{C}$ has then the objects of $\mathscr{C}$ as its vertices and the morphisms (3.2) as arrows from x to y. Note that the canonical projection $\Pi\colon KQ_{\mathscr{C}} \to \mathscr{C}$ is always bijective on the objects.

Proposition 3.25. *If the field K is algebraically closed and the spectroid $\mathscr{C}$ is finite then the canonical projection $\Pi\colon KQ_{\mathscr{C}} \to \mathscr{C}$ is surjective on the morphism spaces.*

Proof. For each object x of $\mathscr{C}$ the algebra $\operatorname{End}_{\mathscr{C}}(x)$ is local and finite-dimensional. Hence $\operatorname{rad}\operatorname{End}_{\mathscr{C}}(x) = \operatorname{rad}\mathscr{C}(x, x)$ is a maximal ideal and therefore $\operatorname{End}_{\mathscr{C}}(x)/\operatorname{rad}\operatorname{End}_{\mathscr{C}}(x)$ a skew field extension of $K\overline{\mathbf{1}_x}$ which is finite-dimensional over K. Since K is assumed to be algebraically closed this implies that each element $a \in \operatorname{End}_{\mathscr{C}}(x)$ can be written uniquely in the form $a = \lambda \mathbf{1}_x + r$ where $\lambda \in K$ and $r \in \operatorname{rad} \operatorname{End}_{\mathscr{C}}(x)$. Observe that this already shows that it suffices to prove that each element of the radical $\operatorname{rad} \mathscr{C}$ belongs to the image of Π.

We now prove by induction on ℓ that for each two objects x, y each element g of $\operatorname{rad}^\ell \mathscr{C}(x,y)$ can be written as $g = g^{(\ell+1)} + \hat{g}^{(\ell)}$ where $g^{(\ell+1)} \in \operatorname{rad}^{\ell+1} \mathscr{C}(x,y)$ and $\hat{g}^{(\ell)}$ is a linear combination of compositions of ℓ arrows of the quiver, that is,

$$\hat{g}^{(\ell)} = \sum_{i=1}^{N} \lambda_i \rho_{i,\ell} \rho_{i,\ell-1} \cdots \rho_{i,1}$$

for some N, some coefficients $\lambda_i \in K$ and some arrows $\rho_{i,j} \in (Q_{\mathscr{C}})_1$ for $i = 1, \ldots, N$ and $j = 1, \ldots, \ell$.

The assertion is true for $\ell = 1$ by definition of the arrows (3.2) of $Q_{\mathscr{C}}$. Now assume the assertion to hold for $\ell - 1$ and let $g \in \operatorname{rad} \mathscr{C}(x,y)$, where x, y are two objects of $\mathscr{C}$. By the definition of the power of an ideal we have

$$g \in \sum_{z \in \mathscr{C}} \operatorname{rad} \mathscr{C}(z,y) \operatorname{rad}^{\ell-1} \mathscr{C}(x,z),$$

that is, there exist elements $\sigma_z \in \operatorname{rad} \mathscr{C}(z,y)$ and $\eta_z \in \operatorname{rad}^{\ell-1} \mathscr{C}$ such that $g_\ell = \sum_{z \in \mathscr{C}} \sigma_z \eta_z$. By induction hypothesis, for each z, we may write $\eta_z = \eta_z^{(\ell)} + \hat{\eta}_z^{(\ell-1)}$ and $\sigma_z = \sigma_z^{(2)} + \hat{\sigma}_z^{(1)}$. Hence

$$\begin{aligned} g &= \sum_{z \in \mathscr{C}} \sigma_z \eta_z \\ &= \underbrace{\sum_{z \in \mathscr{C}} \sigma_z \eta_z^{(\ell)} + \sum_{z \in \mathscr{C}} \hat{\sigma}_z^{(2)} \eta_z^{(\ell-1)}}_{= g^{(\ell+1)} \in \operatorname{rad}^{\ell+1} \mathscr{C}(x,y)} + \underbrace{\sum_{z \in \mathscr{C}} \hat{\sigma}_z^{(1)} \hat{\eta}_z^{(\ell-1)}}_{= \hat{g}^{(\ell)}} . \end{aligned}$$

This proves the induction argument and hence the result. $\square$

Remark 3.26. If K is not algebraically closed, the quiver $Q_{\mathscr{C}}$ does not encode all information of $\mathscr{C}$, because $\mathscr{C}(x,x)/\operatorname{rad} \mathscr{C}(x,x)$ might be some proper skew field extension over K and the quiver should be enhanced to a **species** (to each vertex a division algebra is attached and to each arrow a bimodule over the corresponding skew fields). Dlab and Ringel classified the species of finite representation type in [7] and those which are tame in [8]. $\Diamond$

In the situation of Proposition 3.25, we consider $\mathscr{I} = \operatorname{Ker} \Pi$ given by $\mathscr{I}(x, y) = \{g \in KQ_{\mathscr{C}}(x, y) \mid \Pi(g) = 0\}$. It is easy to see that $\mathscr{I}$ is an ideal of the category $KQ_{\mathscr{C}}$.

We call an ideal $\mathscr{I}$ **admissible** if there exists some $m \geq 2$ such that $\operatorname{rad}^m KQ_{\mathscr{C}} \subseteq \mathscr{I} \subseteq \operatorname{rad}^2 KQ_{\mathscr{C}}$.

Lemma 3.27. *Let K be an algebraically closed field, let $\mathscr{C}$ be a finite spectroid with quiver $Q_{\mathscr{C}}$ and denote by $\Pi \colon KQ_{\mathscr{C}} \to \mathscr{C}$ the canonical projection. Then* $\operatorname{Ker} \Pi$ *is an admissible ideal of the category $KQ_{\mathscr{C}}$.*

Proof. Recall from Exercise 3.6.3 that the radical of $KQ_{\mathscr{C}}$ is generated by the arrows.From the construction of the arrows of $Q_{\mathscr{C}}$, it follows that $\mathscr{I} \subseteq \operatorname{rad}^2 KQ_{\mathscr{C}}$. Also, by construction $\Pi(\operatorname{rad} KQ_{\mathscr{C}}) = \operatorname{rad} \mathscr{C}$ and therefore inductively $\Pi(\operatorname{rad}^m KQ_{\mathscr{C}}) = \operatorname{rad}^m \mathscr{C}$. Since $\operatorname{rad} \mathscr{C}$ is nilpotent, we conclude that there exists an integer $m \geq 2$ such that $\operatorname{rad}^m KQ_{\mathscr{C}} \subseteq \mathscr{I}$. $\square$

Putting together what we already know, we obtain the following result.

Theorem 3.28 (Gabriel)**.** *Let A be a finite-dimensional algebra over an algebraically closed field K. Then A is Morita equivalent to an algebra KQ/I, where I is an admissible ideal in the path algebra KQ of a finite quiver Q.*

Proof. By Proposition 3.16, there exists a basic algebra B such that A and B are Morita equivalent. Since B is basic, there exists a complete set S of pairwise orthogonal, primitive idempotents, such that B_S is a finite spectroid. Let Q be the quiver of the spectroid B_S and $\mathscr{I} = \operatorname{Ker} \Pi$, where $\Pi \colon KQ \to B_S$ is the canonical projection. By Proposition 3.25, Π is surjective on morphisms and by Lemma 3.27 the ideal $\mathscr{I}$ is admissible in KQ. Hence Π induces a functor $KQ/\mathscr{I} \to B_S$ which is bijective on objects and on morphisms, hence an isomorphism of categories.

Consequently we have the following equivalence of categories $\operatorname{mod} KQ/\mathscr{I} \simeq \operatorname{mod} B_S \simeq \operatorname{mod} B \simeq \operatorname{mod} A$, where the second equivalence is due to Proposition 3.11 and the last to Morita equivalence. Finally, if we denote by $A_Q = KQ$ the path algebra and by $I = \bigoplus_{x,y \in Q} \mathscr{I}(x, y) \subset A_Q$ the ideal which corresponds to $\mathscr{I}$, then I is admissible in A_Q and $\operatorname{mod} A_Q/I \simeq \operatorname{mod} KQ/\mathscr{I}$ by Exercsie 3.3.5 and 3.6.4. $\square$

Remark 3.29. Theorem 3.28 explains the expression "let $A = KQ/I$ be an algebra" which is quite common in the literature. $\lozenge$

Exercises

3.7.1 Let Q be a finite quiver. Show that every admissible ideal I is finitely generated.

3.8 Relations

Since admissible ideals are finitely generated, see Exercise 3.7.1, it is enough to give a finite set of generators $\rho_1, \dots, \rho_t$ which are usually written as elements of the path category, that is, each ρ_i is a linear combination of paths with the same starting vertex and the same terminating vertex. Such generators are called **relations**. Relations which are a single path are called **zero relation**. A quiver together with an admissible ideal (or a set of relations defining it) is called **bounded quiver**.

Remark 3.30. By Theorem 3.28 we know how to associate a quiver Q to an algebra A. As we shall see in Chapter 4 the quiver is uniquely determined by A. However, the relations are not uniquely determined by A. For example, let Q be the quiver as shown in the next picture.

$$\begin{tikzcd}
& \bullet \arrow[dr, "\gamma"] & \\
\bullet \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \bullet \\
& \bullet \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Clearly, the two sets of relations $R_1 = \{\gamma\alpha, \delta\beta\}$ and $R_2 = \{\gamma\alpha + \delta\beta, \delta\beta\}$ are different, but they generate the same ideal. But it can even happen that different ideals generate isomorphic algebras, for instance if $I_1 = \langle\gamma\alpha\rangle$ and $I_2 = \langle\delta\beta\rangle$. Then the two algebras KQ/I_1 and KQ/I_2 are isomorphic. $\Diamond$

Theorem 3.28 shows that to study modules over finite-dimensional algebras it is enough to study modules of quotients of path algebras. We can derive even more: it is enough to study quiver representations, as shows the next result. For this, we need some more terminology.

A covariant, K-linear functor $F\colon KQ \to \mathrm{vec}$ is said to **satisfy** the ideal $\mathscr{I}$ if $F(\varphi) = 0$ for each $\varphi \in \mathscr{I}$. If V is a representation of Q, then we say that V **satisfies** the ideal $\mathscr{I}$ if $V_\rho = 0$ for each $\rho \in \mathscr{I}$.

If $\mathscr{D}$ and $\mathscr{C}$ are categories, we call $\mathscr{D}$ a **subcategory** of $\mathscr{C}$ if the objects of $\mathscr{D}$ form a subclass of the objects of $\mathscr{C}$ and for each two objects x, y of $\mathscr{D}$, the set $\mathscr{D}(x,y)$ is a subset of $\mathscr{C}(x,y)$ and the composition of $\mathscr{D}$ is induced by the composition in $\mathscr{C}$, that is, if $f \in \mathscr{D}(x,y)$ and $g \in \mathscr{D}(y,z)$ then $g \circ_{\mathscr{D}} f = g \circ_{\mathscr{C}} f$, where $\circ_{\mathscr{D}}$ (resp. $\circ_{\mathscr{C}}$) denote the composition in $\mathscr{D}$ (resp. in $\mathscr{C}$). We write $\mathscr{D} \subseteq \mathscr{C}$ to indicate that $\mathscr{D}$ is a subcategory of $\mathscr{C}$.

A subcategory $\mathscr{D} \subseteq \mathscr{C}$ is called **full** if $\mathscr{D}(x,y) = \mathscr{C}(x,y)$ for any two objects x, y of $\mathscr{D}$. Note that in order to define a full subcategory $\mathscr{D}$ of $\mathscr{C}$ it is enough to specify the class of objects of $\mathscr{D}$.

Proposition 3.31. *The category* $\operatorname{mod} KQ/\mathscr{I}$ *is isomorphic to the full subcategory* $\operatorname{rep}_{\mathscr{I}} Q$ *of* $\operatorname{rep} Q$ *given by the representations which satisfy the ideal* $\mathscr{I}$.

Proof. The canonical projection functor $p\colon KQ \to KQ/\mathscr{I}$ defines a functor $p^*\colon \operatorname{mod}(KQ/\mathscr{I}) \to \operatorname{mod}(KQ)$, $F \mapsto F \circ p$. Note that p^* induces an isomorphism between $\operatorname{mod} KQ/\mathscr{I}$ and the full subcategory $\operatorname{mod}_{\mathscr{I}}(KQ)$ of $\operatorname{mod}(KQ)$ of functors which satisfy the ideal $\mathscr{I}$.

We know from Exercise 2.4.1 that $\operatorname{mod}(KQ)$ is isomorphic to $\operatorname{rep} Q$. Hence the full subcategory $\operatorname{mod}_{\mathscr{I}}(KQ)$ corresponds to a full subcategory of $\operatorname{rep} Q$, which clearly is $\operatorname{rep}_{\mathscr{I}} Q$. $\square$

Example 3.32. Consider the example, where $B = A/I$ is the quotient of the algebra of lower triangular matrices of size 5×5 and I the ideal generated by $\mathbf{E}^{(i,j)}$ for $i - j > 1$. We have then $I = \operatorname{rad}^2 A$, where $\operatorname{rad} A$ is the Jacobson radical of A. The quiver Q is the same for C_B as for C_A, namely linearly oriented with 5 vertices, and the admissible ideal is generated by the paths $\alpha_i \alpha_{i-1}$ for $i = 2, 3, 4$, indicated in the picture with dotted arcs.

$$Q: \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \xrightarrow{\alpha_3} \underset{4}{\bullet} \xrightarrow{\alpha_4} \underset{5}{\bullet}$$

Thus in $\operatorname{rep}_I Q$, we have only the following indecomposable representations up to isomorphism (with irreducible morphisms between them):

$$\begin{tikzcd}[column sep=small]
& {[4,5]} \arrow[dr] & & {[3,4]} \arrow[dr] & & {[2,3]} \arrow[dr] & & {[1,2]} \arrow[dr] & \\
{[5,5]} \arrow[ur] & & {[4,4]} \arrow[ur] & & {[3,3]} \arrow[ur] & & {[2,2]} \arrow[ur] & & {[1,1]}
\end{tikzcd} \tag{3.3}$$

because the others listed in (2.6) do not satisfy all the relations. $\lozenge$

Proposition 3.31 and the previous example could give the idea that the method of describing the modules over an algebra is straightforward process in three steps: first one determines the quiver Q of A and the admissible ideal I such that KQ/I is Morita equivalent to A, second one calculates all indecomposable representations of Q and in the third step one eliminates those representations which do not satisfy I. Indeed, theoretically this is correct. However, in practice it is not always easy to calculate Q and in most examples it will be completely impossible to achieve the second step: there are only few quivers for which this is possible, the vast majority are wild, that is, there are two-parameter families of pairwise non-isomorphic indecomposable representations for most quivers.

Exercises

3.8.1 How many indecomposable left A-modules exist if $A = K\overrightarrow{\mathbb{A}}_6/I$, where $I = \operatorname{rad}^4 K\overrightarrow{\mathbb{A}}_6$?

3.8.2 Define what formally is a "two-parameter family of indecomposables".

3.9 Summary

We worked quite hard to get Theorem 3.28, which is the main result of this chapter. We even worked harder than really necessary, because we developed ways back and forth between algebras and categories. Although this might seem unnecessary for now, it will pay off later, since we can apply the concepts in much broader context. In particular, we can speak of the radical of any K-category and know that the concept generalizes the Jacobson radical of an algebra.

In the special situation that K is an algebraically closed field and A a basic finite-dimensional K-algebra, Proposition 3.31 resumes that each A-module $M \in \operatorname{mod} A$ can be viewed as a representation of the quiver of A. You will have to wait until the end of the next chapter, see Remark 4.27, to see that this quiver is uniquely determined by A up to isomorphisms of quivers, where an **isomorphism of quivers** $\varphi\colon Q \to Q'$ is a pair of bijections $\varphi_i\colon Q_i \to Q'_i$ for $i = 0, 1$ such that $s_{Q'}(\varphi_1\alpha) = \varphi_0(s_Q\alpha)$ and $t_{Q'}(\varphi_1\alpha) = \varphi_0(t_Q\alpha)$ for all $\alpha \in Q_1$.

We have exposed three different languages for the same subject, and resume them for the case, when K is algebraically closed.

In the classical point of view we have **algebras** (to be thought as finite-dimensional over the field K) together with **modules** (again finite-dimensional). In the categorical point of view we consider finite **spectroids** together with functors (covariant and K-linear into the category of finite-dimensional vector spaces over K). In the combinatorial point of view, we consider a **bounded quiver** (a quiver together with an admissible ideal) and **representations** (of that quiver satisfying the given admissible ideal).

Thus, if A is such an algebra, $\mathscr{C} = \mathscr{C}_A$ the corresponding spectroid with quiver Q and admissible ideal I, we have that

$$\operatorname{mod} A \simeq \operatorname{mod} \mathscr{C} \simeq \operatorname{rep}_I Q$$

are three equivalent categories. It is common to speak of the quiver of an algebra or to look at "full subcategories of the algebra A", which just means that one is adopting the categorical point of view, where others would prefer to look at eAe, for e some idempotent, meaning exactly the same thing. In the forthcoming, we will freely move between these three languages.

If you compare different articles on representation theory you will find that none of what we have said is really standard; it begins with the fact that some authors favour the notation vw for the composition of two paths v from i to j and w from j to h (we denoted it as wv just as functions), some authors even go so far as to denote the composition of functions and homomorphisms in that way, although most of them denote functors and morphisms of categories as we do. It is also a matter of taste whether right or left modules are favoured, whereas most authors prefer covariant functors before contravariant ones, representations before corepresentations and equivalences before antiequivalences. The word "spectroid" is almost completely absent in the literature, but we will use it since it is useful.

As you can see and could have expected, representation theorists have expanded into this vast field of possibilities turning it into a political landscape, where a consensus clearly is not at hand. In some articles definitions are left sufficiently unspecific in order to turn it into a nice exercise for the reader to work his or her way through it to see whether a consistent interpretation is possible.

Chapter 4

Module categories

The focus is now changed towards modules. Several categorical notions for modules are developed. In each case a full characterization of indecomposable modules with that particular property is achieved. This is a first step towards understanding the structure of module categories.

4.1 The categorical point of view

The problem of determining the modules over a given algebra lies in the core of representation theory. It is though often useful to consider all modules or all modules up to isomorphisms together with all morphisms between them. The morphisms give structural insight, they are the flesh of the module category. Without them, the modules would just be a bare list of disconnected entities.

Recall that for a given finite-dimensional algebra A, we denote by $\operatorname{mod} A$ the category of all finite-dimensional left A-modules. We start with some easy remarks on $\operatorname{mod} A$.

In $\operatorname{mod} A$ the surjective and injective homomorphisms satisfy the following cancellation properties, whose proof is straightforward from the definition of surjectivity and injectivity.

Lemma 4.1 (Cancellation properties)**.** *Let $f\colon M \to N$ be a homomorphism of left A-modules. Then f is surjective if and only if for any two homomorphisms $g_1, g_2\colon N \to P$ we have that $g_1 f = g_2 f$ implies $g_1 = g_2$. Similarly, f is injective if and only if for any two homomorphisms $g_1, g_2\colon L \to M$ we have that $f g_1 = f g_2$ implies $g_1 = g_2$.* □

The cancellation conditions stated in Lemma 4.1 do not involve elements of the module M or N. Instead they use the relationship of f with respect to all other morphisms. The cancellation properties can be stated in every category. A morphism $f \in \mathscr{C}(x, y)$ for which $g_1 f = g_2 f$ always implies $g_1 = g_2$ is called an **epimorphism** (or just **epi** for short), whereas if $f g_1 = f g_2$ always implies $g_1 = g_2$ then f is called a **monomorphism** (or **mono**). Thus, "injectivity" is an element-based property whereas mono is a categorical property. We shall meet many categorical properties in this chapter and it will be useful to see what they mean in different categories.

Remark 4.2. In the categorical point of view all properties have to be expressed using morphisms only. While it is instructive to rephrase properties in categorical language, since it broadens the view to greater generality, it also can conduct to excess in abstraction loosing the grip of concrete examples. Caution is thus required to achieve a reasonable balance between both views. $\diamondsuit$

Exercises

4.1.1 Let Q be a finite quiver and $f\colon V \to W$ a morphism of representations of Q. Show that f is mono if and only if f_x is injective for each vertex x of Q. Similarly show that f is epi if and only if f_x is surjective for all x.

4.1.2 Let A be a finite-dimensional K-algebra. Show that a morphism $f\colon M \to N$ is epi (resp. mono) in $\operatorname{mod} A$ if and only if it is surjective (resp. injective). Use that the image $B = f(M)$ is a submodule and that the quotient N/B is again an A-module. Then compare the canonical projection $\pi\colon N \to N/B$ with the zero map $0\colon N \to N/B$.

4.1.3 Prove that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category of rings. This shows, that even in **concrete** categories, where the objects are sets with some structure and morphisms are functions, which preserve that structure, epimorphisms must be surjective.

4.2 Duality

We first look at another categorical notion: an object $x \in \mathscr{C}$ is called **initial object** if for each object $y \in \mathscr{C}$ the morphism set $\mathscr{C}(x, y)$ consists of exactly one element. Similarly, x is called **terminal object** if for each $y \in \mathscr{C}$ the set $\mathscr{C}(y, x)$ has exactly one element.

As you can see, categorical notions come in pairs which correspond to each other: monomorphisms correspond to epimorphims and initial objects to terminal objects. If you look at the definitions, you should see that one can be obtained from the other by somehow reversing the direction of the arrows. In the following we will formalize this.

The **opposite category** $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$ is defined as follows: it has the same objects as $\mathscr{C}$, its morphism sets are $\mathscr{C}^{\mathrm{op}}(x,y) = \mathscr{C}(y,x)$ and the composition is given by

$$\mathscr{C}^{\mathrm{op}}(y,z) \times \mathscr{C}^{\mathrm{op}}(x,y) \to \mathscr{C}^{\mathrm{op}}(x,z), (g,f) \mapsto g \circ^{\mathrm{op}} f = f \circ g.$$

We then see, that a morphism $f\colon x \to y$ is mono in $\mathscr{C}$ if and only if $f\colon y \to x$ is epi in $\mathscr{C}^{\mathrm{op}}$. Similarly an object x is initial in $\mathscr{C}$ if and only it is terminal in $\mathscr{C}^{\mathrm{op}}$.

Defining the opposite category has economical advantages, since we only have to define half of the concepts: instead of stating the definition of terminal objects we could have said that terminal is **the dual concept** to initial.

However, it is not straightforward how we could dualize properties in a concrete category like $\operatorname{mod} A$, as it is not clear that $(\operatorname{mod} A)^{\mathrm{op}}$ is again a module category. The following result solves this problem. The **opposite algebra** A^{op} is A as abelian group, the multiplication $a \cdot^{\mathrm{op}} b = ba$, where the latter denotes the usual product in A. Furthermore, A^{op} has the same inclusion of K in the center $\mathrm{Z}(A^{\mathrm{op}}) = \mathrm{Z}(A)$. Furthermore, for a vector space V we denote by $\mathrm{D}V = \mathrm{D}\operatorname{Hom}_K(V,K)$ the **dual space**.

Proposition 4.3. *Let A be a finite-dimensional K-algebra. Then the dualization yields an isomorphism of categories* $\mathrm{D}\colon (\operatorname{mod} A)^{\mathrm{op}} \to \operatorname{mod}(A^{\mathrm{op}})$.

Proof. For each left A-module M, define $\Phi(M) = \mathrm{D}M = \operatorname{Hom}_K(M,K)$, the dual space of M. Now define the multiplication

$$A^{\mathrm{op}} \times \mathrm{D}M \to \mathrm{D}M, (a,\varphi) \mapsto a\varphi$$

by $a\varphi\colon M \to K, m \mapsto \varphi(am)$. Clearly this multiplication satisfies the distribution laws $(a+b)\varphi = a\varphi + b\varphi$ and $a(\varphi + \psi) = a\varphi + a\psi$. Also

$$(a(b\varphi))(m) = (b\varphi)(am) = \varphi(bam) = (ba)\varphi(m) = (a \cdot^{\mathrm{op}} b)\varphi(m).$$

Note that for each finite-dimensional vector space M the double dual $\mathrm{DD}M$ is canonically isomorphic to M, via the map $M \to \mathrm{DD}M$, $m \mapsto (\varphi \mapsto \varphi(m))$. Hence we see that Φ has an inverse which is also given as dualization. □

Remarks 4.4 (a) Since for infinite-dimensional vector spaces V the double dual $\mathrm{DD}V$ is not isomorphic to V, the category $(\mathrm{Mod}\, A)^{\mathrm{op}}$, that is, the opposite of the category of *all* A-modules, cannot be expected to be isomorphic to $\mathrm{Mod}\, A^{\mathrm{op}}$. In fact, one can show that $(\mathrm{Mod}\, A)^{\mathrm{op}}$ is never a module category.

(b) If we view A as a category A_S for $S = \{e_1, \dots, e_t\}$ some complete set of pairwise orthogonal, primitive idempotents, then we can view A^{op} simply as $(A_S)^{\mathrm{op}}$ which is the same as $(A^{\mathrm{op}})_S$. Therefore, if Q is the quiver of A, then A^{op} has the **opposite quiver** Q^{op} of Q — it has the same vertices as Q but the direction of the arrows is reversed, that is, the two functions s and t of Q are interchanged.

An A^{op}-module $M \in \mathrm{mod}(A^{\mathrm{op}})$, if viewed as representation of Q^{op}, is therefore mapped to the representation $\mathrm{D}M$ of Q which has the space $\mathrm{D}(M_i)$ assigned to the vertex i and if $\alpha\colon i \to j$ is an arrow in Q, then $\alpha^{\mathrm{op}}\colon j \to i$ is an arrow of Q^{op} and

$$(\mathrm{D}M)_\alpha = \mathrm{D}(M_{\alpha^{\mathrm{op}}})\colon \mathrm{D}M_i \to \mathrm{D}M_j, \varphi \mapsto \varphi \circ M_{\alpha^{\mathrm{op}}}$$

describe the linear maps of the representation associated to the arrows of Q. ◊

Exercises

4.2.1 Determine the objects in the category of sets which are initial and also those which are terminal.

4.2.2 Show that the zero module, which is the zero vector space, is both initial and terminal in the category $\mathrm{mod}\, A$.

4.3 Kernels and cokernels

An object z of a category $\mathscr{C}$ is called **zero object** if for each $x \in \mathscr{C}$ the sets $\mathscr{C}(x, z)$ and $\mathscr{C}(z, x)$ contain precisely one element. If there exists a zero object in $\mathscr{C}$, then for any pair x, y of objects in $\mathscr{C}$ there exists a distinguished element, called the **zero morphism** and denoted by 0, which is obtained by composing the unique element of $\mathscr{C}(x, z)$ with that of $\mathscr{C}(z, y)$.

Example 4.5. For each algebra A the module category contains a zero object: the space $\{0\}$, called the **zero module**, denoted by 0. ◊

Let $\mathscr{C}$ be a category with a zero object. Let $f \in \mathscr{C}(x, y)$. Then a **kernel** of f is a map $g \in \mathscr{C}(t, x)$ such that $fg = 0$ satisfying the universal property that, if $g' \in \mathscr{C}(t', x)$ with $fg' = 0$, then there exists a unique $h \in \mathscr{C}(t', t)$ such that $gh = g'$. This is usually expressed as a diagram as shown in the following picture.

$$\begin{tikzcd}
t \arrow[r, "g"] & x \arrow[r, "f"] & y \\
t' \arrow[u, dashed, "\exists! h"] \arrow[ur, "g'"'] & &
\end{tikzcd}$$

The existence of kernels is not generally guaranteed. There are categories where kernels exist always and others where they do not, see Exercise 4.3.1. Let $f\colon x \to z$ is a morphisms in a category $\mathscr{C}$ and y an object of $\mathscr{C}$. We say that f **factors over** y if there exist homomorphisms $g\colon x \to y$ and $h\colon y \to z$ such that $f = hg$. If $g\colon x \to y$ is given, then we say that f **factors through** g and call h a **factorization**.

Example 4.6. In the module category $\operatorname{mod} A$ each morphism $f\colon M \to N$ admits the inclusion $\iota\colon f^{-1}(0) \to M$ as kernel: First, note that the set $f^{-1}(0) = \{m \in M \mid f(m) = 0\}$ is a submodule of M. Clearly $f\iota = 0$ holds and if $g'\colon L' \to M$ satisfies $fg' = 0$ then $f(L) \subseteq f^{-1}(0)$ yields the the desired factorization through ι which is unique since ι is injective. $\Diamond$

The dual concept of kernel is **cokernel**. Using the existence of kernels in $\operatorname{mod} A$, that is, the existence of a kernel for each morphism, we get from Proposition 4.3 that there exists also a cokernel for each morphism. Although this might seem satisfactory from a categorical point of view, we shall describe the construction of cokernels in $\operatorname{mod} A$ explicitely.

Lemma 4.7. *For each morphism $f\colon M \to N$ in $\operatorname{mod} A$ the canonical projection $\pi\colon N \to N/\operatorname{Im} f$ is a cokernel of f, where $\operatorname{Im} f = \{f(m) \mid m \in M\}$ is the* **image** *of f.*

Proof. Clearly $\pi f = 0$. To verify the universal property, let $h\colon N \to P$ be a morphism with $hf = 0$. Then we have $h(n) = 0$ for each $n \in \operatorname{Im} f$. Therefore, the K-linear map $h'\colon N/\operatorname{Im} f \to P$, $h'(n + \operatorname{Im} f) = h(n)$ is well defined. This map is also clearly A-linear and satisfies $h'\pi = h$. That h' is unique with that property follows from the fact that π being surjective is epi. $\square$

Lemma 4.8. *Let $\mathscr{C}$ be a category with zero object and $f \in \mathscr{C}(x,y)$ a morphism. If $g_1 \in \mathscr{C}(t_1,x)$ and $g_2 \in \mathscr{C}(t_2,x)$ are two kernels of f then t_1 and t_2 are isomorphic.*

Proof. By the universal property of g_1 there exists a unique $h_1 \in \mathscr{C}(t_2,t_1)$ such that $g_1h_1 = g_2$. Similarly, there exists a unique $h_2 \in \mathscr{C}(t_1,t_2)$ such that $g_2h_2 = g_1$. Now $h_1h_2 \in \mathscr{C}(t_1,t_1)$ satisfies $g_1(h_1h_2) = g_1 = g_1\mathbf{1}_{t_1}$. Again by the universal property of g_1 we must have $h_1h_2 = \mathbf{1}_{t_1}$. Similarly $h_2h_1 = \mathbf{1}_{t_2}$. This shows that t_1 and t_2 are isomorphic. □

Because of this result, often the object t is called kernel instead of the morphism $g \in \mathscr{C}(t,x)$. Dualizing we obtain that two cokernels are isomorphic. We thus see that in $\operatorname{mod} A$ each morphism has a kernel and a cokernel. It is common to use the notation $\operatorname{Ker} f$ (resp. $\operatorname{Coker} f$) for a chosen kernel (resp. cokernel) of a morphism f.

Lemma 4.9. *Let $f\colon M \to N$ be a morphisms in* $\operatorname{mod} A$*. Then f is injective if and only if* $\operatorname{Ker} f = 0$ *and dually, f is surjective if and only if* $\operatorname{Coker} f = 0$.

Proof. If f is injective, then each fiber consists of at most one element. Since $f(0) = 0$ always holds, $\operatorname{Ker} f = f^{-1}(0) = \{0\} = 0$ follows. Conversely, if $f^{-1}(0) = 0$, then $f(m) = f(m')$ implies $m - m' \in f^{-1}(0)$. This shows that f is injective.

We know from Lemma 4.1 that f is injective if and only if f is a monomorphism. Hence f is a monomorphism if and only if $\operatorname{Ker} f = 0$. Since these are categorical properties we can "dualize" them and get the result: the morphism $\mathrm{D}f\colon \mathrm{D}N \to \mathrm{D}M$ in the category $\operatorname{mod} A^{\mathrm{op}}$ is a monomorphism if and only if $\operatorname{Ker}(\mathrm{D}f) = 0$. But $\mathrm{D}f$ is a monomorphism if and only if f is an epimorphism, which we know is equivalent to f being surjective. Similarly, $\operatorname{Ker}(\mathrm{D}f) = 0$ holds if and only if $\operatorname{Coker} f = 0$. □

For $f\colon M \to N$ we choose a fixed kernel and denote it by $\operatorname{Ker} f$. Similarly, we choose a cokernel and denote it by $\operatorname{Coker} f$.

We now can iterate or combine this two constructions. If $g\colon U \to M$ is a kernel of $f\colon M \to N$ and $h\colon V \to U$ is a kernel of g then we clearly have $f(gh) = 0$. It follows therefore from the universal property of g that $h = 0$. But then $V \simeq 0$ follows, since $h0 = h1_V$ implies by the universal property of h that $0 = 1_V$. This shows that the kernel of a kernel is always zero. Dually the cokernel of a cokernel is zero.

A category $\mathscr{C}$ is **additive** if the morphism sets are abelian groups such that the composition is bilinear, there exists a zero object and for any two objects in $\mathscr{C}$ there exists a direct sum. In an additive category, kernels are mono: if $g\colon U \to M$ is a kernel of $f\colon M \to N$ then for any two morphisms $h, h'\colon W \to U$ with $gh = gh'$ we have $g(h-h') =$ and thus by the uniqueness of the factorization $h - h' = 0$, showing that g is mono. Dually cokernels are epi in an additive category.

Lemma 4.10. *Let $\mathscr{C}$ be a category in which each morphism has a kernel and a cokernel. Then, for each morphism $f\colon M \to N$ there exists a canonical morphism $f_\iota\colon$ $\operatorname{Coker}(\operatorname{Ker} f) \to \operatorname{Ker}(\operatorname{Coker} f)$.*

Proof. Denote by $g\colon L \to M$ a kernel of f and by $h\colon N \to P$ a cokernel of f. Now we use the universal property of the cokernel $\operatorname{Coker} g\colon M \to M'$: since $fg = 0$ there exists a unique $\xi\colon M' \to N$ such that $\xi \circ \operatorname{Coker} g = f$, see the following picture for illustration.

$$\begin{tikzcd}
L \arrow[r, "g"] & M \arrow[r, "f"] \arrow[d, "\operatorname{Coker} g"'] & N \arrow[r, "h"] & P \\
& M' \arrow[ur, "\xi"] \arrow[r, "f_\iota"] & N' \arrow[u, "\operatorname{Ker} h"'] &
\end{tikzcd}$$

Since $\operatorname{Coker} g$ is an epimorphism, we get from $h\xi \operatorname{Coker} g = hf = 0$ that $h\xi = 0$. Using now the universal property of $\operatorname{Coker} h\colon N' \to N$, we obtain a unique morphism $f_\iota\colon M' \to N'$ such that $\operatorname{Coker} h \circ f_\iota = \xi$. $\square$

The morphism (or the object) $\operatorname{Ker}(\operatorname{Coker} f)$ is called **image** of f and denoted by $\operatorname{Im} f$. In the category $\operatorname{mod} A$ the image $\operatorname{Im} f$ of a morphism $f\colon M \to N$ is isomorphic to the inclusion $f(M) \to N$. Similarly $\operatorname{Coker}(\operatorname{Ker} f)$ is called **coimage** and denoted $\operatorname{Coim} f$.

Lemma 4.11. *If $\mathscr{C} = \operatorname{mod} A$ then for each morphism f the canonical morphism of Lemma 4.10 is the isomorphism $f_\iota\colon M/f^{-1}(0) \to \operatorname{Im} f$.*

Proof. We then know that $\operatorname{Ker} f$ is isomorphic to the inclusion of $f^{-1}(0)$ in M and consequently $\operatorname{Coker}(\operatorname{Ker} f)\colon M \to M/f^{-1}(0)$ is the canonical projection. We also know that $\operatorname{Coker} f$ is isomorphic to the canonical projection $N \to N/\operatorname{Im} f$. Therefore $\operatorname{Ker}(\operatorname{Coker} f)$ is isomorphic to the inclusion $\operatorname{Im} f \to N$. By construction, $f_\iota\colon M/f^{-1}(0) \to \operatorname{Im} f$ is given by $f_\iota(m + f^{-1}(0)) = f(m)$. Clearly f_ι is injective and surjective. Hence the result follows. $\square$

A category $\mathscr{C}$ is **abelian** if it is additive and for any morphism exists a kernel, a cokernel and the canonical map from a coimage to an image is an isomorphism. We have seen that the category $\operatorname{mod} A$ is abelian.

Given two morphisms with the same codomain, say $f_X\colon X \to Z$ and $f_Y\colon Y \to Z$. Then a **pull-back** is a triple (U, g_X, g_Y), where U is an A-module and $g_X\colon U \to X$, $g_Y\colon U \to Y$ are morphisms such that $f_X g_X = f_Y g_Y$, with the universal property, that for any other such triple (V, h_X, h_Y) there exists a unique morphism $\zeta\colon V \to U$ such that $h_X = g_X\zeta$ and $h_Y = g_Y\zeta$. The following diagram visualizes the situation.

$$\begin{tikzcd}
V \arrow[drr, "h_X"] \arrow[dr, dashed, "\exists!\zeta"] \arrow[ddr, "h_Y"'] & & \\
& U \arrow[r, "g_X"'] \arrow[d, "g_Y"] & X \arrow[d, "f_X"] \\
& Y \arrow[r, "f_Y"'] & Z
\end{tikzcd}$$

Note that pull-backs exist in $\operatorname{mod} A$, see Exercise 4.3.4. Furthermore, the object U is unique up to isomorphism, see Exercise 4.3.5 and will be denoted by $X \prod_{f,g} Y$. The dual concept of pull-back is called **push-out**, see Exercise 4.3.6.

Exercises

4.3.1 Let $\mathscr{V}$ be the full subcategory of vec given by the vector spaces of even dimension. Show that there exists a morphism $f\colon K^2 \to K^2$ in $\mathscr{V}$ such that a kernel exists in $\mathscr{V}$. Give another morphism where no kernel exists.

4.3.2 Let Q be a finite quiver and $f\colon V \to W$ a morphism of representations of Q. Define the representation L of Q by $L_x = f_x^{-1}(0)$ for each vertex x and $L_\alpha\colon L_x \to L_y$ to be the maps induced by V_α. Show that the inclusion $L \to M$ is a kernel of f. Construct the image and the cokernel of f explicitely.

4.3.3 Let $\mathscr{C}$ be an abelian category and M, N be two objects such that there exist two morphisms $\pi\colon M \to N$ and $\iota\colon N \to M$ satisfying $\pi\iota = \mathrm{id}_N$. Show that $M = N \oplus N'$, where $\iota'\colon N' \to M$ is a kernel of π.

4.3.4 Let $\mathscr{C}$ be an abelian category and $f_X\colon X \to Z$, $f_Y\colon Y \to Z$ be two morphisms. Denote $[f_X\ -f_Y]\colon X \oplus Y \to Z$ the obvious map. Show that $\operatorname{Ker}[f_X\ -f_Y]$ yields a pull-back of f_X and f_Y.

4.3.5 Show that if (U, g_X, g_Y) and (U', g'_X, g'_Y) are two pull-backs of $f_X\colon X \to Z$, $f_Y\colon Y \to Z$, then there exists a unique isomorphism $\varphi\colon U \to U'$ such that $g_X\varphi = g_X$ and $g'_Y\varphi = g_Y$. You may use the uniqueness of kernels and Exercise 4.3.4 or give a direct argument for this.

4.3.6 State the definition of a push-out (including the universal property) and prove existence and uniqueness using dualization.

4.3.7 Let $g\colon Z' \to Z$ and $b\colon Y \to Z$ be two morphisms in an abelian category $\mathscr{C}$. Furthermore, let $a\colon X \to Y$ be a kernel of b. Show that there exists some morphism $\xi\colon Z' \to Y$ with $g = b\xi$ if and only if $(X \oplus Z',\ [0\ 1],\ [a\ \xi])$ is a pull-back of g and b.

4.4 Unique decomposition

As usual A denotes a finite-dimensional K-algebra. Any $M \in \operatorname{mod} A$ admits a decomposition into a direct sum of indecomposable modules $M = \bigoplus_{i=1}^{s} M_i$, since either M is already indecomposable, or it decomposes, say $M \simeq N_1 \oplus N_2$, and then by induction on the dimension we know that N_1 and N_2 admit such a decomposition.

Such a direct sum decomposition $M = \bigoplus_{i=1}^{s} M_i$ comes equipped with canonical inclusions $\iota_i\colon M_i \to M$ and projections $\pi_i\colon M \to M_i$ such that $\pi_i\iota_i = 1_{M_i}$, $\pi_i\iota_{i'} = 0$ for $i \neq i'$ and $\sum_{i=1}^{s} \iota_i\pi_i = 1_M$. Let $M' = \bigoplus_{j=1}^{t} M'_j$ with canonical inclusions ι'_j and projections π'_j. Then each morphism $\varphi\colon M \to M'$ gives rise to a family of morphisms $\varphi_{ji} = \pi'_j\varphi\iota_i\colon M_i \to M'_j$. Note that we can recover φ from this family since $\varphi = \sum_{j,i} \iota'_j\varphi_{ji}\pi_i$.

Now, we want to see how these families behave under the composition of morphisms. For this, let $M'' = \bigoplus_{k=1}^{u} M''_k$ be yet another module with the corresponding canonical inclusions ι''_k and projections π''_k and $\psi\colon M' \to M''$ a morphism and $\psi_{kj} = \pi''_k\psi\iota'_j\colon M'_j \to M''_k$. Then

$$\begin{aligned}
\psi\varphi &= \left(\sum_{k,j'} \iota''_k\psi_{kj'}\pi'_{j'}\right)\left(\sum_{j,i} \iota'_j\varphi_{ji}\pi_i\right) \\
&= \sum_{k,i} \iota''_k \left(\sum_{j',j} \psi_{kj'}\pi'_{j'}\iota'_j\varphi_{ji}\right)\pi_i \\
&= \sum_{k,i} \iota''_k \left(\sum_{j} \psi_{kj}\varphi_{ji}\right)\pi_i,
\end{aligned}$$

which shows that $(\psi\varphi)_{ki} = \sum_j \psi_{kj}\varphi_{ji}$, exactly like matrix multiplication.

Therefore, we can write morphisms between direct sums as matrices between the indecomposables of the given decomposition. The following result shows that the decomposition into indecomposables is unique up to order and isomorphisms.

Theorem 4.12 (Krull-Remak-Schmidt)**.** *If $M = \bigoplus_{i=1}^{s} M_i$ is isomorphic to $N = \bigoplus_{j=1}^{t} N_j$, where all M_i and N_j are indecomposable, then $t = s$ and there exists a permutation σ of $\{1, \ldots, t\}$ such that M_i is isomorphic to $N_{\sigma(i)}$ for $i = 1, \ldots t$.*

Proof. Let $\varphi\colon M \to N$ be an isomorphism. Then φ can be written as matrix $\varphi = (\varphi_{ji})_{j=1}^{t}{}_{i=1}^{s}$, where $\varphi_{ji}\colon M_i \to N_j$. Similarly write $\psi = \varphi^{-1}$ as matrix and observe that $\mathrm{id}_{M_1} = (\psi\varphi)_{11} = \sum_{l=1}^{t} \psi_{1t}\varphi_{t1}$. One summand on the right must then be invertible, since $\mathrm{End}_A(M_1)$ is local. We may assume without loss of generality that $\psi_{11}\varphi_{11}$ is invertible (otherwise reorder $N_1, \ldots, N_t$). But, since both modules, M_1 and N_1, are indecomposable, we have that both homomorphisms, ψ_{11} and φ_{11}, are invertible.

We now exchange the given isomorphism φ by $\varphi' = \alpha\varphi\beta$, where α and β are defined as

$$\alpha = \begin{bmatrix} \mathrm{id}_{N_1} & 0 & \cdots & 0 \\ -\varphi_{21}\varphi_{11}^{-1} & \mathrm{id}_{M_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\varphi_{t1}\varphi_{11}^{-1} & 0 & \cdots & \mathrm{id}_{N_t} \end{bmatrix}$$

and

$$\beta = \begin{bmatrix} \mathrm{id}_{M_1} & -\varphi_{11}^{-1}\varphi_{12} & \cdots & -\varphi_{11}^{-1}\varphi_{1s} \\ 0 & \mathrm{id}_{M_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{id}_{M_s} \end{bmatrix},$$

and observe that $\varphi' = \begin{bmatrix} \varphi_{11} & 0 \\ 0 & \Phi \end{bmatrix}$, where $\Phi\colon \bigoplus_{i=2}^{s} M_i \to \bigoplus_{j=2}^{t} N_j$. Since φ' is bijective, so must be Φ, which is thus an isomorphism again. The result follows now by induction. $\square$

Remark 4.13. Sometimes it is useful to consider just the indecomposable modules together with all irreducible morphisms between them. In the present section we have found that, from a certain perspective, this is enough to know all of the module category. Exercise 4.4.1 gives more evidence for this. That this perspective is too simple shows the following example: consider the quiver Q with a single vertex and no loop, then, from that point of view, the study of $\operatorname{rep} Q \simeq \operatorname{mod} K \simeq \operatorname{vec}$ which is just the theory of linear algebras of finite-dimensional vector spaces, would be reduced to the study of the field K together with all its endomorphisms, which are given by scalar multiplication. $\diamondsuit$

Still in view of the reduction of the module category "up to isomorphism" the following considerations are interesting. We say that a category $\mathscr{C}$ is **small** if its objects form a set. A **skeleton** of a category $\mathscr{C}$ is a full subcategory of $\mathscr{C}$ whose objects consist of representatives of the isomorphism classes of the objects. In other words if we choose for each isomomorphism class a representative and form the full subcategory of $\mathscr{C}$ with these representatives, then we get a skeleton of $\mathscr{C}$. Note that two skeletons of $\mathscr{C}$ are isomorphic, see Exercsie 4.4.5.

Lemma 4.14. *For each finite-dimensional K-algebra A the category* $\operatorname{mod} A$ *has a small skeleton.*

Proof. Every A-module M is a K-vector space. Thus there exists a K-linear bijection $\varphi\colon M \to K^d$, where $d = \dim_K M$. Moreover K^d becomes an A-module isomorphic to M if we define the external multiplication

$$A \times K^d \to K^d,\ (a, m) \mapsto \varphi(a\varphi^{-1}(m)).$$

This shows that each A-module can be represented, up to isomorphism, by a vector space K^d together with a multiplication $A \times K^d \to K^d$. Hence the isomorphism classes of $\operatorname{mod} A$ form a set. $\square$

Exercises

4.4.1 Show that a homomorphism $\varphi\colon \bigoplus_{i=1}^{s} M_i \to \bigoplus_{j=1}^{t} N_j$ is radical if and only if for each i and j the homomorphism $\varphi_{ji}\colon M_i \to N_j$ is radical.

4.4.2 Determine a basis for $\operatorname{Hom}(V, W)$, where $V = [3, 4] \oplus [1, 3]$ and $W = [2, 4] \oplus [4, 5]$ are $A_{\vec{\mathbb{A}}_5}$-modules in the notation established in in Section 2.6.

4.4.3 Let $A = K\vec{\mathbb{A}}_5$ and let $M = \bigoplus_{i=1}^{5}[i, 5]$, where $[i, j]$ denotes an indecomposable A-module in the notation established in Section 2.6. Show that the endomorphism algebra $\operatorname{End}_A(M)$ is isomorphic to the algebra B of upper triangular matrices of size 5×5. Show also that the linear map $B \to A, M \to M^\top$ yields an isomorphism of algebras $B \to A^{\mathrm{op}}$.

4.4.4 Prove that $\operatorname{mod} A$ and $\operatorname{add}(\operatorname{ind} A)$ are equivalent categories by giving equivalences.

4.4.5 Prove that two skeletons of a fixed category $\mathscr{C}$ are isomorphic categories.

4.5 Projectives and injectives

We now consider a new categorical concept with its dual counterpart. An object P of a category $\mathscr{C}$ is called **projective**, if for any epimorphism $f\colon M \to N$ and any morphism $g\colon P \to N$, there exists a (usually not unique) morphism $h\colon P \to M$ such that $g = fh$. It is common to show this property in a diagram as follows.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "\forall g"] \\ M \arrow[r, "\forall f \text{ (epi)}"'] & N \end{tikzcd}$$

In particular, we are interested in the objects of $\operatorname{mod} A$ which are projective, where A is some finite-dimensional K-algebra. Such objects are called **projective** A-modules. The dual concept of projective is **injective**, see also Exercise 4.5.2.

Example 4.15. Let A be a finite-dimensional K-algebra. Then A considered as left A-module is projective in $\operatorname{mod} A$. Indeed, if $f\colon M \to N$ is an epimorphism in $\operatorname{mod} A$, then f is surjective by Exercise 4.1.2. Note also, that any morphism $h\colon A \to M$ is completely determined by the element $m = h(1_A) \in M$, since $h(a) = ah(1_a) = am$. Let $n = g(1_A)$. Then, since f is surjective, the fiber $f^{-1}(n)$ is not empty. Choose $m \in f^{-1}(n)$ and set $h(a) = am$. Then h is a morphism of A-modules and $fh(a) = f(am) = af(m) = an = ag(1_A) = g(a)$. $\lozenge$

Lemma 4.16. *Let $\mathscr{C}$ be a K-category and P an object which is the direct sum $P = P_1 \oplus P_2$. Then P is projective if and only if P_1 and P_2 is projective.*

Proof. As usual, for $i = 1, 2$ denote by $\pi_i\colon P \to P_i$ the canonical projection and by $\iota_i\colon P_i \to P$ the canonical inclusion. Let $f\colon M \to N$ be an epimorphism in $\mathscr{C}$.

Suppose that P is projective. Then for any $g\colon P_i \to N$ we get a morphism $g\pi_i\colon P \to N$, which by hypothesis factors through f, say $g\pi_i = fh$ for some $h\colon P \to M$. Now, consider $h\iota_i\colon P_i \to M$. By construction, $g = g\pi_i\iota_i = f(h\iota_i)$ is the desired factorization of g. This shows that P_i is projective.

Now assume that P_1 and P_2 are both projective and let $g\colon P \to N$ be any morphism. Then, by hypothesis the morphism $g\iota_1\colon P_1 \to N$ factors through M, say $g\iota_1 = fh_1$ for some $h_1\colon P_1 \to M$. Similarly $g\iota_2 = fh_2$ for some $h_2\colon P_2 \to M$. Now define $h\colon P \to M$ by $h = h_1\pi_1 + h_2\pi_2$. Then $fh = fh_1\pi_1 + fh_2\pi_2 = g\iota_1\pi_1 + g\iota_2\pi_2 = g$, which shows that P is projective. □

Proposition 4.17. *Each indecomposable projective A-module is isomorphic to a direct summand of A.*

Proof. Let P be an indecomposable A-module which is projective. Clearly there exists a surjective morphism $f\colon A^n \to P$ for $n = \dim_K P$: take a basis $p_1, \dots, p_n$ of P considered as K-vector space and then define f by $f(a) = \sum_i a_i p_i$ for all $a = (a_1, \dots, a_n) \in A^n$.

Since P is projective there exists a factorization of the identity map $\mathrm{id}_P\colon P \to P$, say $\mathrm{id}_P = fh$ for some $h\colon P \to A^n$. By Exercise 4.3.3, P is a direct summand of A^n. Hence $A^n = P \oplus Q$ and if Q is decomposed into indecomposables, we get a decompositon of A^n into indecomposables. If $A = \bigoplus_{i=1}^t P_i$ is a decomposition into indecomposables we also get that $A^n = \bigoplus_{i=1}^t P_i^n$ is a decomposition into indecomposables. Hence, by the uniqueness of the decomposition, Theorem 4.12, we get that P must be isomorphic to some P_i. □

Corollary 4.18. *Let A be a basic, finite-dimensional K-algebra. If $\{e_1, \dots, e_t\}$ is a complete set of pairwise orthogonal, primitive idempotents of A, then $Ae_1, \dots, Ae_t$ is a complete list of pairwise non-isomorphic indecomposable projective A-modules up to isomorphism.*

Proof. Clearly, the left A-module A can be decomposed as $A = \bigoplus_{i=1}^t Ae_i$. By Example 4.15 and Lemma 4.16 each direct summand Ae_i is projective.

We first show that Ae_i is indecomposable: Each $f \in \mathrm{End}_A(Ae_i)$ is determined by the value $x = f(e_i) \in Ae_i$ and $x = f(e_i) = f(e_i^2) = e_i f(e_i) = e_i x \in e_i A e_i$. Now, if x is not invertible in $e_i A e_i$ then it is nilpotent, since by Corollary 3.15 the algebra $e_i A e_i$ is local. Therefore $x^n = 0$ for some n and thus $f^n(e_i) = x^n = 0$ implies $f^n = 0$. If, on the other hand, $x \in e_i A e_i$ is invertible, say $xy = yx = e_i$, then $g\colon Ae_i \to Ae_i, ae_i \mapsto ay$ defines an inverse of f. Hence $\mathrm{End}_A(Ae_i)$ is local and therefore Ae_i indecomposable, by Proposition 3.13.

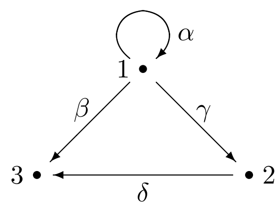
By Proposition 4.17, each indecomposable projective A-module is isomorphic to some direct summand of A and since $A = \bigoplus_{i=1}^{t} Ae_i$ is a decomposition into indecomposables, it follows from the unicity of the decomposition, Theorem 4.12, that each indecomposable projective A-module is isomorphic to Ae_i for some i. It remains to see is that $Ae_1, \ldots, Ae_t$ are pairwise non-isomorphic: to do so, assume otherwise. Then there exist isomorphisms $f\colon Ae_i \to Ae_j$ and $g\colon Ae_j \to Ae_i$ which are inverse to each other. Let $x = f(e_i) \in Ae_j$ and $y = g(e_j) \in Ae_i$. Then $x = xe_j$ and therefore $e_i = gf(e_i) = g(x) = xg(e_j) = xy$ and similarly $yx = e_j$. Then $x\colon e_j \to e_i$ and $y\colon e_i \to e_j$ are inverse isomorphisms in the category A_S, where $S = \{e_1, \ldots, e_t\}$, thus e_i and e_j are isomorphic objects in A_S and hence A is not basic, in contradiction to our assumption. □

Example 4.19. Let A be the matrix of lower triangular matrices in $K^{n\times n}$. Then $\{\mathbf{E}^{(11)}, \ldots, \mathbf{E}^{(nn)}\}$ is a complete set of pairwise orthogonal, primitive idempotents. Therefore $A\mathbf{E}^{(11)}, \ldots, A\mathbf{E}^{(nn)}$ is a complete list of pairwise non-isomorphic indecomposable projective A-modules. Note that $A\mathbf{E}^{(ii)}$ in the language of representations of Section 2.6 is denoted $[i, n]$. ◊

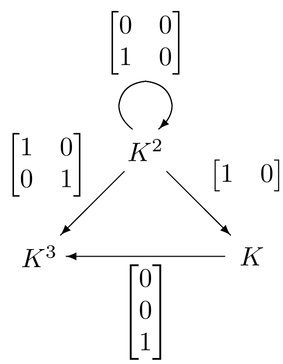
Remarks 4.20 (a) We now translate the description of the indecomposable projective A-modules into the other languages: recall that for the complete set $S = \{e_1, \ldots, e_t\}$ of pairwise orthogonal, primitive idempotents we have associated a spectroid A_S. Now, the module Ae_i corresponds to the functor $A_S(e_i, -)\colon A_S \to \mathrm{vec}$ in the categorical language and to the representation which associates to the vertex e_j the vector space $A_S(e_i, e_j) = e_j A e_i$.

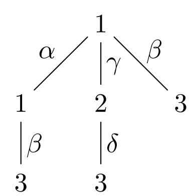
(b) If the algebra A is given by a bounded quiver Q and relations, that is, $A = KQ/I$, then to each vertex x, there corresponds a projective indecomposable representation, which is denoted by P_x. To the vertex y the representation P_x associates the vector space $(P_x)_y = e_y(KQ/I)e_x$ and to an arrow $\alpha\colon y \to z$ the linear map $e_y(KQ/I)e_x \to e_z(KQ/I)e_x, \gamma \mapsto \alpha\gamma$. ◊

Example 4.21. Let $A = KQ/I$, where Q is the following quiver:



with the relations $\alpha^2 = 0$ and $\gamma\alpha = 0$. For the projective P_1, we obtain the vector spaces $P_1(1) = K\mathrm{id}_1 \oplus K\alpha$, $P_1(2) = K\gamma$ and $P_1(3) = K\beta \oplus K\beta\alpha \oplus K\delta\gamma$ and hence for P_1 up to isomorphism the representation as shown below on the left.





It is common and usually more efficient to write down a diagram, as shown above on the right: the vertices are the elements of a basis as representation. In our example these would be $1, \alpha, \beta, \beta\alpha, \delta\gamma, \gamma$. But instead of writing these basis vectors only the corresponding vertex of the quiver is denoted: 1 for $1, \alpha \in (P_1)_1$, then 2 for $\beta, \beta\alpha, \delta\gamma \in (P_1)_2$ and 3 for $\gamma \in (P_1)_3$. Thus in total there are $\dim_K (P_1)_i$ vertices labelled i. The edges of the diagram indicate how these basis vectors are transformed under the linear maps of the representation. Therefore the edges are labelled by the arrows of the quiver. By convention the maps go from upper to lower rows. Thus, for instance the edge labelled α represent the map $(P_1)_\alpha \colon 1 \mapsto \alpha$, whereas all other vectors are sent to zero. There are tow edges labelled β since under the map $(P_1)_\beta$ we have $1 \mapsto \beta$ and $\alpha \mapsto \beta\alpha$, whereas all other basis vectors are sent to zero. If no confusion can arise, the multiplication indicated on the edges and even the edges itself are omitted. $\Diamond$

Proposition 4.22. *If A is a basic, finite-dimensional K-algebra and $S = \{e_1, \ldots, e_t\}$ a complete set of pairwise orthogonal, primitive idempotents,* *then* $\mathrm{De}_1A, \dots, \mathrm{De}_tA$ *is a complete list of pairwise non-isomorphic indecomposable injective* A*-modules up to isomorphism.*

Proof. By Proposition 4.3, the injectives in $\operatorname{mod} A$ are the projectives in $(\operatorname{mod} A)^{\mathrm{op}} \simeq \operatorname{mod}(A^{\mathrm{op}})$, that is, the direct summands of A^{op}. Now the direct summands of A^{op} are $A^{\mathrm{op}} \cdot^{\mathrm{op}} e_i = e_iA$ and therefore the result follows by Remark 4.4 (b). □

Remarks 4.23 (a) If Q is the quiver of the algebra A, then the injective De_xA will be denoted by I_x.

(b) It is useful to translate the description of the injectives into the categorical language. For this, let $\mathscr{C}$ be a finite spectroid. Then, to each object $x \in \mathscr{C}$, there corresponds an indecomposable projective object in $\operatorname{mod}\mathscr{C}$, namely $P_x = \mathscr{C}(x, -)$. The indecomposable injective associated to x is $I_x = \mathrm{D}\mathscr{C}(-, x)$.

Note that here we used tacitly the maps

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(\alpha, y)\colon\ \mathrm{Hom}_{\mathscr{C}}(x', y) &\to \mathrm{Hom}(x, y), \beta \mapsto \beta\alpha, \\ \mathrm{Hom}_{\mathscr{C}}(y, \alpha)\colon\ \mathrm{Hom}_{\mathscr{C}}(y, x) &\to \mathrm{Hom}(y, x'), \gamma \mapsto \gamma\alpha \end{aligned}$$

for $x, x', y \in \mathscr{C}$ and $\alpha\colon x \to x'$. ◊

Lemma 4.24 (Yoneda's Lemma)**.** *Let* A *be a finite-dimensional* K*-algebra and let* $\{e_1, \dots, e_t\}$ *be a complete set of pairwise orthogonal, primitive idempotents. Then for each* A*-module* M *and* $x = 1, \dots, t$ *the following two maps*

$$\Phi_{x,M}\colon\ \mathrm{Hom}_A(P_x, M) \to M_x, f \mapsto f(e_x)$$

and

$$\Psi_{x,M}\colon\ \mathrm{Hom}_A(M, I_x) \to \mathrm{D}M_x, g \mapsto g(?)(e_x)$$

are K*-linear bijections, where* $M_x = e_xM$*.*

Proof. Since $f(e_x) = f(e_x^2) = e_xf(e_x)$ we have that $\Phi_{x,M}$ is well-defined. Clearly $\Phi_{x,M}$ is K-linear and it is straightforward to verify that the inverse of $\Phi_{x,M}$ is given by assigning to $m \in M_x$ the map $Ae_x \to M, ae_x \mapsto ae_xm$.

If $m \in M$ then $g(m) \in I_x = \mathrm{De}_xA$ and hence $g(m)(e_x) \in K$ which shows that $\Psi_{x,M}$ is well defined and K-linear. Again we indicate the inverse of $\Psi_{x,M}$ to show the bijectivity. For $\varphi \in \mathrm{D}M_x$ we define $\Psi'_{x,M}(\varphi) \in \mathrm{Hom}_A(M, I_x)$ by $\Psi'_{x,M}(\varphi)(m)\colon e_xA \to K, e_xr \mapsto \varphi(e_xrm)$. □

Remarks 4.25 (a) If we set $M = P_y$ we get $\mathrm{Hom}_A(P_x, P_y) \simeq (P_y)_x = e_x A e_y$ and hence to every $\alpha \in e_y A e_x$ we get a map $P_\alpha \colon P_x \to P_y$. Similarly if we set $M = I_x$ we get that $\mathrm{Hom}_A(I_x, I_y) \simeq \mathrm{D}(I_x)_y = \mathrm{D}(\mathrm{D}e_x A)_y = \mathrm{DD}e_x A e_y \simeq e_x A e_y$ and to every $\alpha \in e_x A e_y$ we get a homomorphism $I_\alpha \colon I_x \to I_y$.

In particular, if Q is the quiver of A, then for each arrow $\alpha \colon y \to x$ we get homomorphisms $P_\alpha \colon P_x \to P_y$ and $I_\alpha \colon I_x \to I_y$.

(b) The bijections $\Phi_{x,M}$ and $\Psi_{x,M}$ are functorial in M, that is, if $h \colon M \to N$ is a homomorphism of A-modules, then $h_x \Phi_{x,M} = \Phi_{x,N} \mathrm{Hom}_A(P_x, h)$ and $\mathrm{D}h_x \Psi_{x,N} = \Psi_{x,M} \mathrm{Hom}(h, I_x)$.

Note that it follows from Remark 4.25 (a) that for $\alpha \in e_x A e_y$ we get two homomorphisms $P_\alpha \colon P_x \to P_y$ and $I_\alpha \colon I_x \to I_y$. Note that $P_\alpha = ? \cdot \alpha$ and $I_\alpha = \mathrm{D}(\alpha \cdot ?)$. The bijections $\Phi_{x,M}$ and $\Psi_{x,M}$ are also functorial in x in the sense that for each $\alpha \in e_x A e_y$ we have $M_\alpha \Phi_{y,M} = \Phi_{x,M} \mathrm{Hom}(P_\alpha, M)$ and $\mathrm{D}M_\alpha \Psi_{x,M} = \Psi_{y,M} \mathrm{Hom}(M, I_\alpha)$. The proof of these functorialities is left as Exercise 4.5.5. $\Diamond$

There are many applications of the Yoneda Lemma and we mention one at once.

Proposition 4.26. *Let A be a finite-dimensional K-algebra with quiver Q and let $\mathscr{P}$ (resp. $\mathscr{I}$) be the full subcategory of* $\mathrm{mod}\, A$ *composed of all projective (resp. injective) objects. Then there exists a natural equivalence $\nu_A \colon \mathscr{P} \to \mathscr{I}$ defined by $\nu_A P_x = I_x$ for each vertex $x \in Q_0$ and $\nu_A P_\alpha = I_\alpha$ for each arrow $\alpha \in Q_1$.*

Proof. In the following sequence the first (and the last) map is given by the previous lemma and the map in the middle is the natural isomorphism between a vector space and its double dual.

$$\mathrm{Hom}_A(P_x, P_y) \to P_y(x) = e_x A e_y \to \mathrm{DD}e_x A e_y = \mathrm{D}I_x(y) \to \mathrm{Hom}_A(I_x, I_y)$$

Note that all maps are bijections on objects and functorial. $\square$

The functor ν_A is called **Nakayama functor**.

Remark 4.27. Let A be a basic, finite-dimensional K-algebra. Then for any complete set of pairwise orthogonal, primitive idempotents $S = \{e_1, \ldots, e_t\}$

the category $\mathscr{C} = A_S$ is a spectroid, see Example 3.22. Note that $\operatorname{ind}\mathscr{P}$ is a spectroid and isomorphic to $\mathscr{C}^{\mathrm{op}}$, since $\operatorname{Hom}_{\mathscr{C}}(P_x, P_y) \simeq \mathscr{C}(y,x) = \mathscr{C}^{\mathrm{op}}(x,y)$ for all $x, y \in \mathscr{C}$. On the other hand, the arrows of the quiver associated to A via the choice of S correspond to the data $\operatorname{rad}\mathscr{C}/\operatorname{rad}\mathscr{C}^2$, which by the previous is given entirely by the categorical properties of $\operatorname{mod} A$, since the indecomposable projectives in $\operatorname{mod} A$ are completely independent of the choice of S. This shows finally that the algebra A uniquely determines its quiver. $\diamondsuit$

Exercises

4.5.1 Proof that an object P of a category $\mathscr{C}$ is projective if and only if each epimorphism $\psi\colon N \to P$ splits, that is, there exists a morphism $\varphi\colon P \to N$ such that $\psi\varphi = \mathrm{id}_P$. Note that in case $\mathscr{C}$ is additive, it follows then by Exercise 4.3.3 that P is isomorphic to a direct summand of N.

4.5.2 Explicitely state the definition of an injective object in $\operatorname{mod} A$ and draw the corresponding diagram.

4.5.3 Determine all projective A-modules up to isomorphism if $A = KQ$, where Q is the two subspace quiver.

4.5.4 Determine all indecomposable projectives and all indecomposable injectives for the algebra of Example 3.32.

4.5.5 Prove the four statements of functoriality stated in Remark 4.25 (b).

4.5.6 Show the following description of $R = \operatorname{rad} P_x$ as representation of the quiver of A. $R_y = (P_x)_y = e_y A e_x$ holds for all $y \neq x$ and R_x is the maximal ideal of the local algebra $e_x A e_x$.

4.6 Projective covers and injective hulls

We now study the situation where we have an epimorphism $P \to M$ with projective P. First note that, for all A-modules M there always exists such an epimorphism: let $m_1, \ldots, m_n$ be a K-basis of M as K-vector space. Then the morphism $f\colon A^n \to M$ given by $f(a_1, \ldots, a_n) = \sum_{i=1}^n a_i m_i$ is surjective hence epi and A^n is projective.

A **projective cover** of an A-module M is an epimorphism $f\colon P \to M$, where P is projective such that each endomorphism $\rho \in \operatorname{End}_A(P)$ with $f\rho = f$ is an isomorphism. The following result shows that a projective cover is minimal in more than one sense.

Proposition 4.28. *Let A be a finite-dimensional K-algebra. Then for an epimorphism $f\colon P \to M$ with P a projective A-module, the following conditions are equivalent:*

(a) *f is a projective cover of M.*

(b) *P is "closest" to M: for each epimorphism $g\colon Q \to M$ with projective Q there exists an epimorphism $\psi\colon Q \to P$ such that $f\psi = g$.*

(c) *For each epimorphism $g\colon Q \to M$ with projective Q the module P is isomorphic to a direct summand of Q.*

(d) *For each epimorphism $g\colon Q \to M$ with projective Q we have $\dim_K P \leq \dim_K Q$.*

Proof. Since several implications are based on similar arguments, we will start by a preparation, valid for each epimorphism $g\colon Q \to M$ with projective Q. Since P is projective there exists a factorization $\varphi_g\colon P \to Q$ such that $g\varphi_g = f$ and since Q is projective there exists $\psi_g\colon Q \to P$ such that $f\psi_g = g$. Define $\rho_g = \psi_g\varphi_g \in \operatorname{End}_A(P)$ and observe that $f\rho_g = f$. Also recall from Exercise 4.1.2 that epimorphism in $\operatorname{mod} A$ is synonymous with surjective homomorphism.

Assume (a), then ρ_g is an isomorphism and therefore ψ_g surjective and satisfies $f\psi_g = g$ hence (b). Given (b), we get from the projectivity of P a factorization of id_P through the surjective ψ_g, say $\mathrm{id}_P = \psi_g h$. Then by Exercise 4.5.1 the epimorphism ψ_g splits and P is isomorphic to a direct summand of Q. Clearly (c) implies (d).

Finally assume (d) and let $\rho \in \operatorname{End}_A(P)$ be some endomorphism with $f\rho = f$. By Lemma 3.12 there exists an integer $n \geq 1$ such that $P = \operatorname{Im}\rho^n \oplus \operatorname{Ker}\rho^n$. Now $f(\operatorname{Ker}\rho^n) = 0$ since for each $m \in \operatorname{Ker}\rho^n$ we have $f(m) = f\rho^n(m) = 0$. Thus f restricts to a morphism $f'\colon \operatorname{Im}\rho^n \to M$, which is again surjective since $\operatorname{Im} f' = \operatorname{Im} f = M$. By our assumption $\dim_K \operatorname{Im}\rho^n \geq \dim_K P$ and hence we must have $\operatorname{Im}\rho^n = P$. This implies that ρ^n and therefore ρ is an isomorphism. This shows (a). □

Corollary 4.29. *If A is a finite-dimensional K-algebra, then for each A-module $M \in \operatorname{mod} A$ there exists a projective cover. Furthermore, two projective covers are isomorphic, that is, if $P \to M$ and $Q \to M$ are projective covers of M, then $P \simeq Q$.*

Proof. We have already seen, that for each $M \in \operatorname{mod} A$ there exists an epimorphism $A^n \to M$ for $n = \dim_K M$. This shows that there exists an epimorphism $f\colon P \to M$ with projective P and we may assume that the dimension $\dim_K P$ is minimal among all such epimorphisms. Then by the previous result f is a projective cover.

If $f\colon P \to M$ and $g\colon Q \to M$ are two projective covers, then by Proposition 4.28 there exist epimorphisms $\varphi\colon P \to Q$ with $g\varphi = f$ and $\psi\colon Q \to P$ with $f\psi = g$. Since $\psi\varphi \in \operatorname{End}_A(P)$ satisfies $f\psi\varphi = f$ we know that $\psi\varphi$ is an isomorphism, hence φ is injective and therefore an isomorphism. This shows $P \simeq Q$. □

Remark 4.30. Often, only the projective A-module P is given to indicate a projective cover $f\colon P \to M$. Due to Corollary 4.29 this makes sense, since P is uniquely determined up to isomorphism and any surjective morphism $g\colon P \to M$ will give a projective cover. Moreover, there exists $\psi \in \operatorname{End}_A(P)$ such that $g\psi = f$ and each such ψ will be an isomorphism. ◊

The dual concept of projective cover is called **injective hull**. Note that we can dualize all four properties of Proposition 4.28, even the forth one, since the dimension does not change under dualization. Therefore we get that for each A-module $M \in \operatorname{mod} A$ there exists an injective hull $M \to I$ and that I is unique up to isomorphism. Hence $\operatorname{mod} A$ is an abelian category **with enough projectives and injectives**, that is, for each object $M \in \operatorname{mod} A$ there exists a projective cover and an injective hull.

Example 4.31. Let $A = K\overrightarrow{\mathbb{A}}_5$, the path algebra of the quiver whose representations we studied in Section 2.6. To determine the projective cover of $M = [2,3]$, we first observe that by Example 4.19, the indecomposable projective representations are $P_i = [i,n]$ and by Lemma 2.15, we have $\operatorname{Hom}_{K\overrightarrow{\mathbb{A}}_5}([i,n],[2,3]) = 0$ if $i = 1$ or $i = 4, 5$. Note that

$$\iota_{2,5}^{2,3}\colon [2,5] \to [2,3]$$

is surjective and that $\iota_{i,5}^{2,3}$ is not surjective for $i > 2$. Hence $\iota_{2,5}^{2,3}$ is a projective cover of $M = [2,3]$. ◊

Exercises

4.6.1 Show that if $f_1\colon P_1 \to M_1$ and $f_2\colon P_2 \to M_2$ are projective covers, then $f_1 \oplus f_2\colon P_1 \oplus P_2 \to M_1 \oplus M_2$ is also a projective cover.

4.6.2 Conclude from the previous exercise that if $f\colon P \to M_1 \oplus M_2$ is a projective cover, then up to isomorphisms $P = P_1 \oplus P_2$ and $f = f_1 \oplus f_2$, where $f_1\colon P_1 \to M_1$ and $f_2\colon P_2 \to M_2$ are projective covers.

4.6.3 Let Q be the two subspace quiver and M the following indecomposable representation: $K \xrightarrow{1} K \xleftarrow{1} K$. Determine a projective cover of M.

4.7 Simple modules

As in the previous sections, A denotes a finite-dimensional K-algebra. An A-module M is called **simple** or **irreducible** if 0 and M are the only submodules of M.

Example 4.32. Let $P_i = Ae_i$ be an indecomposable projective A-module. Then the quotient $S_i = P_i / \operatorname{rad} P_i$ is a simple module. As a representation, S_i is one-dimensional in the vertex i and zero elsewhere, and $(S_i)_\alpha = 0$ for each arrow α of the quiver of A.

Indeed, it follows from Exercise 4.5.6 that $\operatorname{rad} P_i$ is a complement of Ke_i in P_x viewed as vector spaces. Hence $S_i = P_i / \operatorname{rad} P_i$ is one-dimensional. This shows that S_i is simple since S_i is non-zero and only admits 0 and S_i as submodules. Furthermore $\dim_K (S_i)_i = 1$ and $(S_i)_j = 0$ for all $j \neq i$ and $(S_i)_\alpha = 0$ since $\operatorname{Im}(S_i)_\alpha \subseteq \operatorname{rad} S_i = 0$, where the last equations follows from the Nakayama Lemma. $\Diamond$

Lemma 4.33 (Schur's Lemma)**.** *If S and S' are two simple A-modules then each morphism $S \to S'$ is either zero or an isomorphism.*

Proof. Let $f\colon S \to S'$ be a morphism. Then $\operatorname{Ker} f$ is a submodule of S. If $\operatorname{Ker} f = S$, then $f = 0$. Otherwise $\operatorname{Ker} f = 0$ and then f is injective by Lemma 4.9. Also, the image $f(S)$ is a submodule of S' and cannot be zero. Hence $f(S) = S'$, therefore f is surjective, again by Lemma 4.9, and thus an isomorphism. $\square$

The following result shows that the simples of Example 4.32 are all the simple A-modules.

Proposition 4.34. *Let A be a basic, finite-dimensional K-algebra and $\{e_1, \dots, e_t\}$ a complete set of pairwise orthogonal, primitive idempotents and $P_i = Ae_i$ the corresponding indecomposable projectives. Then $P_1 / \operatorname{rad} P_1$, $\dots$, $P_t / \operatorname{rad} P_t$ is a complete list of pairwise non-isomorphic simple A-modules up to isomorphism.*

Proof. Let S be a simple A-module. By Corollary 4.29 there exists a projective cover $f\colon Q \to S$. Let $Q = Q_1 \oplus \ldots \oplus Q_m$ be a decomposition into indecomposables which are again projective by Lemma 4.16. Let $f_i\colon Q_i \to S$ be the morphism induced by f. Then there exists an index i such that $f_i \neq 0$ and $\operatorname{Im} f_i = S$ follows, since S is simple. This implies that $f_i\colon Q_i \to S$ is already an epimorphism. By the minimality of the projective cover, $Q = Q_i$ is indecomposable and hence isomomorphic to $P_h = Ae_h$ for some h. This shows that the projective cover of S if the form $f\colon P_h \to S$.

Now $\operatorname{rad} P_h$ is a proper submodule of P_h by Nakayama's Lemma (Proposition 3.20) and therefore $P_h/\operatorname{rad} P_h$ a non-zero module and $\operatorname{rad} S = 0$. By Remark 3.19, we have $f(\operatorname{rad} P_h) \subseteq \operatorname{rad} S = 0$. Therefore f induces a morphism $P_h/\operatorname{rad} P_h \to S$ which is still surjective and therefore by Schur's Lemma an isomorphism. □

If Q is the quiver of the finite-dmiensional K-algebra A then the simple module $P_x/\operatorname{rad} P_x$ asscoiated to the vertex x is denoted by S_x.

Note that each non-zero $M \in \operatorname{mod} A$ admits a maximal submodule $M_1 \subsetneq M$: just take M_1 to be a proper submodule of maximal dimension. The quotient M/M_1 must then be a simple module, since otherwise any proper submodule $0 \neq V \subsetneq M/M_1$ could be lifted to $M_1 \subsetneq \pi^{-1}(V) \subsetneq M$, if $\pi\colon M \to M/M_1$ denotes the canonical projection. This can be iterated resulting in a filtration

$$0 = M_\ell \subsetneq M_{\ell-1} \subsetneq \ldots \subsetneq M_1 \subsetneq M_0 = M$$

whose quotients M_i/M_{i+1} are simple. Such a filtration is called **composition series** or **Jordan-Hölder filtration** and the quotients are called **composition factors**. We will later see in Exercise 5.1.2 that each simple S_i must occur precisely $\dim_K M_i$ times in this filtration, where $M_i = e_i M$ is the vector space corresponding to the vertex i. This is the analogon of the classical Theorem of Jordan-Hölder: two composition factors have the same number of factors and they are equal up to order and isomorphisms.

Exercises

4.7.1 Show that an A-module M with $\operatorname{rad} M = 0$ is **semisimple**, that is, it is a direct sum of simple A-modules. The best way to see this is to view M as a representation of the quiver Q of A. Show that the condition $\operatorname{rad} M = 0$ translates into $M_\alpha = 0$ for each arrow α of Q and conclude from this the statement.

4.7.2 Let M be a fixed A-module. Show that if S and S' are two submodules of M which are semisimple, then also $S + S'$ is a semisimple submodule. It therefore makes sense to speak of "the largest submodule of M which is semisimple". This uniquely defined submodule of M is called **socle** of M and denoted by $\operatorname{soc} M$.

Chapter 5

Elements of homological algebra

Homological algebra is one of the most important tools if not the most important of all for the theory of representations of algebras. It gathers information of different point of views about its main subject, which is that of short exact sequences. With this it prepares the basics for the second step for understanding the structure of module categories.

5.1 Short exact sequences

First we develop some tools of homological algebra. Although this is normally done in the context of an arbitrary abelian category (with enough injectives and projectives) we focus only on module categories reaping the benefits of working with functions, their arguments and values. Thus, let A be a finite-dimensional algebra over some algebraically closed field K and $\operatorname{mod} A$ the category of finite-dimensional left A-modules.

A **short exact sequence** in $\operatorname{mod} A$ is an exact sequence of the form

$$0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0, \tag{5.1}$$

with $X, Y, Z \in \operatorname{mod} A$. Note that this means that a is injective, b is surjective and $\operatorname{Ker} b = \operatorname{Im} a$. We abbreviate the sequence (5.1) by (a, b) or (a, Y, b) and often drop the bounding zeros in diagrams. The sequence (5.1) **starts in** X, **ends in** Z and Y is its **middle term**. The two modules X and Z are also called the **end terms** of the sequence (5.1). For fixed modules

$X, Z \in \operatorname{mod} A$ the short exact sequences starting in X and ending in Z do not form a set but a class, which we shall denote by $\mathscr{E}(Z, X)$ (that Z is mentioned before X has its own reason, which will become clear later).

We now introduce an equivalence relation among these short exact sequences: $(a, b) \sim (a', b')$ if there exists an isomorphism ζ such that $a' = \zeta a$ and $b = b'\zeta$. The collection of equivalence classes in $\mathscr{E}(Z, X)$ will be denoted as $\operatorname{Ext}_A^1(Z, X)$. As we will see, $\operatorname{Ext}_A^1(Z, X)$ is a set and it carries the structure of an abelian group, even of a K-vector space. Therefore $\operatorname{Ext}_A^1(Z, X)$ is called **extension group**. Note that by definition, the middle terms of two equivalent short exact sequences must be isomorphic. That the converse does not hold is shown in Example 5.11.

Lemma 5.1. *For each $X, Z \in \operatorname{mod} A$ the class $\operatorname{Ext}_A^1(Z, X)$ is a set.*

Proof. By Exercise 5.1.1, the middle term Y of each exact sequence $(a, b) \in \mathscr{E}(Z, X)$ has dimension $d = \dim_K X + \dim_K Z$. Therefore, there exists a K-linear bijection $\varphi\colon Y \to K^d$ and we can turn K^d into an A-module, isomorphic to Y, via the multiplication

$$A \times K^d \to K^d, (a, v) \mapsto a * v = \varphi(a\varphi^{-1}(v)).$$

The short exact sequence (a, Y, b) is therefore equivalent to $(\varphi a, K^d, b\varphi^{-1})$. Thus to determine $\operatorname{Ext}_A^1(Z, X)$ it is enough to consider short exact sequences of the form (a, K^d, b), where K^d is equipped with some external multiplication $A \times K^d \to K^d$. Such short exact sequences form a set since they are given as triple (a, b, μ), namely two linear maps $a\colon X \to K^d$ and $b\colon K^d \to Z$ and a bilinear map $\mu\colon A \times K^d \to K^d$. Consequently the equivalence classes $\operatorname{Ext}_A^1(Z, X)$ form a set. $\square$

The following result shows that it is enough to have a homomorphism between the middle terms in order to establish an equivalence.

Lemma 5.2. *Let $(a, Y, b), (a', Y', b') \in \mathscr{E}(Z, X)$ be two short exact sequences. If there exists a homomorphism $\zeta\colon Y \to Y'$ such that $\zeta a = a'$ and $b'\zeta = b$, then ζ is an isomorphism and consequently $(a, Y, b) \sim (a', Y', b')$.*

Proof. We exhibit a diagram, since it facilitates the orientation.

$$\begin{tikzcd} & Y \arrow[dd, "\zeta"] \arrow[dr, "b"] & \\ X \arrow[ur, "a"] \arrow[dr, "a'"'] & & Z \\ & Y' \arrow[ur, "b'"'] & \end{tikzcd}$$

Assume $\zeta(y) = 0$ for some $y \in Y$. Then $b(y) = b'\zeta(y) = 0$ and because $\mathrm{Ker}(b) = \mathrm{Im}(a)$ there exists some $x \in X$ such that $a(x) = y$. Now, since $a'(x) = \zeta a(x) = \zeta(y) = 0$ it follows from the injectivity of a' that $x = 0$ and therefore $y = 0$. This shows that ζ is injective. To see the surjectivity of ζ let $y' \in Y'$ be some element and set $z = b'(y')$. Since b is surjective there exists some $y \in Y$ such that $b(y) = z$. The element y is a good candidate for a preimage of y'. However $\zeta(y) \neq y'$ is possible, since we only can ensure that $b'\zeta(y) = z = b'(y')$. Thus we look at the difference $y' - \zeta(y) \in Y'$ and observe that $b'(y' - \zeta(y)) = b'(y') - b(y) = z - z = 0$. Therefore, there exists some $x \in X$ such that $a'(x) = y' - \zeta(y)$. Now let $t = y + a(x) \in Y$ and calculate $\zeta(t) = \zeta(y) + \zeta a(x) = \zeta(y) + a'(x) = y'$. This shows the surjectivity of ζ. □

Remark 5.3. This kind of arguing is called "diagram chasing", since you follow the elements as they are pushed along arrows or "sucked back" if they lie in some image. Note that in the above proof there is no surprise and that, once one is on the right track, one has just hang on and follow to the end.

Each part of algebraic theory has its own taste. In homological algebra it is often the case that reading a proof is equally affording than finding it afresh. It can be more advantageous for the reader to work the details out on his or her own, rather than to follow the detailed descriptions written down in a textbook. However, it is also important to see how the mathematics looks when it is written down and this is why we have given a detailed account of the argumentation in the previous case. In the following we often only outline the things to prove and leave it to the reader to do this properly. ◊

In the following, we define a **left multiplication** on $\mathscr{E}(Z, X)$:

$$\mathrm{Hom}_A(X, X') \times \mathscr{E}(Z, X) \to \mathscr{E}(Z, X'), (h, (a, b)) \mapsto h(a, b). \tag{5.2}$$

Given $(a, Y, b) \in \mathscr{E}(Z, X)$ and a homomorphism $h\colon X \to X'$ we define a new short exact sequence in $\mathscr{E}(Z, X')$, denoted by $h(a, Y, b)$, as follows. First we form the push-out (Y', h', a') of a and h, as shown in the next picture.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "h"'] & Y \arrow[r, "b"] \arrow[d, "h'"] & Z \arrow[d, equal] \\
X' \arrow[r, "a'"'] & Y' \arrow[r, dashed, "\exists! b'"'] & Z
\end{tikzcd}$$

Since $ba = 0$ and $0h = 0$ we have two homomorphisms $b\colon Y \to Z$ and $0\colon X' \to Z$ ending in Z and by the universal property of the push-out, see Exercise 4.3.6, there exists a unique $b'\colon Y' \to Z$ such that $b'h' = b$ and $b'a' = 0$. Thus the right square of the preceding picture commutes.

Lemma 5.4. *The sequence $h(a,b) \in \mathscr{E}(Z, X')$ obtained as explained above is always exact.*

Proof. Let $(a',b') = h(a,b)$. Since b is surjective so must be b'. To see that a' is injective, we recall from Exercise 4.3.6 that we can assume, without loss of generality, that the push-out (Y', h', a') is obtained as cokernel ψ of the map φ:

$$X \xrightarrow{\varphi=\begin{bmatrix} a \\ h \end{bmatrix}} Y \oplus X' \xrightarrow{\psi=[h' \quad -a']} Y'.$$

Hence, if $a'(x') = 0$ then $(0, x') \in Y \oplus X'$ satisfies $\psi(0,x') = h'(0) - a'(x') = 0$. Since $\operatorname{mod} A$ is abelian, we have $\operatorname{Ker}(\psi) = \operatorname{Im}(\varphi) = \varphi(X)$ and therefore there exists an element $x \in X$ such that $0 = a(x)$ and $x' = h(x)$. The injectivity of a implies $x = 0$ and $x' = 0$ follows. This shows that a' is injective.

Since $b'a' = 0$ we have $\operatorname{Im} a' \subseteq \operatorname{Ker} b'$. Now, suppose that some element $y' = \psi(y, x') = h'(y) - a'(x')$ lies in $\operatorname{Ker} b'$. Then $0 = b'(y') = b(y)$ shows that $y = a(x)$ for some $x \in X$ since (a,b) is exact. Hence $y' = h'a(x) - a'(x') = a'(h(x) - x') \in \operatorname{Im}(a')$. Thus, we have shown that $\operatorname{Ker} b' = \operatorname{Im} a'$. Hence (a',b') is a again a short exact sequence. □

Thus, we have defined the left multiplication (5.2).

Proposition 5.5. *The left multiplication defined in* (5.2) *induces a multiplication*

$$\operatorname{Hom}_A(X, X') \times \operatorname{Ext}^1_A(Z, X) \to \operatorname{Ext}^1_A(Z, X'), (h, \varepsilon) \mapsto h\varepsilon \qquad (5.3)$$

which is associative in the sense that for each $h \in \operatorname{Hom}_A(X, X')$ and each $k \in \operatorname{Hom}(X', X'')$ we have $k(h\varepsilon) = (kh)\varepsilon$.

Proof. We have to show that, if (c, T, d) is equivalent to (a, Y, b), then $h(a, Y, b)$ is equivalent to $h(c, T, d)$. Let $(a', Y', b') = h(a, Y, b)$ as above and similarly $(c', T', d') = h(c, T, d)$ with $h''\colon T \to T'$ the corresponding homomorphism of the middle terms.

Since $(a,b) \sim (c,d)$ there exists an isomorphism $\xi\colon Y \to T$ such that $c = \xi a$ and $d\xi = b$. Hence $(h''\xi)a = h''c = c'h$. So, by the push-out property of

Y', there exists a unique homomorphism $\rho\colon Y' \to T'$ with $h''\xi = \rho h'$ and $c' = \rho a'$. Similarly, $(h'\xi^{-1})c = h'a = a'h$ and by the push-out property of T', there exists a homomorphism $\sigma\colon T' \to Y'$ such that $(h'\xi^{-1}) = \sigma h''$ and $a' = \sigma c'$. Hence $(\sigma\rho)a' = a'$ and $(\sigma\rho)h' = h'$. But only the identity map $\mathrm{id}_{Y'}$ has this property. Thus $\sigma\rho = \mathrm{id}_{Y'}$ and similarly one sees that $\rho\sigma = \mathrm{id}_{T'}$. Therefore we have $(c,d) \sim (c',d')$.

This shows, that (5.2) induces a map (5.3). It remains to verify the stated associativity.

Let (Y', h', a') be a push-out of a and h, next let (Y'', k', a'') be a push-out of a' and k and finally let (T, ℓ', c) be a push-out of a and $\ell = kh$, see the next picture for illustration. Then $(k'h')a = k'a'h = a''(kh)$, thus by the universal property of (T, ℓ', c) there exists a unique $\varphi\colon T \to Y''$ such that $\varphi\ell' = k'h'$ and $\varphi c = a''$. By the same property d is uniquely determined by $dc = 0$ and $d\ell' = b$. Therefore, since $b''\varphi c = b''a'' = 0$ and $b''\varphi\ell' = b''k'h' = b'h' = b$ we must have $d = b''\varphi$.

$$\begin{tikzcd}
& X \arrow[r, "a"] \arrow[d, "h"] \arrow[dddl, "\ell = kh"'] & & Y \arrow[r, "b"] \arrow[d, "h'"] \arrow[dddl, "\ell'"] & & Z \arrow[d, equal] \arrow[dddl, equal] \\
& X' \arrow[r, "a'"] \arrow[d, "k"] & & Y' \arrow[r, "b'"] \arrow[d, "k'"] & & Z \arrow[d, equal] \\
& X'' \arrow[r, "a''"] \arrow[dl, equal] & & Y'' \arrow[r, "b''"] & & Z \arrow[dl, equal] \\
X'' \arrow[rr, "c"] & & T \arrow[rr, "d"] \arrow[ur, dashed, "\varphi"'] & & Z &
\end{tikzcd}$$

By Lemma 5.2 the two sequences (c, T, d) and (a'', Y'', b'') are equivalent. $\square$

The construction can be dualized: given a short exact sequence $(a,b) \in \mathscr{E}(X,Z)$ and $g \in \mathrm{Hom}_A(Z', Z)$ we can form the pull-back (Y', g', b') of b and g and obtain an exact sequence (a', b'), which is denoted by $(a,b)g$. Thus a **right multiplication** is defined. The proof of the following result is obtained by dualizing the proof of Proposition 5.5.

Proposition 5.6. *The right multipilication of short exact sequences induces a well defined right multiplication*

$$\mathrm{Ext}^1_A(Z,X) \times \mathrm{Hom}_A(Z',Z) \to \mathrm{Ext}^1_A(Z',X), (\varepsilon, g) \mapsto \varepsilon g. \tag{5.4}$$

Furthermore, this right multiplication is associative in the sense that $(\varepsilon g)g' = \varepsilon(gg')$ *for each* $g' \in \mathrm{Hom}_A(Z'', Z')$. $\square$

The proof of the following compatibility result is left as Exercise 5.1.3 to the reader.

Lemma 5.7. *The left multiplication* (5.3) *is compatible with the right multiplication* (5.4)*, that is, for all* $\varepsilon \in \mathrm{Ext}_A^1(Z, X)$*, all* $h \in \mathrm{Hom}_A(X, X')$ *and all* $g \in \mathrm{Hom}(Z', Z)$ *we have* $(h\varepsilon)g = h(\varepsilon g)$. $\square$

Exercises

5.1.1 Show that for a short exact sequence (5.1) the following dimension formula holds: $\dim_K Y = \dim_K X + \dim_K Z$. Improve this by showing that $\underline{\dim}\, Y = \underline{\dim}\, X + \underline{\dim}\, Z$.

5.1.2 Use the previous exercise and the description of the simple modules of Example 4.32 to show that in a Jordan-Hölder filtration of an A-module M the simple $Ae_i/\operatorname{rad} Ae_i$ occurs precisely $\dim_K M_i$ times.

5.1.3 Prove Lemma 5.7. Draw a diagram of the picture and label the homomorphisms appearing in it. Then work your way through it always using the properties of push-outs and pull-backs.

5.1.4 Show that two consecutive homomorphisms (a, b) form a short exact sequence if and only if a is a kernel of b and b is a cokernel of a.

5.1.5 Show that if $0 \to U \xrightarrow{a} V \xrightarrow{b} W \to 0$ is a short exact sequence of vector spaces, then dualization over the ground field yields a short exact sequence $0 \to \mathrm{D}W \xrightarrow{\mathrm{D}b} \mathrm{D}V \xrightarrow{\mathrm{D}a} \mathrm{D}U \to 0$.

5.1.6 Prove that the short exact sequence $0 \to \operatorname{rad} M \to M \to M/\operatorname{rad} M \to 0$ is dual to $0 \to \operatorname{soc} M \to M \to M/\operatorname{soc} m \to 0$.

5.2 The Baer sum

In the previous section we have gathered all the necessary preliminaries to define a binary operation in the set $\mathrm{Ext}_A^1(Z, X)$, which will turn $\mathrm{Ext}_A^1(Z, X)$ into an abelian group. This operation is called **Baer sum**. For two short exact sequences (a, b) and (c, d) in $\mathscr{E}(Z, X)$, we define their **direct sum** by $(a, b) \oplus (c, d) = (a \oplus c, b \oplus d) \in \mathscr{E}(Z \oplus Z, X \oplus X)$. Then, using the homomorphisms $\Sigma_X\colon X\oplus X \to X, (x,x') \mapsto x+x'$ and $\Delta_Z\colon Z \to Z\oplus Z, z\mapsto (z,z)$, define

$$(a,b)+(c,d) = \Sigma_X(a\oplus c, b\oplus d)\Delta_Z, \tag{5.5}$$

which is again a short exact sequence in $\mathscr{E}(Z,X)$.

In order to see that the binary operation (5.5) satisfies good properties, we need a rather technical result:

Lemma 5.8. *For each $(a,b)\in\mathscr{E}(Z,X)$ we have $\Delta_X(a,b)\sim(a\oplus a, b\oplus b)\Delta_Z$ and $\Sigma_X(a\oplus a,b\oplus b)\sim(a,b)\Sigma_Z$.*

Proof. We first prove that $\Delta_X(a,b)$ is equivalent to the lower row of the next picture.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "\Delta_X = \begin{bmatrix}1\\1\end{bmatrix}"'] & Y \arrow[r, "b"] \arrow[d, "\Delta' = \begin{bmatrix}0\\1\end{bmatrix}"] & Z \arrow[d, equal] \\
X\oplus X \arrow[r, "a' = \begin{bmatrix}-1 & 1\\ a & 0\end{bmatrix}"'] & X\oplus Y \arrow[r, "b' = \begin{bmatrix}0 & b\end{bmatrix}"'] & Z
\end{tikzcd}$$

We start by verifying that $(X\oplus Y, \Delta', a')$ is a push-out of a and Δ_X. Let M be an A-module and further let $g = \begin{bmatrix}g_1 & g_2\end{bmatrix}\colon X\oplus X\to M$ and $h\colon Y\to M$ be two homomorphisms such that $g\Delta_X = ha$, that is, $g_1+g_2 = ha$. Then the homomorphism $f = \begin{bmatrix}g_2 & h\end{bmatrix}\colon X\oplus Y\to M$ satisfies $fa' = \begin{bmatrix}-g_2+ha & g_2\end{bmatrix} = \begin{bmatrix}g_1 & g_2\end{bmatrix} = g$ and clearly $f\Delta' = h$. To see the unicity of f we assume that the homomorphism $f' = \begin{bmatrix}f_1' & f_2'\end{bmatrix}\colon X\oplus Y\to M$ satisfies $f'a' = g$ and $f'\Delta' = h$. Then the first equation implies $f_1' = g$ and the second $f_2' = h$ showing $f' = f$. Hence $(X\oplus Y,\Delta',a')$ is indeed a push-out of a and Δ_X. Since $b = b'\Delta'$ and $b'a' = 0$ we conclude that $\Delta_X(a,b)\sim(a',b')$.

Now we prove that $(a\oplus a, b\oplus b)\Delta_Z \sim (a',b')$. For this, we consider the following picture.

$$\begin{tikzcd}
X\oplus X \arrow[r, "a\oplus a = \begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}"] & Y\oplus Y \arrow[r, "b\oplus b = \begin{bmatrix}b & 0\\ 0 & b\end{bmatrix}"] & Z\oplus Z \\
X\oplus X \arrow[u, equal] \arrow[r, "a' = \begin{bmatrix}-1 & 1\\ a & 0\end{bmatrix}"'] & X\oplus Y \arrow[u, "\Delta'' = \begin{bmatrix}0 & 1\\ a & 1\end{bmatrix}"'] \arrow[r, "b' = \begin{bmatrix}0 & b\end{bmatrix}"'] & Z \arrow[u, "\Delta_Z = \begin{bmatrix}1\\1\end{bmatrix}"']
\end{tikzcd}$$

The proof that $(X \oplus Y, \Delta'', b')$ is a pull-back of $b \oplus b$ and Δ_Z is quite similar to the one above and we therefore leave it to the reader. A direct calculation shows that $\Delta'' a' = a \oplus a$ and $b'a' = 0$, thus $(a \oplus a, b \oplus b)\Delta_Z \sim (a', b')$. The second equivalence follows by dualization. $\square$

Theorem 5.9. *The set* $\mathrm{Ext}^1_A(Z, X)$ *together with the Baer sum, defined in* (5.5)*, is an abelian group. Furthermore, the left- and right-multiplication defined in Propositions 5.5 and 5.6 distribute over the Baer sum. That is, for all* $h, g \in \mathrm{Hom}(X, X')$ *and all* $\varepsilon \in \mathrm{Ext}^1_A(Z, X)$ *the equality* $h\varepsilon + g\varepsilon = (h+g)\varepsilon$ *holds and an analog equation is valid for the right multiplication. As a consequence* $\mathrm{Ext}^1_A(Z, X)$ *is a* K*-vector space.*

Proof. The proof is done in several steps.

(i) The Baer sum is commutative. Clearly $(a \oplus c, b \oplus d)$ and $(c \oplus a, d \oplus b)$ are equivalent short exact sequences, hence by Proposition 5.5 and 5.6 we get $\Sigma_X(a \oplus c, b \oplus d)\Delta_Z \sim \Sigma_X(c \oplus a, d \oplus b)\Delta_Z$.

(ii) The Baer sum is associative. Observe that

$$[(a, b) + (c, d)] + (e, f) = \Sigma_X(\Sigma_X \oplus \mathrm{id}_X)\, S\, (\Delta_Z \oplus \mathrm{id}_Z)\Delta_Z,$$
$$(a, b) + [(c, d) + (e, f)] = \Sigma_X(\mathrm{id}_X \oplus \Sigma_X)\, S\, (\mathrm{id}_Z \oplus \Delta_Z)\Delta_Z,$$

where $S = (a \oplus c \oplus e, b \oplus d \oplus f)$. The associativity follows thus from

$$\Sigma_X(\Sigma_X \oplus \mathrm{id}_X) = \Sigma_X(\mathrm{id}_X \oplus \Sigma_X) \colon X \oplus X \oplus X \to X$$
$$(\Delta_Z \oplus \mathrm{id}_Z)\Delta_Z = (\mathrm{id}_Z \oplus \Delta_Z)\Delta_Z \colon Z \to Z \oplus Z \oplus Z.$$

(iii) The **split** *short exact sequence*

$$\varepsilon_0\colon \quad X \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} X \oplus Z \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}} Z$$

is a neutral element of the Baer sum. Let $(a, Y, b) \in \mathscr{E}(Z, X)$. We first show that the lower sequence of the following diagram is indeed $\Sigma_X(\varepsilon_0 \oplus (a, b))$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
X \oplus X \arrow[r, "a' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & a \end{bmatrix}"] \arrow[d, "\Sigma_X = \begin{bmatrix} 1 & 1 \end{bmatrix}"'] & X \oplus Z \oplus Y \arrow[r, "b' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & b \end{bmatrix}"] \arrow[d, "\Sigma' = \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}"] & Z \oplus Z \arrow[d, equal] \\
X \arrow[r, "a'' = \begin{bmatrix} 0 \\ a \end{bmatrix}"'] & Z \oplus Y \arrow[r, "b'' = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}"'] & Z \oplus Z
\end{tikzcd}$$

A direct inspection shows that the diagram is commutative. We now show that $(Z \oplus Y, \Sigma', a'')$ satisfies the universal property of a push-out of a' and Σ_X. For this let M be an A-module and let $g\colon X \to M$ and $h = \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}\colon X \oplus Z \oplus Y \to M$ be two homomorphisms such that $ha' = \Sigma_X g$. This last equation is:

$$\begin{bmatrix} h_1 & h_3 a \end{bmatrix} = \begin{bmatrix} g & g \end{bmatrix}$$

and hence $h_1 = g = h_3 a$. Let $f = \begin{bmatrix} h_2 & h_3 \end{bmatrix}\colon Z \oplus Y \to M$ and verify directly that $f\Sigma' = \begin{bmatrix} h_3 a & h_2 & h_3 \end{bmatrix} = h$ and $fa'' = h_3 a = h_1 = g$. This shows that f satisfies the required factorization property. Moreover for any $f' = \begin{bmatrix} f_1' & f_2' \end{bmatrix}\colon Z \oplus Y$ with $f'\Sigma' = h$ and $f'a'' = g$, the former equation implies $f_1' = h_2$ and $f_2' = h_3$, that is, $f' = f$. This shows that $(Z \oplus Y, \Sigma', a'')$ is a push-out of a' and Σ_X. Furthermore, we have $b''\Sigma' = b'$ and $b''a'' = 0$ and it follows therefore from the unicity of the push-out, see Exercise 4.3.5 for the dual statement, that $(a'', b'') \sim \Sigma_X(\varepsilon_0 \oplus (a, b))$.

We now show that the lower sequence of the next picture is $\varepsilon_0 + (a, b) = \Sigma_X(\varepsilon_0 \oplus (a, b))\Delta_Z$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
X \arrow[r, "a'' = \begin{bmatrix} 0 \\ a \end{bmatrix}"] \arrow[d, equal] & Z \oplus Y \arrow[r, "b'' = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}"] & Z \oplus Z \\
X \arrow[r, "a"'] & Y \arrow[r, "b"'] \arrow[u, "\Delta' = \begin{bmatrix} b \\ 1 \end{bmatrix}"'] & Z \arrow[u, "\Delta_Z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}"']
\end{tikzcd}$$

First of all, the diagram commutes. We have to verify the universal pull-back property of (Y, Δ', b). Thus assume that M is an A-module and further

$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \colon M \to Z \oplus Y$ and $h\colon M \to Z$ are homomorphisms such that $b''g = \Delta_Z h$. The last equation means $g_1 = h = bg_2$ which can be interpreted as follows: $g_2 \colon M \to Y$ is the factorization which satisfies $\Delta' g_2 = g$ and $bg_2 = h$. Moreover, this is the unique factorization with this property since for any $f\colon M \to Y$ the equation $\Delta' f = g$ implies $f = g_2$. Thus, the lower row of the previous picture is indeed $\varepsilon_0 + (a, b)$. This shows that ε_0 is a neutral element in $\operatorname{Ext}^1_A(Z, X)$.

(iv) The left and right multiplication distributive over the Baer sum. Let $h, g \in \operatorname{Hom}(X, X')$ and $\varepsilon \in \operatorname{Ext}^1_A(Z, X)$. We have to show that $h\varepsilon + g\varepsilon = (h+g)\varepsilon$. For this, observe that $h+g = \Sigma_X(h \oplus g)\Delta_X$ and $(h\oplus g)\varepsilon = h\varepsilon \oplus g\varepsilon$ and therefore, using Lemma 5.8 in equation $(*)$, we get

$$(h+g)\varepsilon = \Sigma_X(h \oplus g)\Delta_X\varepsilon \overset{(*)}{=} \Sigma_X(h \oplus g)\varepsilon\Delta_Z = \Sigma_X(h\varepsilon \oplus g\varepsilon)\Delta_Z = h\varepsilon + g\varepsilon.$$

The distributivity with respect to right multiplication is obtained dually.

(v) Existence of additive inverses. The following equalities, where $\lambda \in K$, are subject of Exercise 5.2.1:

$$\lambda(a,b) = \begin{cases} (\frac{1}{\lambda}a, b), & \text{if } \lambda \neq 0, \\ \varepsilon_0, & \text{if } \lambda = 0. \end{cases} \tag{5.6}$$

Therefore the additive inverse of (a, b) is $(-a, b)$ since

$$(a,b) + (-a,b) = 1(a,b) + (-1)(a,b) = (1 + (-1))(a,b) = 0(a,b) = \varepsilon_0,$$

where we used the distributivity (iv). This concludes the proof. $\square$

Remark 5.10. We can rephrase the statement of Theorem 5.9 as follows: The functor $\operatorname{Ext}^1_A(Z, -)\colon \operatorname{mod} A \to \operatorname{vec}$ is a covariant K-linear functor for each A-module Z and $\operatorname{Ext}^1_A(-, X)\colon \operatorname{mod} A \to \operatorname{vec}$ is a contravariant K-linear functor for each A-module X. Since the left and right action are compatible by Lemma 5.7 we even get that Ext^1_A is a **bifunctor**, that is, a functor

$$\operatorname{mod} A^{\mathrm{op}} \times \operatorname{mod} A \to \operatorname{vec}, (Z, X) \mapsto \operatorname{Ext}^1_A(Z, X),$$

where $\mathscr{C} \times \mathscr{D}$, for two categories $\mathscr{C}$ and $\mathscr{D}$, denotes the **product category**, whose objects are pairs (C, D) with $C \in \mathscr{C}$ and $D \in \mathscr{D}$ and whose morphisms $(C, D) \to (C', D')$ are pairs (f, g) with $f \in \mathscr{C}(C, C')$ and $g \in \mathscr{D}(D, D')$ with the obvious composition and identity morphisms. $\lozenge$

It is not easy to determine the extension groups just using the definition, except in simple examples of low dimensions.

Example 5.11. Let $A = KQ$ be the Kronecker algebra, that is the path algebra of the quiver Q with two vertices 1 and 2 and two arrows $\alpha, \beta \colon 1 \to 2$. Let us calculate $\operatorname{Ext}^1_A(S_1, S_2)$, where S_i denotes the simple A-modules associated to the vertex i, see Example 4.32. Using Exercises 6.3.1 and 5.1.1, we know that the middle term Y of each short exact sequence $0 \to S_2 \to Y \to S_1 \to 0$ has, as representation of Q, dimension one in each of the vertices. Up to an isomorphism of A-modules, or equivalently an equivalence of short exact sequences, we can assume that $Y_1 = Y_2 = K$. Then Y is given by two linear maps $Y_\alpha = [\gamma]$ and $Y_\beta = [\delta]$ for some $\gamma, \delta \in K$. We abbreviate the representation Y with these two linear maps by $(\gamma, \delta) \in K^2$. Thus, we have to consider short exact sequences of representations as shown in the following picture.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & K \arrow[r, "b"] \arrow[d, shift right, "\gamma"'] \arrow[d, shift left, "\delta"] & K \arrow[d, shift left] \arrow[d, shift right] \\
K \arrow[r, "a"'] & K \arrow[r] & 0
\end{tikzcd}$$

Each such exact sequence is given by four K-linear maps $K \to K$, that is, by four scalars $a, b, \gamma, \delta \in K$, where $a \neq 0$ and $b \neq 0$. Note that (γ, δ) and $(\lambda\gamma, \lambda\delta)$ for each $\lambda \in K^* = K \setminus \{0\}$ define isomorphic representations. Thus setting $\varphi_1 = b$ and $\varphi_2 = \frac{1}{a}$ we get an isomorphism $\varphi \colon (\gamma, \delta) \to (\frac{\gamma}{ab}, \frac{\delta}{ab}) = (\gamma', \delta')$ and have then an equivalent short exact sequence, which denote as $\varepsilon(\gamma', \delta')$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & K \arrow[r, "1"] \arrow[d, shift right, "\gamma'"'] \arrow[d, shift left, "\delta'"] & K \arrow[d, shift left] \arrow[d, shift right] \\
K \arrow[r, "1"'] & K \arrow[r] & 0
\end{tikzcd} \tag{5.7}$$

Putting all this together, we get that each equivalence class $\mathscr{E}(S_1, S_2)$ contains an element of the form (5.7). Note that this element is unique. Hence we see that $\operatorname{Ext}^1_A(S_1, S_2)$ has dimension two and can be considered as the set $\{\varepsilon(\alpha, \beta) \mid \alpha, \beta \in K\}$.

The sequence $\varepsilon(0, 0)$ is special: it is the split exact sequence. For all other pairs $(\gamma, \delta) \in K^2$ the set $\{\varepsilon(\lambda\gamma, \lambda\delta) \mid \lambda \neq 0\}$ consists of pairwise non-equivalent short exact sequences which have isomorphic middle terms. In Exercise 5.2.2 the additive behavior is studied further. $\lozenge$

Exercises

5.2.1 Show the equalities stated in the equations (5.6).

5.2.2 Show that in Example 5.11 the following additivity holds: $\varepsilon(\alpha,\beta)+\varepsilon(\gamma,\delta) = \varepsilon(\alpha+\gamma,\beta+\delta)$.

5.2.3 Let $\varepsilon = (X \xrightarrow{a} Y \xrightarrow{b} Z)$ be a short exact sequence and $g\colon Z' \to Z$ some homomorphism. Use Exercise 4.3.7 to show that $\varepsilon g = 0$ if and only if g factors through b, that is, there exists some $\xi\colon Z' \to Y$ such that $g = b\xi$.

5.3 Resolutions

Example 5.11 shows that for a finite-dimensional K-algebra A and two A-modules X, Z it is a difficult problem to compute the extension group $\operatorname{Ext}^1_A(Z,X)$ following the presented definition: we had to consider all possible middle terms Y, from which we only know the dimension (or dimension vector, if the modules are considered as representations of a quiver, see Exercise 6.3.1) and then all possible extensions $X \to Y \to Z$. Afterwards, we had to describe the equivalence classes. That seems like a lot of work.

Therefore we want to present a different approach which is far better for practical purpose and even allows a generalization.

A **projective resolution** of an A-module M is an exact sequence

$$\cdots \xrightarrow{\varphi_4} Q_3 \xrightarrow{\varphi_3} Q_2 \xrightarrow{\varphi_2} Q_1 \xrightarrow{\varphi_1} Q_0 \tag{5.8}$$

where each object Q_i is projective and $\operatorname{Coker}\varphi_1 = M$. Dually, an **injective resolution** of M is an exact sequence

$$J_0 \xrightarrow{\psi_1} J_1 \xrightarrow{\psi_2} J_2 \xrightarrow{\psi_3} \cdots$$

where each J_j is injective and $\operatorname{Ker}\psi_1 = M$.

Remark 5.12. It is common to complete (5.8) to an exact sequence

$$\cdots \xrightarrow{\varphi_4} Q_3 \xrightarrow{\varphi_3} Q_2 \xrightarrow{\varphi_2} Q_1 \xrightarrow{\varphi_1} Q_0 \xrightarrow{\varphi_0} M \to 0$$

and even to call this the projective resolution of M. Similarly an injective resolution can be completed with $\psi_0\colon M \to J_0$. $\lozenge$

Projective resolutions are often easy to calculate.

Example 5.13. Let $A = KQ/I$ where $Q = \overrightarrow{\mathbb{A}}_6$ and the ideal I is generated by $\alpha_3\alpha_2\alpha_1$ and $\alpha_4\alpha_3\alpha_2$, see the next picture, where the two zero relations are indicated by dotted arcs.

$$\begin{tikzcd} 1 \arrow[r, "\alpha_1"'] & 2 \arrow[r, "\alpha_2"'] & 3 \arrow[r, "\alpha_3"'] & 4 \arrow[r, "\alpha_4"'] & 5 \end{tikzcd}$$

We calculate the projective resolution of S_1. For this we start with the projective cover $\varphi_0 \colon P_1 = [1,3] \to S_1$. The kernel of φ is $[2,3]$ and the projective cover of $[2,3]$ is $P_2 = [2,4]$. This yields $\varphi_1 = P_{\alpha_1} \colon P_2 \to P_1$. Proceeding this way we obtain the projective resolution as shown in the top row of the next picture.

$$\begin{tikzcd}[column sep=small] {[5,5]} \arrow[rr, "\varphi_3"] \arrow[dr] & & {[4,5]} \arrow[rr, "\varphi_2"] \arrow[dr] & & {[2,4]} \arrow[rr, "\varphi_1"] \arrow[dr] & & {[1,3]} \arrow[rr, "\varphi_0"] & & {[1,1]} \\ & {[5,5]} \arrow[ur] \arrow[d, equal] & & {[4,4]} \arrow[ur] \arrow[d, equal] & & {[2,3]} \arrow[ur] \arrow[d, equal] & & & \\ & \operatorname{Ker}\varphi_2 & & \operatorname{Ker}\varphi_1 & & \operatorname{Ker}\varphi_0 & & & \end{tikzcd}$$

Note that the projective resolution of S_1 can be rewritten as $0 \to P_5 \xrightarrow{P_{\alpha_4}} P_4 \xrightarrow{P_{\alpha_3\alpha_2}} P_2 \xrightarrow{P_{\alpha_1}} P_1$. $\Diamond$

We now want to use these resolutions to the compute extensions groups. Therefore, let Z be an A-module and let

$$\cdots \xrightarrow{\varphi_4} Q_3 \xrightarrow{\varphi_3} Q_2 \xrightarrow{\varphi_2} Q_1 \xrightarrow{\varphi_1} Q_0 \tag{5.9}$$

be a projective resolution of Z with $\operatorname{Coker}\varphi_1 = \varphi_0 \colon Q_0 \to M$.

For a homomorphism $h \colon Q_1 \to X$, we consider the push-out (Y_h, a_h, h') of h and φ_1, see next picture.

$$\begin{tikzcd} Q_2 \arrow[r, "\varphi_2"] & Q_1 \arrow[r, "\varphi_1"] \arrow[d, "h"'] & Q_0 \arrow[r, "\varphi_0"] \arrow[d, "h'"'] & Z \arrow[d, equal] \\ & X \arrow[r, "a_h"'] & Y_h \arrow[r, "b_h"'] & Z \end{tikzcd} \tag{5.10}$$

Since $\varphi_0\varphi_1 = 0 = 0h$ it follows from the universal property of the push-out (Y_h, a_h, h') that there exists a unique homomorphism $b_h \colon Y_h \to Z$ such that $b_h a_h = 0$ and $b_h h' = \varphi_0$. Thus the diagram commutes.

Lemma 5.14. *The sequence* (a_h, b_h) *as constructed above is a short exact sequence if and only if* a_h *is injective and this happens if and only if* $h\varphi_2 = 0$.

Proof. The exactness of $X \xrightarrow{a_h} Y_h \xrightarrow{b_h} Z \to 0$ follows precisely as in the proof of Lemma 5.4. Therefore (a_h, b_h) is a short exact sequence if an only if a_h is injective.

To see the second equivalence, observe that if $x = h\varphi_2(p) \neq 0$ for some $p \in Q_2$ with $p \neq 0$, then $0 = h'\varphi_1\varphi_2(p) = a_h(x)$ showing that a_h is not injective. Conversely, if $a_h(x) = 0$ for some $x \neq 0$, then there exists a $q \in Q_1$ with $h(q) = x$ and $\varphi_1(q) = 0$ showing that $q = \varphi_2(p)$ and $h\varphi_2 \neq 0$. $\square$

Recall from Remark 4.23 (b) that we denoted by $\operatorname{Hom}_A(\varphi_2, X)$ the map

$$\operatorname{Hom}_A(\varphi_2, X)\colon \operatorname{Hom}_A(Q_1, X) \to \operatorname{Hom}_A(Q_2, X), h \mapsto h\varphi_2.$$

Hence, the above shows that for any homomorphism h in $\operatorname{Ker}\operatorname{Hom}_A(\varphi_2, X)$ some short exact sequence (a_h, b_h) is obtained. We denote by $\varepsilon(h)$ the equivalence class of (a_h, b_h) in $\operatorname{Ext}^1_A(Z, X)$. Hence, the construction yields a map

$$\operatorname{Ker}\operatorname{Hom}_A(\varphi_2, X) \to \operatorname{Ext}^1_A(Z, X), h \mapsto \varepsilon(h). \tag{5.11}$$

Note that we do not distinguish in our notation between the short exact sequence and its equivalence class.

Proposition 5.15. *Let* X, Z *be* A*-modules and let* (5.9) *be a projective resolution of* Z*. Then the map* (5.11) *induces a* K*-linear bijection*

$$\operatorname{Ker}\operatorname{Hom}_A(\varphi_2, X)/\operatorname{Im}\operatorname{Hom}_A(\varphi_1, X) \to \operatorname{Ext}^1_A(Z, X). \tag{5.12}$$

which is functorial in X.

Proof. The proof is rather long and therefore separated in several steps.

(i) The map (5.11) *is surjective.* Let $(a, Y, b) \in \mathscr{E}(Z, X)$ be given. The projectivity of Q_0 implies that there exists $h''\colon Q_0 \to Y$ such that $bh'' = \varphi_0$. Since $0 = \psi\varphi_1 = bh''\varphi_1$, the homomorphism $h''\varphi_1$ factors over $\operatorname{Ker} b = \operatorname{Im} a$ and therefore over a, that is, there exists $h\colon Q_1 \to X$ with $ah = h''\varphi_1$. Let $\varepsilon(h) = (a_h, Y_h, b_h)$ be the short exact sequence obtained as above and $h'\colon Q_0 \to Y_h$ the homomorphism which satisfies $h'\varphi_1 = a_h h$ and $b_h h' = \varphi_0$.

We show that (a, Y, b) is equivalent to (a_h, Y_h, b_h). Since $h''\varphi_1 = ah$, there exists a unique homomorphism $\vartheta\colon Y_h \to Y$ such that $h'' = \vartheta h'$ and $a = \vartheta a_h$. Now, b_h and $b\xi$ are both homomorphisms with $b_h a_h = 0 = ba = (b\vartheta)a_h$

and $b_h h' = \varphi_0 = bh'' = (b\vartheta)h'$, thus it follows from the unicity stated in the universal property of the push-out (Y_h, a_h, h') that $b_h = b\vartheta$. This shows that $(a, b) \sim (a_h, b_h)$.

(ii) The map (5.11) *satisfies* $\varepsilon(h) = \varepsilon(h + \xi\varphi_1)$ *for each* $h \in \mathrm{Hom}_A(Q_1, X)$ *and each* $\xi \in \mathrm{Hom}_A(Q_0, X)$*.* Let $g = h + \xi\varphi_1$. We have to show that $(a_h, Y_h, b_h) \sim (a_g, Y_g, b_g)$. Note that we have homomorphisms $a_g \colon X \to Y_g$ and $g' \colon Q_0 \to Y_g$ satisfying $a_g g = g'\varphi_1$ and therefore $a_g h = a_g(g - \xi\varphi_1) = (g' - a_g\xi)\varphi_1$. By the universal property of the push-out (Y_g, a_g, g') there exists a unique homomorphism $\sigma \colon Y_g \to Y_h$ such that $\sigma a_g = a_h$ and $\sigma g' = h'$. We show that $b_h\sigma = b_g$ holds: indeed, by the universal property the homomorphism b_g is unique with the properties $b_g a_g = 0$ and $b_g g' = \varphi_0$. Since $(b_h\sigma)a_g = b_h a_h = 0$ and $(b_h\sigma)g' = b_h h' = \varphi$ it follows that $b_h\sigma = b_g$ and therefore $(a_h, b_h) \sim (a_g, b_g)$.

(iii) The map (5.12) *is a bijection.* By (i) we already know that the map is surjective. It remains to show that $\epsilon(h) = \epsilon(g)$ implies $h - g \in \mathrm{Im}\,\mathrm{Hom}_A(\varphi_1, X)$. Again we denote $(a_h, Y_h, b_h) = \varepsilon(h)$ and $(a_g, Y_g, b_g) = \varepsilon(g)$ and suppose that there is an isomorphism $\sigma \colon Y_h \to Y_g$ such that $a_g = \sigma a_h$ and $b_h = b_g\sigma$. Then $a_h(h - g) = a_h h - \sigma^{-1}a_g g = (h' - \sigma^{-1}g')\varphi_1$. Since $b_h(h' - \sigma^{-1}g') = \varphi_0 - b_g g' = 0$ there exists a $\xi \colon Q_0 \to X$ such that $a_h\xi = (h' - \sigma^{-1}g')$ and hence $a_h(h - g) = a_h\xi\varphi_1$. By the injectivity of a_h, we conclude $h - g = \xi\varphi_1 \in \mathrm{Im}\,\mathrm{Hom}_A(\varphi_1, X)$.

(iv) The bijection (5.12) *is K-linear.* This follows from (5.6) in the proof of Theorem 5.9 together with the similar equality

$$(a, b)\lambda = \begin{cases} (a, b\frac{1}{\lambda}), & \text{if } \lambda \neq 0, \\ \varepsilon_0, & \text{if } \lambda = 0 \end{cases}$$

and the elementary observation that $(a, b\frac{1}{\lambda}) \sim (\frac{1}{\lambda}a, b)$.

(v) The bijection (5.12) *is functorial in X.* For given $h \in \mathrm{Ker}\,\mathrm{Hom}_A(\varphi_2, X)$ and $\xi \colon X \to X'$, we have $\xi h \in \mathrm{Ker}(\varphi_2, X')$ and it follows $\xi\epsilon(h) = \epsilon(\xi h)$ by the push-out property precisely as the associativity in Proposition 5.5. □

Remark 5.16. The result shows that $\mathrm{Ker}\,\mathrm{Hom}_A(\varphi_2, X)/\,\mathrm{Im}\,\mathrm{Hom}_A(\varphi_1, X)$ is independent of the choice of a projective resolution of Z. Even more, one can show that this quotient is functorial in Z. Since we will not use this, we leave the proof to the interested reader. ◊

The dual construction starts with an injective resolution of the A-module X:

$$J_0 \xrightarrow{\psi_1} J_1 \xrightarrow{\psi_2} J_2 \xrightarrow{\psi_3} \cdots$$

and using pull backs along homomorphisms $Z \to J_1$ the following bijection is obtained

$$\operatorname{Ker}\operatorname{Hom}_A(Z, \psi_2)/\operatorname{Im}\operatorname{Hom}_A(Z, \psi_1) \to \operatorname{Ext}^1_A(Z, X). \tag{5.13}$$

In the forthcoming we will not distinguish between any of these three descriptions and uniformly denote them by $\operatorname{Ext}^1_A(Z, X)$.

Example 5.17. Continuing Example 5.13, we can calculate the extension group $\operatorname{Ext}^1_A(S_1, M)$ for various indecomposable A-modules $M = [i, j]$. By Proposition 5.15, we only have $\operatorname{Ext}^1_A(S_1, M) \neq 0$ if $\operatorname{Hom}_A(P_2, M) \neq 0$, hence by Yoneda's Lemma 4.24 if $M_2 \neq 0$. Furthermore, we only need to consider $h\colon P_2 \to M$ if $h\varphi_2 = 0$. We conclude $i \leq 2 \leq j$. Since $\varphi_2 = P_{\alpha_3\alpha_2}$ and all indecomposable representations are of the form $[i, j]$ we must have $M_4 = 0$, thus $j \leq 4$. It remains to determine which homomorphisms h factor through $\varphi_1 = P_{\alpha_1}$. Clearly this is the case if and only if $i = 1$. Thus we conclude that $\operatorname{Ext}^1_A(S_1, [i, j]) \neq 0$ if and only if $i = 2$ and $j = 2, 3$. $\lozenge$

The approach with resolutions has the advantage that we can define **higher extension groups** by setting

$$\operatorname{Ext}^i_A(Z, X) = \operatorname{Ker}\operatorname{Hom}_A(\varphi_{i+1}, X)/\operatorname{Im}\operatorname{Hom}_A(\varphi_i, X).$$

The **length** of the projective resolution (5.8) of M is just the maximal number i such that $Q_i \neq 0$ if such a number exists and otherwise infinite. We define the **projective dimension** $\operatorname{pdim} M$ of an A-module M as the minimal **length** of any possible projective resolution of M. Hence $\operatorname{pdim}_A M$ is either non-negative integer or ∞.

Similarly the **injective dimension** of M is the minimal length occurring in the injective resolutions of M, it is denoted by $\operatorname{idim} M$. The supremum of the projective dimensions of all A-modules is called the **global dimension**. We denote the global dimension of A by $\operatorname{gldim} A$ and observe that it coincides with the supremum of all injective dimensions of all A-modules, since it is just the supremum of all i such that there exists some A-modules M and N with $\operatorname{Ext}^i(M, N) \neq 0$.

The **finitistic dimension** $\operatorname{findim} A$ of the algebra A is by definition the supremum of all projective dimensions which are finite, that is,

$$\operatorname{findim} A = \sup\{\operatorname{pdim} M \mid M \in \operatorname{mod} A, \operatorname{pdim} M < \infty\}.$$

There is an interesting open problem about the finitistic dimension.

Conjecture 5.18 (Finitisitc dimension coinjecture)**.** The finitistic dimension of each finite-dimensional K-algebra is finite.

Exercises

5.3.1 Let A be the algebra considered in Example 5.13. Determine the projective resolutions of all simple A-modules S_i for $i > 1$.

5.3.2 Prove that the dimension of the space $\mathrm{Ext}_A^1(S_i, S_j)$ is precisely the number of arrows in the quiver of A.

5.3.3 Show that by definition $\mathrm{Ext}_A^0(M, X)$ is isomorphic to $\mathrm{Hom}_A(M, X)$ for any two A-modules M, X.

5.3.4 Show that an A-module M is projective if and only if $\mathrm{Ext}^1(M, N) = 0$ holds for all A-modules N.

5.3.5 Show that the projective dimension of an A-module M is precisely the maximal i such that $\mathrm{Ext}^i(M, A) \neq 0$.

5.4 Long exact sequences

As usual let A be a finite-dimensional K-algebra. Fix a short exact sequence $X \xrightarrow{a} Y \xrightarrow{b} Z$. Then for any A-module M we can compare the three extension groups $\mathrm{Ext}_A^i(M, X)$, $\mathrm{Ext}_A^i(M, Y)$ and $\mathrm{Ext}_A^i(M, Z)$. For this, we fix a projective resolution of M

$$\cdots Q_3 \xrightarrow{\varphi_3} Q_2 \xrightarrow{\varphi_2} Q_1 \xrightarrow{\varphi_1} Q_0.$$

Note that for each $i \geq 0$ we have a sequence

$$\mathrm{Hom}(Q_i, X) \xrightarrow{\mathrm{Hom}(Q_i, a)} \mathrm{Hom}(Q_i, Y) \xrightarrow{\mathrm{Hom}(Q_i, b)} \mathrm{Hom}(Q_i, Z). \tag{5.14}$$

Note that we have dropped the subscripts indicating the algebra A to simplify the notation. If $f \in \mathrm{Ker}\,\mathrm{Hom}(\varphi_{i+1}, X)$, that is, $f\colon Q_i \to X$ satisfies $f\varphi_{i+1} = 0$, then $af\varphi_{i+1} = 0$, that is $af \in \mathrm{Ker}\,\mathrm{Hom}(\varphi_{i+1}, Y)$. Hence, the sequence (5.14) induces a sequence

$$\mathrm{Ker}\,\mathrm{Hom}(\varphi_{i+1}, X) \to \mathrm{Ker}\,\mathrm{Hom}(\varphi_{i+1}, Y) \to \mathrm{Ker}\,\mathrm{Hom}(\varphi_{i+1}, Z).$$

Similarly, if $f \in \mathrm{Im}\,\mathrm{Hom}(\varphi_i, X)$, that is, $f = h\varphi_i$ for some $h \in \mathrm{Hom}(Q_{i-1}, X)$, then $af = ah\varphi_i \in \mathrm{Im}\,\mathrm{Hom}(\varphi_i, Y)$. This shows that the sequence (5.14) induces also the sequence

$$\mathrm{Im}\,\mathrm{Hom}(\varphi_i, X) \to \mathrm{Im}\,\mathrm{Hom}(\varphi_i, Y) \to \mathrm{Im}\,\mathrm{Hom}(\varphi_i, Z).$$

Thus, we get a sequence of the quotients which by Proposition 5.15 is

$$\mathrm{Ext}^i(M,X) \xrightarrow{\mathrm{Ext}^i(M,a)} \mathrm{Ext}^i(M,Y) \xrightarrow{\mathrm{Ext}^i(M,b)} \mathrm{Ext}^i(M,Z). \tag{5.15}$$

We now present a construction which relates $\mathrm{Ext}^i(M,Z)$ with $\mathrm{Ext}^{i+1}(M,X)$. For this we assume that $\varepsilon \in \mathrm{Ext}^i(M,Z)$ is represented by $f\colon Q_i \to Z$ with $f\varphi_{i+1} = 0$. Since $b\colon Y \to Z$ is surjective and Q_i projective there exists a factorization $\ell\colon Q_i \to Y$ such that $f = b\ell$.

$$\begin{tikzcd}
Q_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, dashed, "h"'] & Q_i \arrow[d, dashed, "\ell"'] \arrow[dr, "f"] & \\
X \arrow[r, "a"'] & Y \arrow[r, "b"'] & Z
\end{tikzcd}$$

Note that $f\varphi_{i+1} = 0$ implies that $b(\ell\varphi_{i+1}) = 0$ and since a is a kernel of b, see Exercise 5.1.4, there exists a factorization $h\colon Q_{i+1} \to X$ such that $\ell\varphi_{i+1} = ah$. Hence $ah\varphi_{i+2} = \ell\varphi_{i+1}\varphi_{i+2} = 0$ follows and since a is injective this implies $h\varphi_{i+2} = 0$, that is, $h \in \mathrm{Ker}\,\mathrm{Hom}(\varphi_{i+2}, X) \subseteq \mathrm{Hom}(Q_{i+1}, X)$. By taking the equivalence class of h we get an element of $\mathrm{Ext}^{i+1}(M,X)$ which is denoted by $\delta_i(\varepsilon)$.

Lemma 5.19. *The construction explained above yields a well-defined map* $\delta_i\colon \mathrm{Ext}^i_A(M,Z) \to \mathrm{Ext}^{i+1}_A(M,X)$ *for each* $i \geq 0$.

Proof. We verify that the equivalence class of h does not depend on the possible choices. If f' represents the same equivalence class as f, then $f' = f + g\varphi_i$ for some $g\colon Q_{i-1} \to Z$. Now let $\ell'\colon Q_i \to Y$ be chosen such that $b\ell' = f'$ and let $h'\colon Q_{i+1} \to X$ be the unique factorization such that $ah' = \ell'\varphi_{i+1}$. We have to show that h and h' define the same equivalence class in $\mathrm{Ext}^{i+1}(M,X)$.

Again, since Q_{i-1} is projective and b surjective, we get $m\colon Q_{i-1} \to Y$ such that $bm = g$. Hence $bm\varphi_i = g\varphi_i = f' - f = b(\ell' - \ell)$. Therefore $b(\ell' - \ell - m\varphi_i) = 0$ and this implies that there exists $k\colon Q_i \to X$ such that $\ell' - \ell - m\varphi_i = ak$. Consequently $a(h' - h) = (\ell' - \ell)\varphi_{i+1} = (ak + m\varphi_i)\varphi_{i+1} = ak\varphi_{i+1}$. By the injectivity of a follows $h' - h = k\varphi_{i+1}$, that is, h' and h define the same class in $\mathrm{Ext}^{i+1}(M,X)$. $\square$

The maps δ_i are called **connecting maps**. The next result is very important for dealing with extension groups.

Theorem 5.20. *Let $X \xrightarrow{a} Y \xrightarrow{b} Z$ be a short exact sequence of A-modules. Then for each A-module M the infinite sequence*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(M,X) \arrow[r] & \operatorname{Hom}_A(M,Y) \arrow[r] & \operatorname{Hom}_A(M,Z) \arrow[dll, "\delta_0"] \\
& \operatorname{Ext}^1_A(M,X) \arrow[r] & \operatorname{Ext}^1_A(M,Y) \arrow[r] & \operatorname{Ext}^1_A(M,Z) \arrow[dll, "\delta_1"] \\
& \operatorname{Ext}^2_A(M,X) \arrow[r] & \cdots & \\
& & \cdots \arrow[r] & \operatorname{Ext}^{i-1}_A(M,Z) \arrow[dll, "\delta_{i-1}"] \\
& \operatorname{Ext}^i_A(M,X) \arrow[r] & \operatorname{Ext}^i_A(M,Y) \arrow[r] & \operatorname{Ext}^i_A(M,Z) \arrow[dll, "\delta_i"] \\
& \operatorname{Ext}^{i+1}_A(M,X) \arrow[r] & \cdots &
\end{tikzcd}$$

is exact, where all the horizontal maps are as in (5.15).

Proof. Note, that we already have used Exercise 5.3.3 to replace Ext^0 by Hom. In the sequel we will drop again the subscript indicating the algebra A. The proof is done in three steps.

(i) The sequence (5.15) *is exact for each $i \geq 0$.* Let $\varepsilon \in \operatorname{Ext}^i(M,Y)$ be represented by $f \in \operatorname{Hom}(Q_i, Y)$ with $f\varphi_{i+1} = 0$. Now $\operatorname{Ext}^1(M,b)(\varepsilon)$ is represented by the homomorphism bf, thus $\operatorname{Ext}^1(M,b)(\varepsilon) = 0$ means that bf represents the equivalence class which contains the zero map. In other words we have $bf = m\varphi_i$ for some $m\colon Q_{i-1} \to Z$. Since Q_{i-1} is projective and b surjective, there exists $\ell\colon Q_{i-1} \to Y$ such that $b\ell = m$. This implies $b(f - \ell\varphi_i) = 0$ and consequently there exists some $h\colon Q_i \to X$ such that $f - \ell\varphi_i = ah$, that is, f and ah represent the same equivalence class in $\operatorname{Ext}^i(M,Y)$. If η is the equivalence class of h in $\operatorname{Ext}^i(M,X)$, then $\varepsilon = a\eta$. This shows $\operatorname{Ker}\operatorname{Ext}^i(M,b) \subseteq \operatorname{Im}\operatorname{Ext}^i(M,a)$. The other inclusion follows directly from $b(a\eta) = (ba)\eta = 0\eta = 0$.

(ii) For each $i \geq 0$, the sequence

$$\operatorname{Ext}^i(M,Y) \xrightarrow{\operatorname{Ext}^i(M,b)} \operatorname{Ext}^i(M,Z) \xrightarrow{\delta_i} \operatorname{Ext}^{i+1}(M,X)$$

is exact. Let $\varepsilon \in \operatorname{Ext}^i(M,Z)$ be represented by $f \in \operatorname{Hom}(Q_i,Z)$ with $f\varphi_{i+1} = 0$. Now, if $\varepsilon \in \operatorname{Im}\operatorname{Ext}^1(M,b)$ then $f = b\ell' + m\varphi_i$ for some $\ell' \in \operatorname{Hom}(Q_i,Y)$ with $\ell'\varphi_{i+1} = 0$ and some $m \in \operatorname{Hom}(Q_{i-1},Z)$. By Lemma 5.19

we have $\delta_i(f) = \delta_i(b\ell)$ and by construction $\delta_i(b\ell) = 0$ follows since the factorization $h\colon Q_{i+1} \to X$ with $ah = \ell\varphi_{i+1} = 0$ can be chosen to be $h = 0$. This shows $\operatorname{Im}\operatorname{Ext}^i(M, b) \subseteq \operatorname{Ker}\delta_i$.

To see the converse we recall that the construction of $\delta_i(\varepsilon)$ yields two homomorphisms, namely $\ell\colon Q_i \to Y$ such that $f = b\ell$ and $h\colon Q_{i+1} \to X$ such that $ah = \ell\varphi_{i+1}$. Now, $\delta_i(\varepsilon) = 0$ holds if and only if the factorization h belongs to $\operatorname{Im}\operatorname{Hom}(\varphi_{i+1}, X)$, that is, $h = g\varphi_{i+1}$ for some $g \in \operatorname{Hom}(Q_i, X)$. Consequently, for $\ell' = ag$ we have $f = b\ell = b(\ell - \ell')$ since $b\ell' = bag = 0$ and $(\ell - \ell')\varphi_{i+1} = ah - ah = 0$. This shows that $\ell - \ell' \in \operatorname{Ker}\operatorname{Hom}(\varphi_{i+1}, Y)$ and $f \in \operatorname{Im}\operatorname{Ext}^i(M, b)$.

(iii) For each $i \geq 0$, the sequence

$$\operatorname{Ext}^i(M, Z) \xrightarrow{\delta_i} \operatorname{Ext}^{i+1}(M, X) \xrightarrow{\operatorname{Ext}^{i+1}(M,a)} \operatorname{Ext}^{i+1}(M, Y)$$

is exact. Let $\eta \in \operatorname{Ext}^{i+1}(M, X)$ be represented by $h\colon Q_{i+1} \to X$ satisfying $h\varphi_{i+2} = 0$. Assume that $h = \delta_i(\varepsilon)$ for some $\varepsilon \in \operatorname{Ext}^i(M, Z)$, that is, ε is represented by a homomorphism $f\colon Q_i \to Z$ and by construction we have $\ell\colon Q_i \to Y$ such that $f = b\ell$ and $ah = \ell\varphi_{i+1}$. The latter equation shows $ah \in \operatorname{Im}\operatorname{Hom}(\varphi_{i+1}, Y)$ and therefore $\operatorname{Ext}^{i+1}(M, a)(h) = 0$ showing $\operatorname{Im}\delta_i \subseteq \operatorname{Ker}\operatorname{Ext}^{i+1}(M, a)$.

For the converse inclusion we assume that η lies in $\operatorname{Ker}\operatorname{Ext}^{i+1}(M, a)$, that is, $ah = \ell'\varphi_{i+1}$ for some $\ell'\colon Q_i \to Y$. Then set $f' = b\ell'$ with equivalence class $\varepsilon' \in \operatorname{Ext}^i(M, X)$ and observe that by construction of $\delta_i(\varepsilon')$ and Lemma 5.19 we have $\eta = \delta_i(\varepsilon')$. This shows that shows $\operatorname{Ker}\operatorname{Ext}^{i+1}(M, a) \subseteq \operatorname{Im}\delta_i$. $\square$

The infinite exact sequence of Theorem 5.20 is called the **long exact sequence obtained by applying** $\operatorname{Hom}(M, ?)$ **to** (5.1). Taking an injective resolution of M we get connecting maps $\delta_i'\colon \operatorname{Ext}^i_A(M, X) \to \operatorname{Ext}^{i+1}_A(M, Z)$ and then a second long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \operatorname{Ext}^{i-1}_A(X, M) \arrow[dll, "\delta_i'"] \\
\operatorname{Ext}^i_A(Z, M) \arrow[r] & \operatorname{Ext}^i_A(Y, M) \arrow[r] & \operatorname{Ext}^i_A(X, M) \arrow[dll, "\delta_{i-1}'"] \\
\operatorname{Ext}^{i+1}_A(Z, M) \arrow[r] & \cdots &
\end{tikzcd}$$

which starts with $0 \to \operatorname{Hom}(Z, M) \to \operatorname{Hom}(Y, M) \to \cdots$, and is called **the long exact sequence obtained from** (5.1) **by applying** $\operatorname{Hom}(?, M)$.

Exercises

5.4.1 Use the long exact sequences to prove that for the A-modules of a short exact sequence (5.1) the following inequalities hold:

$$\begin{aligned}\operatorname{pdim} Y &\leq \max\{\operatorname{pdim} X, \operatorname{pdim} Z\},\\ \operatorname{idim} Y &\leq \max\{\operatorname{idim} X, \operatorname{idim} Z\}.\end{aligned}$$

5.4.2 Show by induction on the dimension of an A-module M that $\operatorname{pdim} M \leq \max\{\operatorname{pdim} S_1, \ldots, \operatorname{pdim} S_n\}$, where $S_1, \ldots, S_n$ are the simple A-modules, see Proposition 4.34.

Chapter 6

The Auslander-Reiten theory

There is a series of notions and results named after Auslander and Reiten, all of them constitute the Auslander-Reiten theory. They concern the structure of module categories and yield a remarkable deep insight. Since it is valid in full generality it can be considered as the crown jewel of representation theory.

6.1 The Auslander-Reiten quiver and the infinite radical

As in the previous section, A denotes a finite-dimensional algebra over some field K. For any isomorphism class of indecomposables in $\operatorname{mod} A$, fix one representative and denote by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ given by those representatives.

For a full subcategory $\mathscr{D} \subseteq \mathscr{C}$ we denote by $\operatorname{add} \mathscr{D}$ the **additive closure** of $\mathscr{D}$, that is, the smallest full subcategory of $\mathscr{C}$, which contains each object of $\mathscr{D}$ and is **closed** under direct sums and isomorphisms: if x and y are objects in $\mathscr{D}$ and $x \oplus y \in \mathscr{C}$ a direct sum, then $x \oplus y \in \operatorname{add} \mathscr{D}$; similarly, if $x \in \mathscr{D}$ and $x' \in \mathscr{C}$ is isomorphic to x, then $x' \in \operatorname{add} \mathscr{D}$. Note that the conditions refer only to objects, since only full subcategories of $\mathscr{C}$ are involved.

By Lemma 4.14 the category $\operatorname{ind} A$ is a small category and therefore $\operatorname{ind} A$ is a spectroid and although it is usually not a finite one (namely if A is not of finite representation type), we can still associate the quiver $Q_{\operatorname{ind} A}$ to it, which is called the **Auslander-Reiten quiver** of A or sometimes **representation quiver** and denoted by Γ_A.

The objects of the quiver Γ_A are the chosen representatives of isomorphism classes of indecomposables which constitute the objects of $\operatorname{ind} A$. The morphisms between two such indecomposables lie in $\operatorname{rad}(\operatorname{ind} A)$ but not in $\operatorname{rad}^2(\operatorname{ind} A)$. We have called such morphisms **irreducible** in Section 2.6. Note that a homomorphism is radical in $\operatorname{ind} A$ if and only if it is radical in $\operatorname{mod} A$. Therefore $\operatorname{rad}(\operatorname{ind} A)(M,N) = \operatorname{rad}(\operatorname{mod} A)(M,N)$.

Examples 6.1 (a) The quiver given in (2.6), at the end of Section 2.6, shows the Auslander-Reiten quiver Γ_A for the path algebra $A = K\overrightarrow{\mathbb{A}}_5$ of the linearly oriented quiver $\overrightarrow{\mathbb{A}}_5$, which is, as we know from Example 3.2 (a), isomorphic to the algebra A of lower triangular matrices.

(b) Example 3.32 shows a quotient of $K\overrightarrow{\mathbb{A}}_5$. The Auslander-Reiten quiver is shown in (3.3). ◊

The quiver Γ_A decomposes into **components** in a straightforward way: two vertices X and Y lie in the same component if there is a "unoriented path" of irreducible morphisms between them, more precisely, if there are objects $X = Z_0, Z_1, \ldots, Z_{t-1}, Z_t = Y$ such that for each $i = 1, \ldots, t$ there is an arrow (an irreducible morphism) $Z_{i-1} \to Z_i$ or an arrow $Z_i \to Z_{i-1}$. A component is **finite** if it only contains finitely many vertices, otherwise it **infinite**.

Now, Γ_A encodes many information of $\operatorname{mod} A$, but in the case of infinite representation type not all information. Remember that by Corollary 3.24 the radical of $\operatorname{ind} A$ is nilpotent if $\operatorname{ind} A$ is finite, that is, if A is of finite representation type. This no longer holds if A is of infinite representation type. One defines the **infinite radical** $\operatorname{rad}_A^\infty$ by

$$\operatorname{rad}_A^\infty(X,Y) = \bigcap_{n>0} \operatorname{rad}^n(\operatorname{ind} A)(X,Y).$$

It contains some information not given by Γ_A, namely the morphisms between different components or those which start in a component and end in the same component but still cannot be written as composition of finitely many irreducible morphisms.

Exercises

6.1.1 Let $M, N \in \operatorname{mod} A$ be two indecomposable A-modules and $f\colon M \to N$ a radical homomorphism. Prove the following characterization: f does not belong to rad(mod A) if and only if each factorization $f = hg$ with $g\colon M \to L$ and $h\colon L \to N$ (where L is not necessarily indecomposable) implies that either g is a **section** (sections are also called **split mono**) (that is, g admits a left inverse) or h is a **retraction** (retractions are also called **split epi**) (that is, h admits a right inverse).

6.1.2 Calculate the Auslander-Reiten quiver of the two subspace algebra, that is, the path algebra of the quiver $Q\colon 1 \to 3 \leftarrow 2$. Recall that the indecomposable representations are given in Proposition 2.5. For this you have to calculate first all morphisms between the indecomposables. Note that all calculations can be done directly in rep Q since this category is equivalent to mod A.

6.1.3 Let Q be the quiver of an algebra A. The algebra is called **connected** if Q is a connected. Show, that if Q is not connected then $A = A_1 \times A_2$ and Γ_A is the disjoint union of Γ_{A_1} and Γ_{A_2}.

6.2 Example: the Kronecker algebra

We want to calculate the Auslander-Reiten quiver for a very interesting algebra: the Kronecker algebra. For this, all morphism spaces are calculated explicitely. The calculations are rather lengthy and thus given only partially but with sufficient details for the reader to be able to fill in all the missing parts. If you do not want to follow all these calculations, you should at least appreciate the taste of pure linear algebra in them and read the main result Theorem 6.5 carefully before moving on to the next section.

The Kronecker algebra shall be denoted by A for the sake of simplicity. By Theorem 3.3, the category mod A is equivalent to rep Q, where Q is the Kronecker quiver, see Example 2.1 (b).

Recall also from Proposition 2.13 that rep Q is equivalent to the category $\mathscr{M}_Q$ which, in this case, is the Kronecker problem studied extensively in Section 1.4. The indecomposables of the Kronecker problem were classified in Proposition 1.6. We translated these solutions to the language of representations, see Example 2.2 and Exercise 2.2.2.

Hence, the following representations for $n \in \mathbb{N}$ and $m \in \mathbb{N}_{>0}$ give a complete set of pairwise non-isomorphic indecomposable representations of Q:

$$\begin{array}{llll} R_{m,\lambda}\colon & K^m \overset{\mathbf{1}_m}{\underset{J(m,\lambda)}{\rightrightarrows}} K^m & R_{m,\infty}\colon & K^m \overset{J(m,0)}{\underset{\mathbf{1}_m}{\rightrightarrows}} K^m \\ Q_n\colon & K^n \overset{\begin{bmatrix}\mathbf{1}_n \\ z^\top\end{bmatrix}}{\underset{\begin{bmatrix}z^\top \\ \mathbf{1}_n\end{bmatrix}}{\rightrightarrows}} K^{n+1} & J_n\colon & K^{n+1} \overset{[\mathbf{1}_n\ z]}{\underset{[z\ \mathbf{1}_n]}{\rightrightarrows}} K^n \end{array} \tag{6.1}$$

where z denotes a zero vector with n rows.

In order to determine Γ_A we have to calculate the morphism spaces between all these representations. The calculations are based on induction and are rather tedious. After you have seen some of them you should be able to work out the rest by your own, which is why we present only some of them.

Lemma 6.2. *If M is an indecomposable representation of Q such that there exists a non-zero morphism $f\colon J_\ell \to M$, then M is isomorphic to J_m for some $m \leq \ell$. If $m \leq \ell$, the space $\mathrm{Hom}_Q(J_\ell, J_m)$ has dimension $\ell - m + 1$ and consists morphisms in $\mathrm{rad}^{\ell-m}$. As a consequence, there are two arrows $J_\ell \to J_{\ell-1}$ in the Auslander-Reiten quiver Γ_A for each $\ell \geq 1$. Furthermore, if $J_\ell \to J_m$ is an arrow in Γ_A, then $m = \ell - 1$.*

Proof. A morphism $f\colon J_\ell \to J_m$ is given by two matrices $A \in K^{(m+1)\times(\ell+1)}$ and $B \in K^{m\times\ell}$ such that

$$B \cdot [\mathbf{1}_\ell\ z] = [\mathbf{1}_m\ z] \cdot A \quad \text{and} \tag{6.2}$$
$$B \cdot [z\ \mathbf{1}_\ell] = [z\ \mathbf{1}_m] \cdot A \tag{6.3}$$

Note that the right hand side of (6.2) (resp. of (6.3)) is obtained by eliminating the last (resp. first) row from A whereas the left hand side is simply adding a zero column to B after the last column (resp. before the first column). Now consider the first column of A. From equation (6.2) we get that the first column of A looks as shown in the following on the left, whereas equation (6.3) gives the right:

$$[B_{11}\ B_{21}\ \cdots\ B_{m1}\ ?]^\top = [?\ 0\ \cdots\ 0\ 0]^\top,$$

hence $B_{i1} = 0$ for $i > 1$. If we look at the second column of A, we get:

$$[B_{12}\ B_{22}\ \cdots\ B_{m2}\ ?]^\top = [?\ B_{11}\ B_{21}\ \cdots\ B_{m1}]^\top = [?\ B_{11}\ 0\ \cdots\ 0]^\top,$$

which implies that $B_{22} = B_{11}$ and $B_{i2} = 0$ for $i > 2$. Inductively, we get $B_{ij} = B_{i-1,j-1}$ for $i \geq j$ and $B_{ij} = 0$ for $i > j$. Now, if we start comparing the last column of A we get

$$[0\ 0\ \cdots\ 0\ ?] = [?\ B_{1m}\ B_{2m}\ \cdots\ B_{\ell m}],$$

hence $B_{im} = 0$ for $i < \ell$. Therefore, we also have $A_{ij} = 0$ if $m - i > \ell - j$. This shows that if $m > \ell$, then both matrices A and B are identically zero. If $m \leq \ell$, then the matrix A is of the form $\beta_{\ell+1}(B_{11}, B_{12}, \ldots, B_{1,\ell-m+1})$, where

$$\beta_u(a_0, a_1, \ldots, a_r) = \begin{bmatrix} a_0 & a_1 & \cdots & a_r & & & \bigcirc \\ & a_0 & a_1 & \cdots & a_r & & \\ & & \ddots & \ddots & & \ddots & \\ \bigcirc & & & a_0 & a_1 & \cdots & a_r \end{bmatrix} \in K^{(u-r)\times u} \tag{6.4}$$

(the big circles indicate that all entries in that region are zero). Furthermore $B = \beta_m(B_{11}, B_{12}, \ldots, B_{1,\ell-m+1})$. As a result, we obtain that $\operatorname{Hom}_Q(J_\ell, J_m) = 0$ for $\ell < m$ and $\dim_K \operatorname{Hom}_Q(J_\ell, J_m) = \ell - m + 1$ for $\ell \geq m$.

We now show that each morphism $f\colon J_\ell \to Q_m$ is zero. If f is given by the two matrices $A \in K^{m\times(\ell+1)}$ and $B \in K^{(m+1)\times\ell}$, then they must satisfy

$$B \cdot [\mathbf{1}_\ell\ z] = \begin{bmatrix} \mathbf{1}_m \\ z^\top \end{bmatrix} \cdot A \quad \text{and} \tag{6.5}$$

$$B \cdot [z\ \mathbf{1}_\ell] = \begin{bmatrix} z^\top \\ \mathbf{1}_m \end{bmatrix} \cdot A. \tag{6.6}$$

If we extract the first row of (6.6), we see that on the left hand side we have a zero row, thus the first row of B is zero. Now the first row of (6.5) looks as follows

$$[A_{11}\ \cdots\ A_{1,\ell+1}] = [B_{11}\ \cdots\ B_{1\ell}\ 0] = [0\ \cdots\ 0\ 0]$$

and hence also the first row of A is zero. Now repeat the same for the second rows. Iteratively, we find that both matrices A and B must be zero.

The proof that all morphisms $f\colon J_\ell \to R_{m,\lambda}$ are zero is similar and left to the reader in Exercise 6.2.1. Therefore, the only non-zero morphisms starting in J_ℓ must end in some J_m for $m \leq \ell$.

Note also that $\operatorname{Hom}_Q(J_\ell, J_\ell) = K \cdot 1_{J_\ell}$. Therefore, and since $\operatorname{Hom}_Q(J_\ell, M) = 0$ for each indecomposable representation except for $M = J_m$ with $\ell \geq m$

we have

$$\operatorname{rad}^2(J_\ell, J_m) = \sum_{a\colon \ell>a>m} \operatorname{Hom}_Q(J_a, J_m) \operatorname{Hom}_Q(J_\ell, J_a).$$

A direct matrix calculation shows that

$$\beta_{u+r}(b_0, \ldots, b_s)\beta_u(a_0, \ldots, a_r) = \beta_u(c_0, \ldots, c_{r+s}), \tag{6.7}$$

where $c_n = \sum_{i=0}^{n} b_{n-i}a_i$ for $n = 0, \ldots, r+s$ with the convention that $a_i = 0$ for $i > r$ and similarly $b_j = 0$ for $j > s$. Since morphisms between non-isomorphic indecomposables are always radical, we have that $\beta_\ell(1, 0)$ and $\beta_\ell(0, 1)$ are radical. Moreover, it follows from (6.7), that

$$\beta_{\ell-r-s}(0,1)\cdots\beta_{\ell-r-1}(0,1)\beta_{\ell-r}(1,0)\cdots\beta_\ell(1,0) = \beta_\ell(\underbrace{0,\ldots,0}_{s \text{ zeros}},1,\underbrace{0,\ldots,0}_{r \text{ zeros}}).$$

This implies that

$$\operatorname{Hom}(J_\ell, J_m) = \operatorname{rad}^{\ell-m}(J_\ell, J_m).$$

Since the space $\operatorname{Hom}_Q(J_\ell, J_{\ell-1})$ has dimension two and $\operatorname{rad}^2(J_\ell, J_{\ell-1}) = 0$, we get two arrows $J_\ell \to J_{\ell-1}$ in Γ_A, namely $\beta_\ell(1, 0)$ and $\beta_\ell(0, 1)$. □

The proof of the following result is completely analogous and therefore left to the reader in Exercise 6.2.2.

Lemma 6.3. *If M is an indecomposable representation of Q such that there exists a non-zero morphism $f\colon M \to Q_m$ then M is isomorphic to Q_ℓ for some $\ell \leq m$. If $\ell \leq m$, the space $\operatorname{Hom}_Q(Q_\ell, Q_m)$ has dimension $m - \ell + 1$ and consists morphisms in $\operatorname{rad}^{m-\ell}$. As a consequence, there are two arrows $Q_{m-1} \to Q_m$ in the Auslander-Reiten quiver Γ_A for each $m \geq 1$. Furthermore, if $Q_\ell \to Q_m$ is an arrow in Γ_A, then $\ell = m - 1$.* □

The next is the third and last auxiliary result needed for the full determination of Γ_A.

Lemma 6.4. *If $f\colon R_{\ell,\lambda} \to R_{m,\mu}$ is a non-zero morphism, then $\lambda = \mu$. If f is irreducible, then $m = \ell \pm 1$. Moreover, for each $\ell \geq 1$ there exists precisely one arrow $R_{\ell,\lambda} \to R_{\ell+1,\lambda}$ and one arrow $R_{\ell+1,\lambda} \to R_{\ell,\lambda}$.*

Proof. Assume first that $\lambda, \mu \neq \infty$. A morphism $f\colon R_{\ell,\lambda} \to R_{m,\mu}$ consists of two matrices $A, B \in K^{m\times\ell}$ and the first condition is $A = B$, so the second is

$$AJ(\ell, \lambda) = J(m, \mu)A. \tag{6.8}$$

Now assume additionally that $\lambda \neq \mu$. Then the last entry in the first row is $\lambda A_{m1} = \mu A_{m1}$, hence $A_{m1} = 0$. Now look at the entry in position $(m-1, 1)$. We find $\lambda A_{m-1,1} + A_{m1} = \mu A_{m-1,1}$ and hence $A_{m-1,1} = 0$. Proceeding this way we obtain $A_{i1} = 0$ for all i. Observe that therefore $(J(m, \mu)A)_{i2} = \mu A_{i2}$ for all i and we may repeat our procedure for the second column showing $A_{i2} = 0$ for all i. Inductively, we get that A is the zero matrix.

If $\lambda = \infty \neq \mu$ or $\lambda \neq \infty = \mu$ a very similar argument shows the same result. Hence

$$\mathrm{Hom}(R_{\ell,\lambda}, R_{m,\mu}) = 0, \qquad \text{if } \lambda \neq \mu.$$

Now, if $\lambda = \mu \neq \infty$, then each morphism $f\colon R_{\ell,\lambda} \to R_{m,\lambda}$ is given by a pair of matrices (A, A) which satisfies

$$A(\lambda \mathbf{1}_\ell + J(\ell, 0)) = (\lambda \mathbf{1}_m + J(m, 0))A.$$

This is equivalent to $AJ(\ell, 0) = J(m, 0)A$. Proceeding from the lower left corner up and to the right we conclude that $A_{ij} = 0$ if one of the following conditions is satisfied: $i > j$ or $m - i < \ell - j$. For all other entries $A_{ij} = A_{i+1,j+1}$ holds. Hence, if $\ell \leq m$, the matrix A is of the form $A = \gamma_{m,\ell}(A_{11}, \dots, A_{1\ell})$ where

$$\gamma_{m,\ell}(a_0, \dots, a_{\ell-1}) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{\ell-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_1 \\ & & & a_0 \\ & \bigcirc & & \\ & & & \end{bmatrix} \in K^{m\times\ell}$$

and if $\ell > m$, then A is of the form $A = \gamma_{m,\ell}(A_{m\ell}, A_{m-1,\ell}, \dots, A_{1\ell})$ where in this case the matrix $\gamma_{m,\ell}(a_0, \dots, a_{m-1})$ is defined as

$$\gamma_{m,\ell}(a_0, \dots, a_{m-1}) = \begin{bmatrix} & & a_0 & a_1 & \cdots & a_{m-1} \\ & & & \ddots & \ddots & \vdots \\ \bigcirc & & & & \ddots & a_1 \\ & & & & & a_0 \end{bmatrix} \in K^{m\times\ell}.$$

A similar calculation deals with the case when $\lambda = \infty$ or $\mu = \infty$. We conclude that $\mathrm{Hom}(R_{\ell,\lambda}, R_{m,\lambda})$ has dimension $\min(\ell, m)$.

A note of caution: As we see, the same pair (A, A) of matrices constitutes a morphism $R_{\ell,\lambda} \to R_{m,\lambda}$, regardless of the value $\lambda \in K \cup \{\infty\}$. However, as morphisms they are different, as a morphism always "knows" to which pair of objects it belongs.

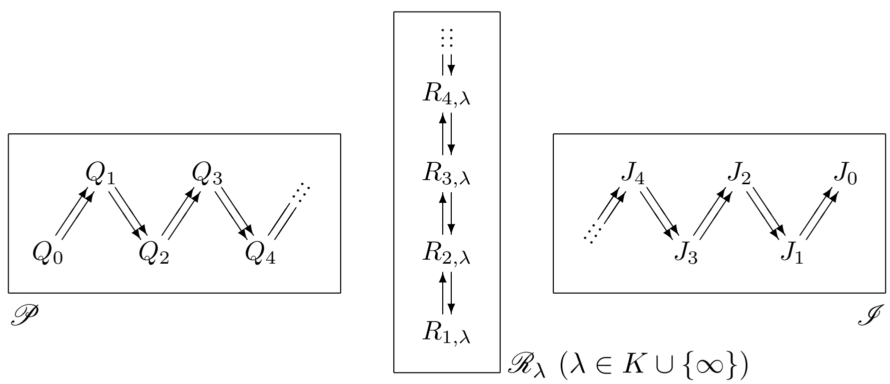
Now, observe that in the sum on the right hand side of

$$\operatorname{rad}(R_{\ell,\lambda}, R_{m,\lambda}) = \sum_{M \in \operatorname{ind} Q} \operatorname{rad}(M, R_{m,\lambda}) \operatorname{rad}(R_{\ell,\lambda}, M)$$

most of the summands are zero, since by Lemma 6.2 we have $\operatorname{Hom}(J_n, R_{m,\lambda}) = 0$ for all n, by Lemma 6.3, $\operatorname{Hom}(R_{\ell,\lambda}, Q_n) = 0$ for all n and by the above $\operatorname{Hom}(R_{\ell,\lambda}, R_{n,\nu}) = 0$ for all $\nu \neq \lambda$. Hence only the summands for $M = R_{n,\lambda}$ may contribute.

Similarly, as in the proof of Lemma 6.2, we see that the morphisms given by the matrices $\gamma_{m,\ell}(a_0, \dots, a_r)$ behave well under composition and conclude that $\operatorname{Hom}(R_{\ell,\lambda}, R_{m,\lambda}) = \operatorname{rad}^{|\ell-m|}(R_{\ell,\lambda}, R_{m,\lambda})$ and $\operatorname{rad}^{|\ell-m|+1}(R_{\ell,\lambda}, R_{m,\lambda}) = 0$. Hence, there is exactly one arrow $R_{\ell,\lambda} \to R_{\ell+1,\lambda}$ and one $R_{\ell+1,\lambda} \to R_{\ell,\lambda}$ and no other arrow between two vertices of the form $R_{\ell,\lambda}$ and $R_{m,\mu}$. $\square$

Theorem 6.5. *The Auslander-Reiten quiver Γ_A of the Kronecker algebra A looks as follows:*



More precisely, there are two special components: the **preprojective component** $\mathscr{P}$ *which contains all A-modules Q_n and two arrows $Q_n \to Q_{n+1}$ for $n \geq 0$, and the* **preinjective component** $\mathscr{I}$ *which contains all J_n and two arrows $J_{n+1} \to J_n$ for $n \geq 0$. Moreover, there is a family $(\mathscr{R}_\lambda)_{\lambda \in K \cup \{\infty\}}$ of* **regular components**, *each $\mathscr{R}_\lambda$ contains all $R_{m,\lambda}$ and two arrows between $R_{m,\lambda}$ and $R_{m+1,\lambda}$, one in each direction for $m \geq 1$. And these are all arrows in Γ_A.*

All morphisms which go in the picture from right to left are zero and each morphism $Q_\ell \to J_n$ factors, for each λ, through some $R_{m,\lambda}$. Furthermore, for $\lambda \neq \mu$ all morphisms $R_{\ell,\lambda} \to R_{m,\mu}$ are zero.

Proof. We have already seen in Lemma 6.2, 6.3 and 6.4 that in Γ_A, there are arrows as described in the statement and that the morphisms from right to left and between different $\mathscr{R}_\lambda$ and $\mathscr{R}_\mu$ are zero.

It remains to see that there are no further arrows and to prove the stated factorization property. For this, we first show these properties for the case $\lambda = 0$, which makes calculations easy. A morphism $f\colon Q_\ell \to R_{m,0}$ is given by two matrices $A \in K^{m\times\ell}$ and $B \in K^{m\times(\ell+1)}$ which satisfy

$$A = B \begin{bmatrix} \mathbf{1}_\ell \\ z^\top \end{bmatrix},$$
$$J(m,0)A = B \begin{bmatrix} z^\top \\ \mathbf{1}_\ell \end{bmatrix}.$$

Substitution of the first equation in the second and an inductive argument yield that B satisfies $B_{ij} = 0$ if $i + j \geq m$ and $B_{ij} = B_{i-1,j+1}$ for all $1 < i \leq m$, $1 \leq j \leq \ell$. Typical examples of such matrices are $\sigma^{(i)}_{m,\ell+1} \in K^{m\times(\ell+1)}$ with entries given by

$$\left(\sigma^{(i)}_{m,\ell+1}\right)_{ab} = \begin{cases} 1, & \text{if } a+b-1 = i, \\ 0, & \text{else.} \end{cases} \tag{6.9}$$

The pairs

$$s^{(i)}_{m,\ell} = \left(\sigma^{(i)}_{m,\ell}, \sigma^{(i)}_{m,\ell+1}\right), \qquad \text{for } i = 1, \ldots, m \tag{6.10}$$

give a basis of $\operatorname{Hom}_Q(Q_\ell, R_{m,0})$.

Furthermore, a direct calculation shows that

$$\sigma^{(i)}_{m,\ell} = \gamma_{m,m+1}(1,0,\ldots,0)\sigma_{m+1,\ell},$$

and therefore $s^{(i)}_{m,\ell}$ belongs to rad^2. Hence none of the morphisms $s^{(i)}_{m,\ell}$ can be irreducible and consequently, in Γ_A, there is no arrow from Q_ℓ to $R_{m,0}$ for any ℓ, m.

Now, a completely analogous calculation shows that in Γ_A there is no arrow from $R_{m,0}$ to J_n for any m, n. Moreover, if $(C,D)\colon R_{m,0} \to J_n$ is a morphism, then C satisfies $C_{ij} = 0$ for $i > j$ and $C_{ij} = C_{i+1,j+1}$ for $1 \leq i < n$, $1 \leq j \leq m$. Typical examples of such matrices are $\rho^{(j)}_{n+1,m} \in K^{(n+1)\times m}$ for $j = 1, \ldots, m$ whose entries are given by

$$\left(\rho^{(j)}_{n+1,m}\right)_{cd} = \begin{cases} 1, & \text{if } d-c+1 = j, \\ 0, & \text{else.} \end{cases} \tag{6.11}$$

Furthermore, we define the morphisms $r_{m,n}^{(j)} = (\rho_{n+1,m}^{(j)}, \rho_{n,m}^{(j)})$ for $j = 1, \dots, m$. Note that they form a basis of $\operatorname{Hom}(R_{m,0}, J_n)$.

Finally, we determine all morphisms $(E, F)\colon Q_\ell \to J_n$. With similar arguments as seen before, we conclude that the matrix E satisfies $E_{ij} = E_{i-1,j+1}$ (for all i and j, where this makes sense). Typical such matrices are $\sigma_{n+1,\ell}^{(i)}$ given as in (6.9), with the difference that all indices $i = 1, \dots, \ell + n$ are possible. The pairs

$$t_{n,\ell}^{(i)} = \left(\sigma_{n+1,\ell}^{(i)}, \sigma_{n,\ell+1}^{(i)}\right), \qquad \text{for } i = 1, \dots, \ell + n \tag{6.12}$$

form a basis of $\operatorname{Hom}(Q_\ell, J_n)$.

Now it is easy to see that for $m = \ell + n$, we have $\rho_{n+1,m}^{(1)} \sigma_{m,\ell}^{(i)} = \sigma_{n+1,\ell}^{(i)}$ and $\rho_{n,m+1}^{(1)} \sigma_{m+1,\ell+1}^{(i)} = \sigma_{n,\ell+1}^{(i)}$ and therefore

$$t_{n,\ell}^{(i)} = r_{n,m}^{(1)} s_{m,\ell}^{(i)}, \qquad \text{for } i = 1, \dots, m = \ell + n \tag{6.13}$$

This shows that each morphism $Q_\ell \to J_n$ factors through $R_{\ell+n,0}$, that is the factorization property for $\lambda = 0$.

To obtain the analogous statements for all other possible values of λ we use a trick. Recall that A is the path algebra $A = KQ$, where Q is the Kronecker quiver:

$$Q: \qquad 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2\,.$$

Consider the K-linear map $H^\lambda\colon A \to A$ defined by $H^\lambda(e_i) = e_i$ for $i = 1, 2$, $H^\lambda(\alpha) = \alpha$ and $H^\lambda(\beta) = \beta + \lambda\alpha$. Then it follows that $H^\lambda(\delta\gamma) = H^\lambda(\delta) H^\lambda(\gamma)$ for any two paths δ, γ and hence for any two elements of the path algebra. Consequently H^λ is a homomorphism of K-algebras. Since H^λ has an inverse, namely $H^{-\lambda}$, it is an isomorphism of K-algebras.

Now, if M is an A-module with structure map $A \times M \to M, (a, m) \mapsto a \cdot m$, we can define a second structure map $A \times M \to M, (a, m) \mapsto a *^\lambda m := H^\lambda(a) \cdot m$ and shall denote by M^λ the abelian group M together with this map. If $f\colon M \to N$ is a homomorphism of A-modules, then also $f^\lambda = f\colon M^\lambda \to N^\lambda$, since

$$a *^\lambda f(m) = H^\lambda(a) \cdot f(m) = f(H^\lambda(a) \cdot m) = f(a *^\lambda m).$$

Hence, we get a functor $?^\lambda\colon \operatorname{mod} A \to \operatorname{mod} A$ which has $?^{-\lambda}$ as inverse. These functors are therefore isomorphisms, in particular they send indecomposables to indecomposables. Hence $(Q_\ell)^\lambda$ is again indecomposable and its vector spaces have dimensions ℓ and $\ell+1$ in the vertices 1 and 2 respectively. Since there is only one such indecomposable in $\operatorname{ind} A$, we must have $(Q_\ell)^\lambda \simeq Q_\ell$. Similarly $(J_n)^\lambda \simeq J_n$. We now describe $V = (R_{m,0})^\lambda$. It is clear that we only have to determine the multiplication with β, and if (v_1, v_2) denotes a general element in V, then only $\beta *^\lambda v_1$ has to be considered:

$$\beta *^\lambda v_1 = H^\lambda(\beta) \cdot v_1 = (\beta + \lambda\alpha) \cdot v_1 = J(m, \lambda) v_1,$$

that is, $(R_{m,0})^\lambda = R_{m,\lambda}$. Since $?^\lambda$ is a functor, we get the statements for any $\lambda \neq \infty$. For this last case the interchange automorphism $I\colon A \to A$, which interchanges α and β may be considered covering this last case in the same way. □

Remark 6.6. We have presented a "low-tech" proof. Other approaches can for example be found in Ringel's book, [22], where much more theory is developed before and then used in the proof. We took this rather bumpy ride to show that the representation's point of view reduces many questions effectively to linear algebra, where they might be solved. In the following, we present gradually more theory, as it offers more insight, and often come back to this particular example. ♢

Exercises

6.2.1 Verify that all morphisms $f\colon J_\ell \to R_{m,\lambda}$ are zero.

6.2.2 Prove Lemma 6.3.

6.3 The Auslander-Reiten translate

As usual, let A be a finite-dimensional K-algebra. A sequence of morphisms in $\operatorname{mod} A$

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_\to \ldots \to m_{t-1} \xrightarrow{f_t} M_t \tag{6.14}$$

is called **exact** if $\operatorname{Ker} f_i = \operatorname{Im} f_{i-1}$ holds for all $i = 1, \ldots t$.

Example 6.7. The sequence

$$0 \to M \xrightarrow{\mathrm{id}_M} M \to 0$$

is always exact for each module $M \in \operatorname{mod} A$. Also, for each morphism $f\colon M \to N$ the sequence

$$0 \to \operatorname{Ker} f \xrightarrow{\iota} M \xrightarrow{f} N \xrightarrow{\pi} \operatorname{Coker} f \to 0,$$

where ι and π denote the canonical inclusion and projection respectively, is an exact sequence. $\lozenge$

By Lemma 4.9 a sequence $0 \to M \xrightarrow{f} N$ (resp. $M \xrightarrow{f} N \to 0$) is exact if and only if f is injective (resp. surjective).

A **projective presentation** of a module $M \in \operatorname{mod} A$ is an exact sequence

$$Q \xrightarrow{\mu} P \xrightarrow{\varepsilon} M \to 0 \tag{6.15}$$

where P and Q are projectives. Such a projective presentation is **minimal** if ε is a projective cover of M and μ is a projective cover of the object $\operatorname{Ker} \varepsilon$. This shows that a projective presentation always exists. The dual concept is called **injective presentation**.

Example 6.8. We continue Example 4.31, where we calculated the projective cover of $M = [2,3]$ to be $\varepsilon = \iota_{2,5}^{2,3} \colon [2,5] \to [2,3]$. Its kernel is $[4,5]$ which is projective. Hence

$$[4,5] \to [2,5] \to [2,3]$$

is a minimal projective presentation of $[2,3]$. $\lozenge$

Note that in a projective presentation (6.15) it is enough to give the morphism μ, since $\varepsilon = \operatorname{Coker}(\mu)$ is determined by μ up to isomorphism. Now, if $\varphi_1 \colon Q_1 \to P_1$ is a projective presentation of M_1 and $\varphi_2 \colon Q_2 \to P_2$ a projective presentation of M_2 then clearly $\varphi_1 \oplus \varphi_2 \colon Q_1 \oplus Q_2 \to P_1 \oplus P_2$ is a projective presentation of $M_1 \oplus M_2$. Conversely, if a projective presentation φ admits a decomposition $\varphi = \varphi_1 \oplus \varphi_2$, then the presented module decomposes.

We are now ready to describe an important construction for the study of the module category $\operatorname{mod} A$, namely that of the **Auslander-Reiten translate**, which will be denoted by τ_A: given an module $M \in \operatorname{mod} A$, we first take a minimal projective presentation (6.15), apply the Nakayama functor ν_A to the morphism $\mu \colon Q \to P$ and then take the kernel

$$N \xrightarrow{\operatorname{Ker}(\nu_A \mu)} \nu_A Q \xrightarrow{\nu_A \mu} \nu_A P. \tag{6.16}$$

We then define the **Auslander-Reiten translate** $\tau_A M = N$ of M, where N is the kernel in (6.16). The important properties of the Auslander-Reiten translate, which is also called **AR-translate** for short, are gathered in the following result.

Proposition 6.9. *Let A be a finite-dimensional K-algebra. The AR-translate $\tau_A M$ of $M \in \operatorname{mod} A$ is independent on the choice of the minimal projective presentation of M. We have $\tau_A M = 0$ if and only if M is projective. Furthermore, the AR-translate induces a bijection between the indecomposable non-projectives and the indecomposable non-injectives, up to isomorphism.*

Proof. The independence of the choice of projective presentation follows from Corollary 4.29. Now, if M is projective then $\mathrm{id}_M \colon M \to M$ is a projective cover and $0 \to M \to M$ a projective presentation of M. Consequently $\tau_A M = 0$. Conversely, if $\operatorname{Ker} \nu_A \mu = 0$, then $\nu_A \mu$ is injective and therefore splits by the dual of Exercise 4.5.1, that is,

$$\nu_A \mu \simeq \begin{bmatrix} \mathrm{id}_{\nu_A Q} \\ 0 \end{bmatrix} : \nu_A Q \to \nu_A Q \oplus \nu_A Q' = \nu_A P.$$

Consequently $\varepsilon = \operatorname{Coker} \mu = [0 \;\; \mathrm{id}_{Q''}] \colon Q \oplus Q' \to Q' = M$ and M is projective.

Now, suppose that M is an indecomposable non-projective A-module and (6.15) a minimal projective presentation of M. Then μ cannot decompose since otherwise M decomposes. Therefore $\nu_A \mu$ does not decompose showing that $\tau_A M$ is indecomposable or zero. However, the latter is not possible since M was assumed to be non-projective.

Using dual arguments, we can start with any A-module L and construct the minimal injective presentation

$$0 \to L \xrightarrow{\varphi} I \xrightarrow{\lambda} J,$$

then apply the inverse of the Nakajama functor ν_A^{-1} and construct $\tau_A^- L = \operatorname{Coker}(\nu_A^{-1} \lambda)$. Dually, we have $\tau_A^- L = 0$ if and only if L is injective. Hence for M an indecomposable non-projective $\tau_A M$ is indecomposable and non-injective. Consequently, τ_A gives a bijection between the isomorphism classes of indecomposable non-projectives and the isomorphism classes of indecoposables non-injectives. $\square$

Remark 6.10. The notation τ_A^- might seem strange at first sight, but it remains us of the fact that τ_A^- is not really an inverse of τ_A: for each projective A-module M we have $\tau_A M = 0$ and there exists no A-module N such that $\tau^- N = M$. Therefore τ_A and τ_A^- are not properly inverse to each other. However, we will use $\tau^{-\ell}$ instead of the clumsy $(\tau^-)^\ell$. $\lozenge$

Example 6.11. As in the Examples 4.31 and 6.8 let $A = K\overrightarrow{\mathbb{A}}_5$ and $M = [2,3]$. Then $\iota\colon P_4 = [4,5] \to [2,5] = P_2$ is the projective presentation. Hence we obtain $\nu_A\iota\colon I_4 = [1,4] \to [1,2] = I_2$ by applying the Nakayama functor and $\tau_A M = \operatorname{Ker}\nu_A\iota = [3,4]$. We can do this with all indecomposables and obtain the following picture:

$$
\begin{tikzcd}[column sep=small]
 & & & & {[1,5]} \arrow[dr] & & & & \\
 & & & {[2,5]} \arrow[ur] \arrow[dr] & & {[1,4]} \arrow[ll, dashed, "\tau_A"'] \arrow[dr] & & & \\
 & & {[3,5]} \arrow[ur] \arrow[dr] & & {[2,4]} \arrow[ll, dashed, "\tau_A"'] \arrow[ur] \arrow[dr] & & {[1,3]} \arrow[ll, dashed, "\tau_A"'] \arrow[dr] & & \\
 & {[4,5]} \arrow[ur] \arrow[dr] & & {[3,4]} \arrow[ll, dashed, "\tau_A"'] \arrow[ur] \arrow[dr] & & {[2,3]} \arrow[ll, dashed, "\tau_A"'] \arrow[ur] \arrow[dr] & & {[1,2]} \arrow[ll, dashed, "\tau_A"'] \arrow[dr] & \\
{[5,5]} \arrow[ur] & & {[4,4]} \arrow[ll, dashed, "\tau_A"'] \arrow[ur] & & {[3,3]} \arrow[ll, dashed, "\tau_A"'] \arrow[ur] & & {[2,2]} \arrow[ll, dashed, "\tau_A"'] \arrow[ur] & & {[1,1]} \arrow[ll, dashed, "\tau_A"']
\end{tikzcd}
\tag{6.17}
$$

The Auslander-Reiten quiver should always be considered as equipped with the AR-translate. But there is still more to explore and this will be done in the next chapter. $\diamondsuit$

Exercises

6.3.1 Show that a sequence $U \xrightarrow{f} V \xrightarrow{g} W$ of representations of a quiver Q is exact if and only if for each vertex i the sequence $U_i \xrightarrow{f_i} V_i \xrightarrow{g_i} W_i$ is exact.

6.3.2 Calculate the Auslander-Reiten translate of all indecomposable non-projective A-modules, where A is the algebra considered in Example 3.32.

6.3.3 Calculate the Auslander-Reiten translate of the indecomposable non-projectives over the two subspace algebra.

6.3.4 Show that if in the exact sequence (6.14) the two terms M_{i-1} and M_{i+1} are zero, then also M_i.

6.4 Auslander-Reiten sequences

Let A be a finite-dimensional K-algebra. Note that for each A-module M the space $\operatorname{Hom}_A(M, A)$ is a right A-module, where the right multiplication fr for $f \in \operatorname{Hom}_A(M, A)$ and $r \in A$ is given as $fr\colon M \to A, s \mapsto f(s)r$. In other words, $\operatorname{Hom}_A(M, A)$ is a left A^{op}-module and therefore $\mathrm{D}\operatorname{Hom}_A(M, A)$ is a left A-module. The following result shows that $\mathrm{D}\operatorname{Hom}_A(?, A)$ generalizes the Nakayama functor ν_A defined in Section 4.5.

Lemma 6.12. *Let A be a finite-dimensional K-algebra. For each projective A-module P we have $\nu_A P \simeq \mathrm{D}\,\mathrm{Hom}_A(P, A)$ as A-modules and for each homomorphism $\varphi\colon P \to P'$ we have $\nu_A\varphi \simeq \mathrm{D}\,\mathrm{Hom}_A(\varphi, A)$.*

Proof. For the indecomposable projective P_x associated to the vertex x we get by Yoneda's Lemma 4.24

$$\mathrm{Hom}_A(P_x, A) \xrightarrow{\Phi_{x,A}} A_x = e_x A, f \mapsto f(e_x).$$

Since $\Phi_{x,A}(fr) = fr(e_x) = f(e_x)r = \Phi_{x,A}(f)r$ for each $r \in A$ this bijection preserves the right multiplication. Dualization yields an isomorphism of left A-modues

$$\nu_A P_x = I_x = \mathrm{D}e_x A \xrightarrow{\mathrm{D}\Phi_{x,A}} \mathrm{D}\,\mathrm{Hom}_A(P_x, A).$$

This isomorphism is functorial in x, that is, if $\alpha\colon x \to y$ is an arrow of the quiver of A then $P_\alpha\colon P_y \to P_x$ is a homomorphism and $\nu_A P_\alpha = I_\alpha = \mathrm{D}(\alpha\cdot?)$ corresponds to $\mathrm{D}\,\mathrm{Hom}_A(P_\alpha, A)$, see Remark 4.25 (b). The result follows now, as each projective A-module is decomposed into a direct sum of indecomposables of the P_x. $\square$

We can now define the **Nakayama functor** for the whole module category $\nu_A\colon \bmod A \to \bmod A$ by

$$\nu_A = \mathrm{D}\,\mathrm{Hom}_A(?, A). \tag{6.18}$$

We give one nice property of the Nakayama functor at once.

Lemma 6.13. *For each exact sequence $X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$ the Nakayama functor yields an exact sequence*

$$\nu_A X \xrightarrow{\nu_A a} \nu_A Y \xrightarrow{\nu_A b} \nu_A Z \to 0.$$

Proof. We know from Theorem 5.20 that

$$0 \to \mathrm{Hom}(Z, A) \xrightarrow{\mathrm{Hom}(b,A)} \mathrm{Hom}(Y, A) \xrightarrow{\mathrm{Hom}(a,A)} \mathrm{Hom}(X, A)$$

is exact. Applying the dualization over the ground field reverses the arrows and preserves exactness, see Exercise 5.1.5, hence the result. $\square$

With the new definition of the Nakayama functor (6.18) we can also generalize Lemma 4.24. For this let L, M be A-modules. Note that any homomorphism $A \to L$ is given as $\mathrm{mul}_\ell \colon r \mapsto r\ell$ for some $\ell \in L$. Hence for a given homomorphism $g \colon M \to A$ and a given element $\ell \in L$, we can define the homomorphism

$$\mathrm{mul}_\ell \circ g \colon M \to L, m \mapsto g(m)\ell. \tag{6.19}$$

Proposition 6.14. *For any two A-modules L and M there exists a natural map*

$$\alpha_{L,M} \colon \mathrm{D}\,\mathrm{Hom}_A(M, L) \to \mathrm{Hom}_A(L, \nu_A M), \tag{6.20}$$

given by $\alpha_{L,M}(\varphi) = \varphi(\mathrm{mul}_\ell \circ ?)$. This map is functorial and additive in both arguments and it is a bijection if M is projective. Furthermore, the kernel of $\alpha_{L,M}$ can be described explicitely

$$\mathrm{Ker}\,\alpha_{L,M} = \{\varphi \mid \varphi(\mathrm{proj}_A(M, L)) = 0\},$$

where $\mathrm{proj}_A(M, L)$ is the subset of $\mathrm{Hom}_A(M, L)$ given by those homomorphisms which factor over some projective A-module.

Proof. We start by looking that the explicit description makes sense. For a given $\varphi \in \mathrm{D}\,\mathrm{Hom}_A(M, L)$ the value $\alpha_{L,M}(\varphi)$ has to be a homomorphism $L \to \mathrm{D}\,\mathrm{Hom}_A(M, A)$, that is, for each $\ell \in L$ a linear map $\mathrm{Hom}_A(M, A) \to K$ has to be given. This is done by $g \mapsto \varphi(\mathrm{mul}_\ell \circ g)$.

Now, we look at the case where M is a projective indecomposable $M = P_x = Ae_x$. By Yoneda's Lemma 4.24 we have the following sequence of functorial K-linear bijections

$$\mathrm{D}\,\mathrm{Hom}_A(P_x, L) \xrightarrow{F=\mathrm{D}\Phi_{x,L}^{-1}} \mathrm{D}L_x \xrightarrow{G=\Psi_{x,L}^{-1}} \mathrm{Hom}_A(L, \mathrm{D}e_x A). \tag{6.21}$$

Using Lemma 6.12 we have a bijection $\mathrm{D}\Phi_{x,A} \colon \mathrm{D}e_x A \to \mathrm{D}\,\mathrm{Hom}_A(P_x, A)$ and hence can prolongate the sequence (6.21) one step further:

$$\mathrm{Hom}_A(L, \mathrm{D}e_x A) \xrightarrow{H=\mathrm{Hom}_A(L,\mathrm{D}\Phi_{x,A})} \mathrm{Hom}_A(L, \mathrm{D}\,\mathrm{Hom}(P_x, A)). \tag{6.22}$$

Therefore the composition HGF is K-linear, bijective and functorial. We now show that HGF is in fact α_{L,P_x}.

The inverse of $\Phi_{x,L}$ was described in the proof of Yoneda's Lemma 4.24 and therefore for $\varphi \in \mathrm{Hom}_A(P_x, L)$ we have

$$F(\varphi) \colon L_x \to K, \ell \mapsto \varphi(\mathrm{mul}_\ell),$$

see the following picture.

$$\begin{tikzcd}
\operatorname{Hom}_A(Ae_x, L) \arrow[dd] & & L_x \arrow[ll] \arrow[ddll] \\
(r \mapsto r\ell) \arrow[dd, mapsto] & & \ell \arrow[ll, mapsto] \arrow[ddll, mapsto] \\
K & & \\
\varphi(r \mapsto r\ell) & &
\end{tikzcd}$$

For $\rho \in \mathrm{D}L_x$ we first observe that

$$HG(\rho) = \mathrm{D}\Phi_{x,A} \circ G(\rho) \colon L \to \mathrm{D}\operatorname{Hom}(Ae_x, A)).$$

The functor $G = \Psi_{x,L}^{-1}$ was described in the proof of Yoneda's Lemma 4.24. The evaluation $G(\rho)$ for $\rho \in \mathrm{D}L_x$ is given as shown in the next picture on the left-hand side. The right-hand side is obtained by evaluating the effect of $\mathrm{D}\Phi_{x,A}$.

$$\begin{tikzcd}
L \arrow[r, "G(\rho)"] & \mathrm{D}e_x A \arrow[r, "\mathrm{D}\Phi_{x,A}"] & \mathrm{D}\operatorname{Hom}(Ae_x, A) \\
\ell \arrow[r, mapsto] & \begin{pmatrix} e_x A & \longrightarrow & K \\ s & \longmapsto & \rho(s\ell) \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} \operatorname{Hom}(Ae_x, A) & \longrightarrow & K \\ f & \longmapsto & \rho(f(e_x)\ell) \end{pmatrix}
\end{tikzcd}$$

Now, if $\rho = \varphi(r \mapsto r\ell)$, then

$$\rho(f(e_x)\ell) = \varphi(r \mapsto rf(e_x)\ell),$$

but since $r = re_x$ we have $rf(e_x)\ell = f(r)\ell$. This shows that $HGF(\varphi)$ is the map $L \to \mathrm{D}\operatorname{Hom}(Ae_x, A), \ell \mapsto \varphi(r \mapsto f(r)\ell)$ exactly as does $\alpha_{L,P_x}(\varphi)$.

For a general projective A-module the result follows now immediately since HGF is functorial and therefore preserves direct sums. It remains to see that for any M the map $\alpha_{L,M}$ has the kernel as indicated in the statement.

Thus, assume that $\alpha_{L,M}(\varphi) = 0$. This means that for all $\ell \in L$ and all $g \colon M \to A$ the value $\varphi(\mathrm{mul}_\ell \circ g)$ equals zero. Now, let $f \colon M \to L$ be any homomorphism which factors over some projective A-module Q, say $f = f''f'$ for some $f' \colon M \to Q$ and some $f'' \colon Q \to L$. Take any surjective homomorphism $h = [h_1 \ \cdots \ h_m] \colon A^m \to L$ and use the projectivity of Q to factor f'' through h, that is, $f'' = hk$ for some $k = [k_1 \ \cdots \ k_m]^\top \colon Q \to A^m$. Then $f = hkf' = \sum_{i=1}^m h_i(k_i f')$ is the sum of homomorphisms which factor over A. As we have observed before $h_i = \mathrm{mul}_{\ell_i}$ for some $\ell_i \in L$. Since φ is additive we see that indeed $\varphi(f) = \sum_{i=1}^m \varphi(\mathrm{mul}_{\ell_i} \circ k_i f') = 0$ and therefore

$\varphi(\mathrm{proj}(M,L)) = 0$. The converse follows at once, since $\mathrm{mul}_\ell \circ g$ factors over the projective A and therefore $\alpha_{L,M}(\varphi) = 0$. □

Remark 6.15. For two A-modules X and Y we denote by $\mathrm{proj}_A(X,Y)$ the subspace of $\mathrm{Hom}_A(X,Y)$ given by all homomorphisms which factor over a projective A-module. It is straightforward that proj_A forms an ideal of the category $\mathrm{mod}\, A$. The respective quotients $\underline{\mathrm{Hom}}_A(X,Y) = \mathrm{Hom}_A(X,Y)/\mathrm{proj}_A(X,Y)$ form a category which is denoted by $\underline{\mathrm{mod}}\, A$.

Similarly, inj_A is the ideal of all homomorphisms which factor over an injective module and the quotients $\overline{\mathrm{Hom}}_A(X,Y) = \mathrm{Hom}_A(X,Y)/\mathrm{inj}_A(X,Y)$. form a category which is denoted by $\overline{\mathrm{mod}}\, A$. Both quotients, $\underline{\mathrm{mod}}\, A$ and $\overline{\mathrm{mod}}\, A$, are called **stable module categories**.

This allows to describe the kernel of $\alpha_{L,M}$ in the following way

$$\mathrm{Ker}\, \alpha_{L,M} = \mathrm{D}\, \underline{\mathrm{Hom}}_A(M,L).$$

◊

We have now gathered sufficient observations from homological algebra to set out and prove the most important result on the structure of the Auslander-Reiten quiver of A.

An exact sequence $0 \to L \xrightarrow{a} E \xrightarrow{b} M \to 0$ is called a **Auslander-Reiten sequence** if it is a short exact non-split sequence with indecomposable end terms and if any radical homomorphism $a' \colon L \to L'$ factors through a and any radical homomorphism $b' \colon M' \to M$ factors through b. Note that since L and M are indecomposable a homormorphism $a' \colon L \to L'$ is radical if and only if it is not a section, see Exercise 6.1.1 and similarly $b' \colon M' \to M$ is radical if and only if b' is not a retraction. The following result not only asserts the existence of such sequences, but it also implies that the end terms of such a sequence are always linked by the Auslander-Reiten translate.

Theorem 6.16 (Auslander-Reiten)**.** *Let A be a finite-dimensional algebra over K.*

(a) *For any two modules $L, M \in \mathrm{mod}\, A$, there are bijections*

$$\mathrm{Ext}^1_A(L, \tau_A M) \longrightarrow \mathrm{D}\underline{\mathrm{Hom}}_A(M,L),$$
$$\mathrm{Ext}^1_A(\tau_A^- L, M) \longrightarrow \mathrm{D}\overline{\mathrm{Hom}}_A(M,L)$$

which are functorial in both arguments.

(b) *For any non-projective indecomposable module M there exists an Auslander-Reiten sequence* $0 \to \tau_A M \to E \to M \to 0$.

(c) *For any non-injective indecomposable module L there exists an Auslander-Reiten sequence* $0 \to L \to E \to \tau_A^- L \to 0$.

Proof. Clearly, it is enough to prove (a) for indecomposable M because both sides preserve direct sums. If M is in addition projective, we trivially have equality since both sides are zero. So, suppose that M is non-projective and indecomposable.

Recall from Section 6.3 the construction of $\tau_A M$: We have to take a minimal projective presentation $Q \xrightarrow{\mu} P \xrightarrow{\varepsilon} M \to 0$ of M. Using Lemma 6.13 we get an exact sequence

$$0 \to \tau_A M \to \nu_A Q \xrightarrow{\nu_A \mu} \nu_A P \xrightarrow{\nu_A \varepsilon} \nu_A M \to 0 \tag{6.23}$$

since by definition $\tau_A M = \operatorname{Ker} \nu_A \mu$. If we compose $\nu_A \varepsilon \colon \nu_A P \to \nu_A M$ with the injective hull $\iota \colon \nu_A M \to J$ of $\nu_A M$, we get the starting of an injective resolution of $\tau_A M$ and hence can calculate by (5.12)

$$\operatorname{Ext}^1_A(L, \tau_A M) = \operatorname{Ker} \operatorname{Hom}_A(L, \iota \circ \nu_A \varepsilon) / \operatorname{Im} \operatorname{Hom}_A(L, \nu_A \mu). \tag{6.24}$$

Note that $\operatorname{Ker} \operatorname{Hom}_A(L, \iota \circ \nu_A \varepsilon) = \operatorname{Ker} \operatorname{Hom}_A(L, \nu_A \varepsilon)$ since ι is injective. It follows from the functoriality of $\alpha_{L,?}$ that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{D}\operatorname{Hom}_A(Q,L) \arrow[r, "\mathrm{D}\operatorname{Hom}_A(\mu, L)"] \arrow[d, "\alpha_{L,Q}"] & \mathrm{D}\operatorname{Hom}_A(P,L) \arrow[r, "\mathrm{D}\operatorname{Hom}_A(\varepsilon, L)"] \arrow[d, "\alpha_{L,P}"] & \mathrm{D}\operatorname{Hom}_A(M,L) \arrow[r] \arrow[d, "\alpha_{L,M}"] & 0 \\
\operatorname{Hom}_A(L, \nu_A Q) \arrow[r, "\operatorname{Hom}_A(L, \nu_A \mu)"'] & \operatorname{Hom}_A(L, \nu_A P) \arrow[r, "\operatorname{Hom}_A(L, \nu_A \varepsilon)"'] & \operatorname{Hom}_A(L, \nu_A M) &
\end{tikzcd}$$

Furthermore, the rows are exact. In the following sequence of K-linear bijections, the first is due to the bijectivity of $\alpha_{L,Q}$ and $\alpha_{L,P}$, the second and third to the exactness of the upper row in the diagram.

$$\begin{aligned}
\operatorname{Hom}_A(L, \nu_A P) / \operatorname{Im} \operatorname{Hom}_A(L, \nu_A \mu) &\simeq \mathrm{D}\operatorname{Hom}_A(P, L) / \operatorname{Im} \mathrm{D}\operatorname{Hom}(\mu, L) \\
&\simeq \mathrm{D}\operatorname{Hom}_A(P, L) / \operatorname{Ker} \mathrm{D}\operatorname{Hom}(\varepsilon, L) \\
&\simeq \mathrm{D}\operatorname{Hom}_A(M, L).
\end{aligned}$$

The following sequence of K-linear bijections yields the first formula of (a). The first bijection is due to (6.24), the last to Remark 6.15.

$$\begin{aligned}
\operatorname{Ext}^1_A(L, \tau_A M) &= \operatorname{Ker}\operatorname{Hom}(L, \nu_A \varepsilon) / \operatorname{Im}\operatorname{Hom}(L, \nu_A \mu) \\
&\simeq \operatorname{Ker}\left(\alpha_{L,M} \circ \operatorname{D}\operatorname{Hom}(\varepsilon, L)\right) / \operatorname{Ker}\operatorname{D}\operatorname{Hom}(\varepsilon, L) \\
&\simeq \operatorname{Ker}\alpha_{L,M} \\
&= \operatorname{D}\underline{\operatorname{Hom}}(M, L)
\end{aligned}$$

Observe that all bijections are functorial in both arguments. The second statement is completely dual to the first. This shows part (a).

For part (b), we choose $L = M$. Since M is not projective the identity homomorphism id_M does not factor over a projective A-module and since M is indecomposable $\operatorname{Hom}_A(M, M)$ is local with maximal ideal $\operatorname{rad}_A(M, M)$. We therefore can define a non-zero linear map $\Omega\colon \underline{\operatorname{Hom}}_A(M, M) \to K$ by $\Omega(\overline{\mathrm{id}}_M) = 1$ and $\Omega(\overline{r}) = 0$ for each radical morphism $r \in \operatorname{rad}_A(M, M)$.

According to part (a) the linear map Ω corresponds to a short exact sequence

$$\varepsilon_M\colon 0 \to \tau_A M \xrightarrow{a} E \xrightarrow{b} M \to 0 \tag{6.25}$$

which has indecomposable end terms and does not split since $\Omega \neq 0$. We will show that this is indeed an Auslander-Reiten sequence. Let $g\colon L \to M$ be a non-retraction. If we decompose L into indecomposables $L = \bigoplus_{i=1}^t L_i$ and write $g = [g_1 \ \dots \ g_t]$, we have that each g_i is a radical homomorphism. Therefore, if we consider

$$\operatorname{D}\underline{\operatorname{Hom}}_A(M, g_i)\colon \operatorname{D}\underline{\operatorname{Hom}}_A(M, M) \to \operatorname{D}\underline{\operatorname{Hom}}_A(M, L_i),$$

then $\Omega \in \operatorname{D}\underline{\operatorname{Hom}}(M, M)$ is mapped to $\Omega \circ (g_i \circ ?)$. Certainly $\Omega \circ (g_i \circ f) = 0$ holds for each endomorphism f of M since the composition $g_i f$ is always radical. This shows that $\operatorname{D}\underline{\operatorname{Hom}}_A(M, g)(\Omega) = 0$. Exploiting the functoriality of the formula given in part (a) we get that $\varepsilon_M g = \operatorname{Ext}^1_A(g, \tau_A M)(\varepsilon_M) = 0$, that is, the sequence obtained from (6.25) by taking the pull-back along g and b splits. Thus by Exercise 5.2.3 the homomorphism g factors through b. The rest namely that every non-section $\tau_A M \to L$ factors over a follows dually. $\square$

Remark 6.17. The Auslander-Reiten Theorem is one of the most important results in representation theory of finite-dimensional algebras. It is valid in broader generality, see [**?**, ARS] Note that the **Auslander-Reiten formulas**, as part (a) is often called, yields a forth and fifth way to calculate extension groups. We will see in the next section that the Auslander-Reiten sequences may be used to effectively calculate parts of the AuslanderReiten quiver, a technique which is known as knitting. ♢

Exercises

6.4.1 Let $\mathscr{C}$ and $\mathscr{D}$ are abelian categories. A functor $F\colon \mathscr{C} \to \mathscr{D}$ is called **exact** if each short exact sequence is mapped to a sequence which is again short exact. Show that the dualization $\mathrm{D}\colon \operatorname{vec} \to \operatorname{vec}$ is an exact contravariant functor.

6.4.2 Show that in the quotient category $\underline{\operatorname{mod}}\, A$ each projective A-module is a zero object.

6.4.3 Let $\varepsilon \in \operatorname{Ext}^1_A(M, \tau_A M)$ be the class of the Auslander-Reiten sequence ending in M. Show that for each radical homomorphism $f\colon \tau_A M \to X$ (resp. for each radical $g\colon Y \to M$) the sequence $f\varepsilon$ (resp. εg) splits. This is why AR-sequences are also called **almost split sequences**.

Chapter 7

Knitting

The Auslander-Reiten Theorem can be translated into a combinatorial technique called knitting. It yields in many concrete cases large parts or all of the Auslander-Reiten quiver including the structure of the contained indecomposable modules.

7.1 Source and sink maps

In this chapter we first develop a technique from Auslander-Reiten's Theorem 6.16 which is of great importance because it allows to compute the Auslander-Reiten quiver Γ_A of a finite dimensional K-algebra A in many examples. Later in this chapter other combinatorial concepts are introduced which are useful in the classification of indecomposable A-modules.

We assume that all indecomposables appearing in the considerations are vertices of Γ_A. This has the advantage that two indecomposables which are isomorphic are equal, thereby simplifying the arguments. Recall from Example 3.23 that for non-isomorphic, indecomposable A-modules M and N we have $\mathrm{rad}_A(M, N) = \mathrm{Hom}_A(M, N)$. Moreover $\mathrm{Hom}_A(M, M)$ is a finite-dimensional, local algebra with radical $\mathrm{rad}_A(M, M)$ consisting of all homomorphisms which are nilpotent, or equivalently, which are not isomorphisms; see Proposition 3.13.

As we will see the two maps occurring in an Auslander-Reiten sequence can be characterised in categorical terms. To do so we introduce some new terminology. Let M be an indecomposable A-module. A homomorphism $a\colon M \to N$ is a **source map** for M if it is radical, every radical homomorphism $g\colon M \to X$ factors through a (that is, there exists $f\colon N \to X$ such that $fa = g$) and every $\varphi\colon N \to N$ with $\varphi a = a$ is an isomorphism. The dual concept is called **sink map**. The source and sink maps for an indecomposable A-module are unique, see Exercise 7.1.1.

We first show that the third condition for a source or a sink map can be replaced.

Lemma 7.1. *Let M be an indecomposable A-module and $N = \bigoplus_{i=1}^{t} N_i$ an A-module decomposed into indecomposables. Then a homomorphism $f = \begin{bmatrix} f_1 & \cdots & f_t \end{bmatrix}^\top\colon M \to N$ is a source map if and only if the following three conditions hold: (i) f is radical, (ii) every radical homomorphism $g\colon M \to N'$ factors through f and (iii) for each $Y \in \Gamma_A$ the set $\{f_i \mid N_i = Y\}$ is mapped to a basis of the space $\operatorname{rad}_A(M,Y)/\operatorname{rad}_A^2(M,Y)$ under the canonical projection.*

Dually a homomorphism $f\colon N \to M$ is a sink map if and only if (i) f is radical, (ii) every radical homomorphism $g\colon N' \to M$ factors through f and (iii) for every $Y \in \Gamma_A$ the set $\{f_i \mid N_i = Y\}$ yields a basis of the space $\operatorname{rad}_A(Y,M)/\operatorname{rad}_A^2(Y,M)$.

Proof. Assume first that the homomorphism f staisfies (i), (ii) and (iii). Further let $\varphi\colon N \to N$ be a homomorphism such that $\varphi f = f$. We write φ as matrix with entries $\varphi_{ji}\colon N_i \to N_j$. Now, we fix $Y \in \Gamma_A$ and observe that we may assume without loss of generality that $N_i = Y$ for $1 \le i \le s$ and $N_i \ne Y$ for $s < i \le t$. Then for $i = 1, \ldots, s$ we have modulo rad_A^2 that

$$f_i = \sum_{j=1}^{t} \varphi_{ij} f_j \equiv \sum_{j=1}^{s} \varphi_{ij} f_j$$

holds because φ_{ij} is radical for all $i \le s$ and $j > s$. For each $i, j = 1, \ldots, s$ we have that $\varphi_{ij} = \lambda_{ij} \operatorname{id}_Y + \rho_{ij}$, where $\lambda_{ij} \in K$ and ρ_{ij} is radical. Thus it follows that modulo rad_A^2 we have

$$f_i \equiv \sum_{j=1}^{s} \lambda_{ij} f_j.$$

These equations indicate a base change in the space $\operatorname{rad}_A(M,Y)/\operatorname{rad}_A^2(M,Y)$ showing that the matrix $\lambda = (\lambda_{ij})_{i,j=1}^{s} \in K^{s\times s}$ is invertible. Therefore $(\varphi)_{i,j=1}^{s}$ is also invertible and by varying Y we see that φ itself is an isomorphism.

Now assume that $\varphi f = f$ implies that φ is an isomorphism. We fix $Y \in \Gamma_A$ and assume again that $N_i = Y$ for $1 \leq i \leq s$ and $N_i \neq Y$ for $s < i \leq t$. To prove property (iii) assume that $\lambda_i \in K$ are scalars for $i = 1, \dots s$ such that $\psi = \sum_{i=1}^{s} \lambda_i f_i$ belongs to $\operatorname{rad}_A^2(M, Y)$. We have to show that $\lambda_i = 0$ for all $i = 1, \dots, s$. So assume otherwise: $\lambda_j \neq 0$ for some $j \leq s$. Since $\psi \in \operatorname{rad}_A^2(M, Y)$ there are two radical homomorphisms $g\colon M \to X$ and $h\colon X \to Y$ with $\psi = hg$. By property (ii) there exists a factorization $g'\colon N \to X$ such that $g'f = g$. Consequently

$$\psi = \Lambda f = hg'f\colon M \to Y,$$

where $\Lambda\colon \bigoplus_{i=1}^{t} N_i \to Y$ is given by $\Lambda_i = \lambda_i \operatorname{id}_Y$ for $i \leq s$ and $\Lambda_i = 0$ for $i > s$. Now let $\iota_j\colon Y \to N$ be the inclusion into the j-th summand $N_j = Y$. Then

$$f = (\operatorname{id}_N - \iota_j \frac{1}{\lambda_i}(\Lambda - hg'))f$$

and therefore $\varphi = \operatorname{id}_N - \iota_j \frac{1}{\lambda_i}(\Lambda - hg')$ is an isomorphism. But an inspection of the j-th column of $(\varphi_{ih})_{ih=1}^{t}$ will show the contrary: clearly $\varphi_{ij} = 0$ holds for all $i \neq j$ and $\varphi_{jj} = \operatorname{id}_Y - \frac{1}{\lambda_j}(\lambda_j \operatorname{id}_Y + hg'_j) = \frac{1}{\lambda_j} hg'_j$ is radical. Thus we have reached a contradiction and shown that $f_1, \dots, f_s$ are linearly independent modulo $\operatorname{rad}_A^2(M, Y)$.

This concludes the proof of the statement for source maps and the statement for sink maps is obtaind by dualization. □

Proposition 7.2. *For every indecomposable A-module a source map and a sink map exists. More precisely: if M is indecomposable non-projective, then the homomorphism b in the Auslander-Reiten sequence $\tau_A M \to E \xrightarrow{b} M$ is a sink map for M. If M is indecomposable projective, then the canonical inclusion $\operatorname{rad} M \to M$ is a sink map for M. Dually, if M is indecomposable non-injective, then the map b in the Auslander-Reiten sequence $M \xrightarrow{b} F \to \tau_A^- M$ is a source map for M. If M is indecomposable injective, then the canonical projection $M \to M/\operatorname{soc} M$ is a source map for M.*

Proof. Let M be an indecomposable A-module. If M is non-projective, then by the Auslander-Reiten Theorem 6.16 there exists an AR-sequence starting in M. If we decompose the middle term into indecomposables $E = \bigoplus_{i=1}^{t} E_i$ this sequence looks as follows:

$$0 \to \tau_A M \xrightarrow{a=[a_1 \ \cdots \ a_t]^\top} \bigoplus_{i=1}^{t} E_i \xrightarrow{b=[b_1 \ \cdots \ b_t]} M \to 0 \tag{7.1}$$

First note that the homomorphisms b_i are radical since otherwise they would be invertible and the Auslander-Reiten sequence (7.1) would split in contradiction to the definition. Furthermore, by the definition of AR-sequences each radical homomorphism $g\colon X \to M$ factors through b which shows the second property. Now, if $\varphi\colon E \to E$ satisfies $b\varphi = b$, then by Lemma 3.12 there exists some integer n such that $E = \mathrm{Im}(\varphi^n) \oplus \mathrm{Ker}(\varphi^n)$. Since $b(\mathrm{id}_E - \varphi^n) = 0$ and a is a kernel of b, there exists a homomorphism $c\colon E \to L$ such that

$$ac = \mathrm{id}_E - \varphi^n.$$

If we restrict this equality to $\mathrm{Ker}\,\varphi^n$, then on the right we get a section whereas on the left we get a radical homomorphism. This can only happen if this restriction is zero. Consequently $\mathrm{Ker}\,\varphi^n = 0$. This shows that φ^n and therefore also φ itself is injective and hence bijective. It follows that the map b of the AR-sequence (7.1) is a sink map.

Now we assume that M is indecomposable and projective, say $M = P_x$. We have to show that the inclusion $\iota\colon \mathrm{rad}\, P_x \to P_x$ is a sink map. Let $\mathrm{rad}\, P_x = \bigoplus_j N_j$ be a decomposition into indecomposable summands. Then ι induces injective maps $N_j \to P_x$ which never can be an isomorphism since $\dim_K N_j \leq \dim_K \mathrm{rad}\, P_x < \dim_K P_x$. Therefore ι is radical. Let $\pi\colon P_x \to S_x$ be the canonical projection and observe that if some homomorphism $g\colon Y \to P_x$ satisfies $\pi g \neq 0$, then there exists some $y \in Y$ such that $g(y) = e_x - r$ for some $r \in \mathrm{rad}\, P_x = \mathrm{rad}\, Ae_x \subset Ae_x$. Thus $g(y+ry) = e_x - r + rg(y) = e_x - r^2$, where we used $r = re_x$. Inductively we get $g(y+ry+\ldots+r^{n-1}y) = e_x - r^n$. Observe that r^n belongs to $\mathrm{rad}^n P_x$. Since $\mathrm{rad}^n P_x = 0$ for large n we get that there exists an element $y' \in Y$ such that $e_x = g(y')$. Consequently for all $a \in P_x$ we have $a = ae_x = ag(y') = g(ay')$ showing that g is surjective and hence splits by Exercise 4.5.1. This shows that every radical homomorphism $g\colon X \to P_x$ factors over ι. Finally, if $f\colon \mathrm{rad}\, P_x \to \mathrm{rad}\, P_x$ satisfies $\iota f = \iota$, then $f = \mathrm{id}_{\mathrm{rad}\, P_x}$ since ι is injective.

The existence of source maps and the statement about their shape follows by dualization. □

Remark 7.3. It follows from the previous description of source and sink maps that in the Auslander-Reiten quiver Γ_A of a finite-dimensional K-algebra there exist only finitely many arrows with a fixed start vertex or a fixed end vertex. Moreover, if X and $\tau_A X$ are both indecomposable A-modules then the number of arrows $Y \to X$ equals the number of arrows $\tau_A X \to Y$. ◊

Exercises

7.1.1 Show that if $a\colon L \to E$ and $a'\colon L \to E'$ are two source maps, then there exists an isomorphism $\varphi\colon E \to E'$ such that $a' = \varphi a$. Prove the dual statement about sink maps.

7.1.2 Show that for some indecomposable A-module M the map $0 \to M$ is a sink map if and only if M is simple and projective if and only if $M = P_x$ for some **sink** x of the quiver Q of A, that is, a vertex in which no arrows starts. A vertex in which no arrow ends is called a **source**.

7.1.3 Show the statement about the indecomposable projective A-module P_y formulated in Remark **??**.

7.2 The knitting technique

In the following we describe the mechanism of the procedure of **knitting** by constructing source and sink maps inductively.

For this we define for indecomposable A-modules X, Y the number

$$\delta_{X,Y} = \dim_K \left(\operatorname{rad}_A(X,Y)/\operatorname{rad}_A^2(X,Y)\right).$$

Note that $\delta_{X,Y}$ is precisely the number of arrows $X \to Y$ in the AR-quiver Γ_A of A. Moreover, for each indecomposable A-module X we denote

$$X^{\leftarrow} = \{Z \in \Gamma_A \mid \delta_{Z,X} > 0\}$$

and

$$X^{\rightarrow} = \{Z \in \Gamma_A \mid \delta_{X,Z} > 0\}.$$

It follows from the existence of source and sink maps given in Proposition 7.2 and the characterization given in Lemma 7.1 that these sets are finite.

For $Y = P_y$, the projective indecomposable associated to the vertex y, the number δ_{X,P_y} is the maximal power ℓ such that X^ℓ is a direct summand of $\operatorname{rad} P_y$. We define the homomorphism

$$\alpha_{X,P_y} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{\delta_{X,P_y}} \end{bmatrix} \colon\ X \to P_y^{\delta_{X,P_y}},$$

where $\alpha_i\colon X \to P_y$ is the composition of inclusion in the i-th coordinate $X \to X^{\delta_{X,P_y}}$ with the canonical inclusions $X^{\delta_{X,P_y}} \to \operatorname{rad} P_y \to P_y$.

The next result is the basic tool for knitting.

Proposition 7.4. *Let A be a finite-dimensional K-algebra and X an indecomposable A-module. Let*

$$\rho_X = \begin{bmatrix}\cdots & \rho_{Z,X} & \cdots\end{bmatrix}_Z \colon \bigoplus_{Z\in X^{\leftarrow}} Z^{\delta_{Z,X}} \longrightarrow X$$

be the sink map for X and further for each indecomposable $Z \in X^{\leftarrow}$ let

$$\sigma_Z = \begin{bmatrix}\vdots \\ \sigma_{Z,E} \\ \vdots\end{bmatrix}_E \colon\ Z \longrightarrow \bigoplus_{E\in Z^{\rightarrow}} E^{\delta_{Z,E}}$$

be the source map for Z. For each non-injective summand $Z \in X^{\leftarrow}$ let

$$\mu_{\tau_A^- Z} = \begin{bmatrix}\cdots & \mu_{E,\tau_A^- Z} & \cdots\end{bmatrix}_E \colon \bigoplus_{E\in Z^{\rightarrow}} E^{\delta_{Z,E}} \longrightarrow \tau_A^- Z,$$

be the cokernel of σ_Z and define

$$\mu^{\top}_{X,\tau_A^- Z} = \begin{bmatrix}\mu_1 \\ \vdots \\ \mu_{\delta_{Z,X}}\end{bmatrix} \colon X \longrightarrow (\tau_A^- Z)^{\delta_{Z,X}},$$

where $\mu_1, \ldots, \mu_{\delta_{Z,X}}$ are the entries of $\mu_{X,\tau_A^- Z}$, that is,

$$\mu_{X,\tau_A^- Z} = \begin{bmatrix}\mu_1 & \cdots & \mu_{\delta_{Z,X}}\end{bmatrix} \colon X^{\delta_{Z,X}} \longrightarrow \tau_A^- Z.$$

Then, the homomorphism

$$\xi_X = \begin{bmatrix}\vdots \\ \mu^{\top}_{X,\tau_A^- Z} \\ \vdots \\ \alpha_{X,P_y} \\ \vdots\end{bmatrix}_{Z,y} \colon\ X \longrightarrow \bigoplus_{Z\in X^{\leftarrow}_{\mathrm{ni}}} (\tau_A^- Z)^{\delta_{Z,X}} \oplus \bigoplus_{y\in Q_0} P_y^{\delta_{X,P_y}} \tag{7.2}$$

is a source map for X, where $X^{\leftarrow}_{\mathrm{ni}} = \{Z \in X^{\leftarrow} \mid Z \text{ is not injective}\}$.

Proof. If $Y \in \Gamma_A$ is indecomposable and non-projective then $Z = \tau_A Y$ belongs to $X^{\leftarrow}$ and is not injective. Furthermore, by considering the AR-sequences ending in Y we see by Lemma 7.1 and Proposition 7.2 that the entries of $\mu_{X,Y}$ induce a basis of the space $\operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y)$.

Now, if Y is projective, then $\iota\colon \operatorname{rad} Y \to Y$ is a sink map by Proposition 7.2. Thus if we decompose $\operatorname{rad} Y$ into indecomposables $\operatorname{rad} Y = \bigoplus_{R \in \Gamma_A} R^{\delta_{R,Y}}$, we get homomorphisms $\iota_X\colon X^{\delta_{X,Y}} \to Y$ whose entries yield a basis for the space $\operatorname{rad}(X, Y)/\operatorname{rad}^2(X, Y)$ by Lemma 7.1.

Thus by Lemma 7.1 the map ξ_X is a source map for X. $\square$

We give a small example to exhibit the use of Proposition 7.4.

Example 7.5. Consider the path algebra $A = K\overrightarrow{\mathbb{A}}_3$. The vertex 3 is a sink of the quiver Q of A thus $0 = \operatorname{rad} P_3$ and $0 \to P_3$ is a sink map for P_3. Furthermore $\operatorname{rad} P_2 = P_3$. Thus by Proposition 7.4 the source map for P_3 is $a\colon P_3 \to P_2$, that is, the inclusion of the radical. By Proposition 7.2 this is also the sink map for P_2. Moreover $P_2 = \operatorname{rad} P_1$ and once again by the same result, we get that the source map for P_2 is of the form $\begin{bmatrix} b \\ c \end{bmatrix} : P_2 \to \tau_A^- P_3 \oplus P_1$. Therefore we can calculate its cokernel and get a sink map $\begin{bmatrix} d & e \end{bmatrix} : \tau_A^- P_3 \oplus P_3 \to \tau_A^- P_2$. Note that we also obtain that $b\colon P_2 \to \tau_A^- P_3$ is a sink map for $\tau_A^- P_3$ and therefore $e\colon \tau_A^- P_3 \to \tau_A^- P_2$ a source map for $\tau_A^- P_3$. Now P_1 and $\tau_A^- P_2$ are both injective, thus the Proposition 7.4 yields that the source map for $\tau_A^- P_2$ is obtained as cokernel of e and has the form $f\colon \tau_A^- P_2 \to \tau^{-2} P_3$.

$$\begin{tikzcd}
 & & P_1 \arrow[dr, "e"] & & \\
 & P_2 \arrow[ur, "c"] \arrow[dr, "b"] & & \tau_A^- P_2 \arrow[ll, dashed, "\tau"'] \arrow[dr, "f"] & \\
P_3 \arrow[ur, "a"] & & \tau_A^- P_3 \arrow[ll, dashed, "\tau"'] \arrow[ur, "d"] & & \tau_A^{-2} P_3 \arrow[ll, dashed, "\tau"']
\end{tikzcd} .$$

Altogether we have knitted the Auslander-Reiten quiver Γ_A as shown in the picture above. $\lozenge$

Exercises

7.2.1 Formulate the dual of Proposition 7.4 and use it to knit Γ_A of Example 7.5 starting with the injectives.

7.3 Knitting with dimension vectors

Knitting with modules is still rather complicated since cokernels have to be computed. A reduction of the information yields a far better way to knit in many interesting examples. For this we consider the dimension vectors of the modules involved in the computations considered as representations of the quiver of the algebra, see Section 2.2. This has the advantage that the computation of cokernels is replaced by the calculation of its dimension vector as a difference of known vectors using the fact that AR-sequences are short exact sequences and hence the sum of dimension vectors of the end terms equals the dimension vector of the middle term; see Exercise 5.1.1. We exhibit this in an example.

Example 7.6. We consider the algebra $A = KQ/I$, where

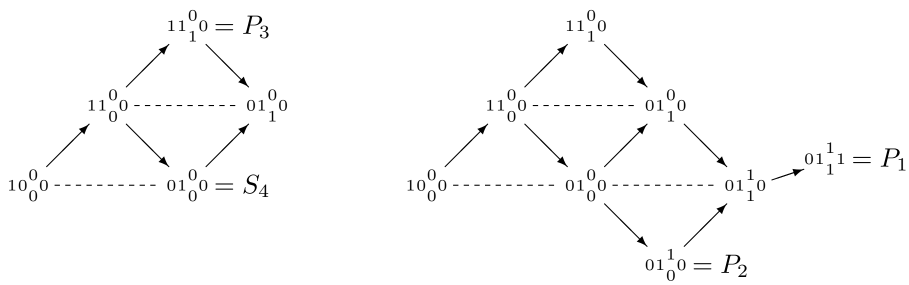
$$Q: \quad \begin{tikzcd} & & 2 \arrow[dl, "\delta"'] & \\ 5 & 4 \arrow[l, "\varepsilon"'] & & 1 \arrow[ul, "\beta"'] \arrow[dl, "\alpha"] \\ & & 3 \arrow[ul, "\gamma"] & \end{tikzcd} \qquad \text{and} \qquad I = \langle \gamma\alpha - \delta\beta, \varepsilon\delta \rangle. \tag{7.3}$$

We start by calculating the indecomposable projective and the indecomposable injective A-modules, but we indicate only their dimension vectors:

$$\begin{aligned}
\underline{\dim}\, P_1 &= \begin{smallmatrix}&&1&\\0&1&&1\\&&1&\end{smallmatrix}, & \underline{\dim}\, I_1 &= \begin{smallmatrix}&&0&\\0&0&&1\\&&0&\end{smallmatrix},\\
\underline{\dim}\, P_2 &= \begin{smallmatrix}&&1&\\0&1&&0\\&&0&\end{smallmatrix}, & \underline{\dim}\, I_2 &= \begin{smallmatrix}&&1&\\0&0&&1\\&&0&\end{smallmatrix},\\
\underline{\dim}\, P_3 &= \begin{smallmatrix}&&0&\\1&1&&0\\&&1&\end{smallmatrix}, & \underline{\dim}\, I_3 &= \begin{smallmatrix}&&0&\\0&0&&1\\&&1&\end{smallmatrix},\\
\underline{\dim}\, P_4 &= \begin{smallmatrix}&&0&\\1&1&&0\\&&0&\end{smallmatrix}, & \underline{\dim}\, I_4 &= \begin{smallmatrix}&&1&\\0&1&&1\\&&1&\end{smallmatrix},\\
\underline{\dim}\, P_5 &= \begin{smallmatrix}&&0&\\1&0&&0\\&&0&\end{smallmatrix}, & \underline{\dim}\, I_5 &= \begin{smallmatrix}&&0&\\1&1&&0\\&&1&\end{smallmatrix}.
\end{aligned}$$

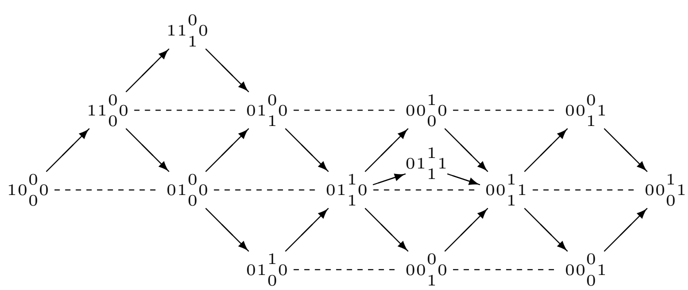
Notice that $I_5 = P_3$ and $I_4 = P_1$. Since P_5 is simple its radical is zero and therefore there is no non-zero radical homomorphism ending in P_5. Further, we have $\operatorname{rad} P_4 = P_5$ and therefore the inclusion $P_5 \to P_4$ is the sink map for P_4 and the source map for P_5. We have an Auslander-Reiten sequence $P_5 \to P_4 \to \tau_A^- P_5$ and by inspecting the dimension vectors we conclude that $\underline{\dim}\, \tau_A^- P_5 = \begin{smallmatrix}&&0&\\1&1&&0\\&&0&\end{smallmatrix} - \begin{smallmatrix}&&0&\\1&0&&0\\&&0&\end{smallmatrix} = \begin{smallmatrix}&&0&\\0&1&&0\\&&0&\end{smallmatrix}$. Consequently $\tau_A^- P_5$ is the simple S_4.

Similarly, $\operatorname{rad} P_3 = P_4$ and therefore we have an Auslander-Reiten sequence $P_4 \to S_4 \oplus P_3 \to \tau_A^- P_4$ where $\underline{\dim}\, \tau_A^- P_4 = 01{}^{0}_{1}0$. So far we have constructed the part shown in the next picture on the left:



The dotted lines join the end terms of the Auslander-Reiten sequences we have constructed so far and the morphisms are exactly the ones appearing in those two sequences. Proceeding we first observe that $P_3 = I_5$ is not the beginning of an Auslander-Reiten sequence. Further, the simple module S_4 is the radical of P_2 and hence we can determine the dimension vector $\underline{\dim}\, \tau_A^- S_4 = 01{}^{1}_{1}0$. From theory we know that $\tau_A^- S_4$ is indecomposable so, there is only one choice: the radical of P_1. The result is depicted in the previous picture on the right.

Now, no surprise can occur, since we already have found all indecomposable projectives, that is, we just have to go on and calculate the Auslander-Reiten sequences starting in any non-projective.



The previous picture shows the Auslander-Reiten quiver of A obtained by knitting. $\Diamond$

Example 7.6 shows that knitting with dimension vectors is both: easy and tremendously powerful. It enabled us to produce easily all indecomposable modules, since for each dimension vector appearing only one possible choice (up to isomorphism) for an indecomposable module is possible.

However, the attentive reader should have noted that it is not clear that we found *all* indecomposables with the knitting procedure. We only know that the constructed part is a component of the AR-quiver Γ_A since we have constructed all arrows starting or ending in any of the constructed modules. But there could be many undetected components out there. As to that the next result will help us out.

Theorem 7.7 (Auslander)**.** *If A is a connected algebra such that its Auslander-Reiten quiver Γ_A contains a finite component C, then $C = \Gamma_A$.*

Proof. Let M be the direct sum of all (indecomposable) A-modules in C and $B = \mathrm{End}_A(M)$ the finite-dimensional K-algebra which by Corollary 3.21 has a nilpotent radical given by the radical homomorphisms in C. Hence there is an upper bound, say $d \geq 0$, for which $g_d \circ \ldots \circ g_1 = 0$ for any radical homomorphisms $g_1, \ldots, g_d$ in C.

We show that this implies that for any $X \in \Gamma_A \setminus C$ and any $Y \in C$ we have $\mathrm{Hom}_A(X,Y) = 0$. Suppose that $f\colon X \to Y$ is a non-zero homomorphism. Then f factors through the sink map $\pi_Y : E \to Y$ ending in Y. Therefore, there is at least one indecomposable direct summand $Y_1 \in C$ of E and two radical homomorphisms $f_1\colon X \to Y_1$ and $g_1\colon Y_1 \to Y$ such that $g_1 \circ f_1 \neq 0$. Iterating the argument for f_1, we obtain an indecomposable $Y_2 \in C$ and further two homomorphisms $f_2\colon X \to Y_2$ and $g_2\colon Y_2 \to Y_1$ such that $g_1 \circ g_2 \circ f_2 \neq 0$. Hence $\mathrm{rad}^2(Y_2, Y) \neq 0$. Inductively we find objects $Y_n \in C$ such that $\mathrm{rad}^n(Y_n, Y) \neq 0$ for any n, in contradiction to $\mathrm{rad}^d(Y_d, Y) = 0$. Similarly, we have $\mathrm{Hom}_A(Y, X) = 0$ for any $Y \in C$ and any $X \in \Gamma_A \setminus C$.

Now it follows from the existence of projective covers, Corollary 4.29, that for a fixed $Y \in C$ there exists a projective indecomposable module P admitting a non-zero homomorphism $P \to Y$. Hence by the above P lies in C. Since A is connected inductively all other projective indecomposable modules belong to C. But then again, for any indecomposable A-module X there exists a non-zero homomorphism $Q \to X$ for some projective indecomposable A-module Q and therefore X lies in C. □

Exercises

7.3.1 Knit the AR-quiver of the two subspace algebra $A = KQ$ by using dimension vectors.

7.3.2 Knit the AR-quiver of the three-subspace algebra $A = KQ$, where Q is the following quiver

$$\begin{tikzcd}
1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\
& 4 &
\end{tikzcd}$$

and compare the effort needed now with the one necessary to do Exercise 1.3.3.

7.3.3 Knit the AR-quiver of the algebra $A = KQ/I$, where the quiver Q and the ideal I are given as follows

$$Q: \quad \begin{tikzcd}
1 \arrow[dr, "\alpha"] & & \\
& 3 \arrow[r, "\gamma"] & 4 \\
2 \arrow[ur, "\beta"'] & &
\end{tikzcd} \qquad \text{and} \qquad I = \langle \gamma\alpha \rangle.$$

7.3.4 Determine the AR-quiver of $B = KQ/J$, where Q is the quiver of the Exercise 7.3.3 but the ideal is $J = \langle \gamma\alpha,\ \gamma\beta \rangle$. You may do this either by knitting or by the idea explained in Section 3.8, that is, by considering $B = A/\langle \gamma\beta \rangle$.

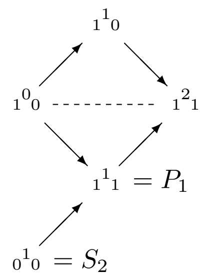
7.4 Limits of knitting

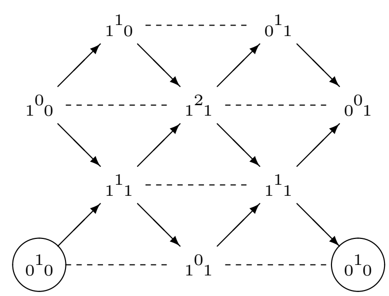
Knitting is the promised tool which effectively can replace the matrix problem approach – at least in some cases. It is really "self correcting" since an error in the calculation does not stand long undetected. Unfortunately it is not that universally applicable as we see in the forthcoming considerations. The orbits under the action of the Auslander-Reiten translate τ_A are called **Auslander-Reiten orbits** or **τ_A-orbits** for short. Clearly, in each indecomposable projective such a τ_A-orbit starts and in each indecomposable injective a τ_A-orbit ends. So the question arises whether there might be other orbits at all. For this we work out another example.

Example 7.8. Let $A = KQ/I$, where

$$Q: \quad \begin{tikzcd}
& 2 \arrow[dl, "\beta"'] & \\
3 & & 1 \arrow[ul, "\alpha"'] \arrow[ll, "\gamma"]
\end{tikzcd} \qquad \text{and} \qquad I = \langle \beta\alpha \rangle.$$

We calculate the indecomposable projectives and their radicals. We obtain: $\operatorname{rad} P_3 = 0$, $\operatorname{rad} P_2 = P_3$ and $\operatorname{rad} P_1 = P_3 \oplus S_2$. Thus we may start knitting with the inclusions $P_3 \to P_2$ and $P_3 \to P_1$ and obtain the part shown on the left in the following picture.



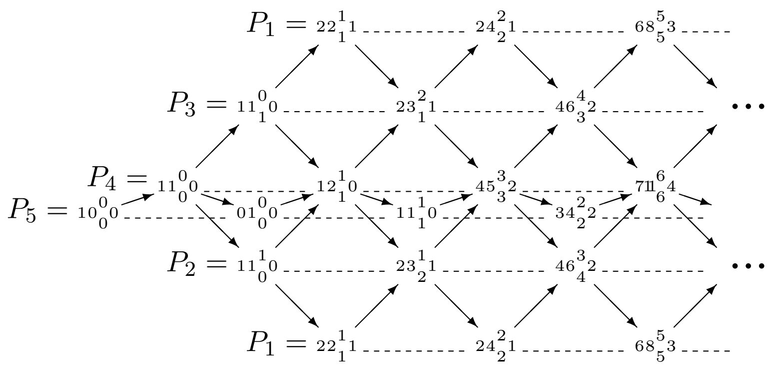


However, it is not clear whether the inclusion $\iota\colon S_2 \to P_1$ is the source map for S_2 and we cannot proceed until this question is not settled. So, suppose that $f\colon S_2 \to M$ is a radical non-zero homomorphism. Then $f_2\colon K \to M_2$ is not zero, $f_2(1) = m \neq 0$. Therefore $M_\beta(m) = M_\beta \circ f_2(1) = f_1 \circ (S_2)_\beta(1) = 0$ since $(S_2)_\beta = 0$. We have $m \in \operatorname{Im} M_\alpha$ because otherwise $Km \subset M$ is a submodule isomorphic to S_2 and f a section. So, there exists $x \in M_1$ such that $M_\alpha(x) = m$. Now, let $g\colon P_1 \to M$ be such that $g_1(1) = x$ and note that by the Yoneda Lemma 4.24 this defines g uniquely. Clearly, we have $f = g \circ \iota$. Thus, we really can proceed knitting and obtain the Auslander-Reiten quiver on the right above.

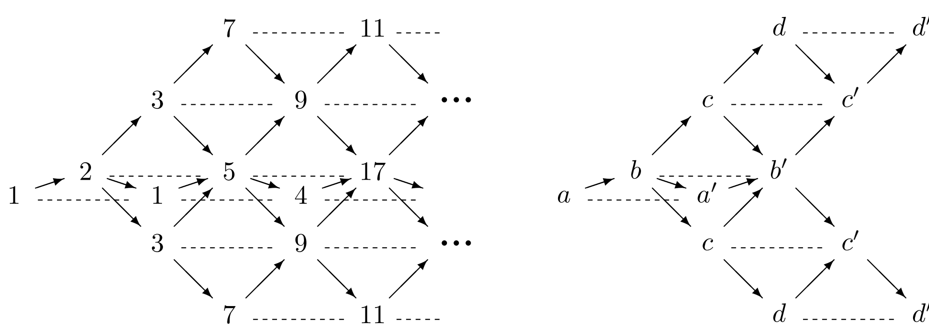
Observe that two dimension vectors, namely ${}_{0}{}^{1}{}_{0}$ and ${}_{1}{}^{1}{}_{1}$ occur twice in the picture. However, there is only one module with dimension vector ${}_{0}{}^{1}{}_{0}$, namely the simple S_2. The two encircled vertices are thus the same and have to be identified. It follows that $\tau_A^2 S_2 = S_2$ and therefore the two A-modules S_2 and $\tau_A S_2$ form a τ_A-orbit not ending in an injective nor beginning in a projective. The case is different with the second dimension vector ${}_{1}{}^{1}{}_{1}$. It clearly occurs as dimension vector of P_1, but also as the one of I_3. But I_3 and P_1 cannot be isomorphic, since $(P_1)_\beta = 0$ and $(I_3)_\beta \neq 0$, so the second occurrence could be the one coming from I_3. If $\tau_A^- P_1$ is not injective, then $\tau_A^{-2} P_1$ would be indecomposable, but if we proceed knitting we obtain negative numbers which clearly is absurd for a dimension vector hence $\tau_A^- P_1 = I_3$. $\Diamond$

Remember, that we reduced to dimension vectors since knitting is so easy with them. Example 7.8 shows the drawback of this reduction: we have to argue in order to understand which modules correspond to the dimension vectors we found. As we see, knitting sometimes does not work so smoothly and some ad hoc arguments have to be used. A strategy to attack this problem is using coverings, see for example [5]. Unfortunately this is beyond the scope of this notes. The next example shows a different problem of knitting.

Example 7.9. Let $A = KQ$, where Q is the quiver (7.3) used in Example 7.6. Since there are no relations the projectives and their radicals have to be computed afresh. It turns out that the radical of each projective A-module is projective again. So knitting becomes really easy. We show the resulting part of the Auslander-reiten quiver after some steps in the knitting proces in the next picture, where the top and the bottom row have to be identified.



It seems that this will go on forever. But how can we be shure about this? It could be that after several thousand steps the dimension vectors start to decrease and finally the injectives appear? If we only look at the total dimension of the modules only, then the beginning of the AR-quiver looks as shown in the next picture on the left.



Observe that we have an obvious symmetry with respect to the horizontal axis. We suppose that at some stage ℓ the dimensions are

$$\begin{aligned}
a &= \dim_K \tau_A^{-\ell} P_5, \\
b &= \dim_K \tau_A^{-\ell} P_4, \\
c &= \dim_K \tau_A^{-\ell} P_3 = \dim_K \tau_A^{-\ell} P_2, \\
d &= \dim_K \tau_A^{-\ell} P_1,
\end{aligned}$$

see the previous picture on the right for illustration. We denote the dimensions for the τ_A^--translated modules by a', b', c' and d' respectively. Using that the Auslander-Reiten sequences are short exact sequences we get the following relations:

$$\begin{aligned}
a' &= b - a, \\
b' &= 2c + a' - b = 2c - a, \\
c' &= d + b' - c = d + c - a, \\
d' &= 2c' - d = d + 2c - 2a.
\end{aligned}$$

We show that for all ℓ the following inequalities hold:

$$d > c > b \geq 2a. \tag{7.4}$$

Clearly they are valid for $\ell = 0$. We suppose they hold for some ℓ and calculate

$$d' - c' = d + 2c - 2a - (d + c - a) = c - a > 0$$

hence $d' > c'$. Similarly the inequalities $c' > b' \geq 2a'$ can be shown. Thus by induction (7.4) hold for all ℓ. Furthermore the argument shows also that $a' \geq a$, $b' > b$, $c' > c$ and $d' > d$. Consequently the dimensions proceed to increase forever. A similar picture is obtained by knitting in the opposite direction starting with the injective indecomposables to the left.

Are this all modules? Clearly not, since the simples S_3 and S_2 do not appear. Notice that $f\colon P_4 \to P_3$ is a projective presentation for S_3, hence $\tau_A S_3$ is the kernel of $\nu_A f\colon I_4 \to I_3$, where ν_A is the Nakayama functor. Therefore $\operatorname{Ker}(\nu_A f)$ has dimension vector $01{}^{1}_{0}1$. Hence $P_5 \oplus P_3 \to P_1$ is the projective presentation of $\tau_A S_3$ and we calculate $\underline{\dim}\, \tau_A^2 S_3 = 11{}^{1}_{2}2$. Similarly, we can calculate $\tau_A^- S_3, \tau_A^{-2} S_3$ and thus the sequence

$$11{}^{1}_{2}2 \;\text{-------}\; 01{}^{1}_{0}1 \;\text{-------}\; 00{}^{0}_{1}0 \;\text{-------}\; 11{}^{1}_{0}1 \;\text{-------}\; 01{}^{1}_{0}0$$

is obtained. $\Diamond$

Example 7.9 shows two problems: First, knitting may continue forever or it may stop, but the process itself does not tell us which one occurs in a given example. Second, as we have seen in the calculation of the τ_A-translates of S_3 that we really have to know the module structure for calculating the projective presentations, the dimension vector is not enough. Thus we cannot knit as we would like to.

Exercises

7.4.1 Determine the AR-quiver of $A = KQ/I$, where the quiver Q and the ideal I are as follows.

$$Q: \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\ 4 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"] \end{tikzcd} \qquad \text{and} \qquad I = \langle \beta\alpha,\ \gamma\beta,\ \delta\gamma,\ \alpha\delta \rangle.$$

7.4.2 Let $A = KQ$, where Q is the **four subspace quiver**:

$$\begin{tikzcd} 1 \arrow[drr] & 2 \arrow[dr] & & 3 \arrow[dl] & 4 \arrow[dll] \\ & & 5 & & \end{tikzcd}$$

Start knitting from the projectives. Prove that the component you are knitting is infinite. Do the same with the component obtained by knitting from the injectives and prove that there is some indecomposable which does not belong to either component.

7.5 Preprojective components

A component C of the Auslander-Reiten quiver Γ_A of a finite-dimensional algebra A is called **preprojective** or sometimes **postprojective** if each τ_A-orbit contains a projective indecomposable (and consequently there are only finitely many τ_A-orbits) and there is no cycle in C. Dually, a component without cycle in which each τ_A-orbit contains an injective indecomposable is called **preinjective** (nobody calls them "postinjective"). A component without any projective or injective A-module is called **regular**. An A-module contained in a preprojective, resp. preinjective, resp. regular component is called **preprojective**, resp. **preinjective**, resp. **regular**.

Knitting of a preprojective component C is always successful: if we start with a simple projective A-module in it, either the knitting process goes on forever for some τ_A-orbits or in each τ_A-orbit an injective indecomposable A-module is reached and in that case the component C is finite and hence $C = \Gamma_A$ by Theorem 7.7.

Note that not every algebra admits a preprojective component. For instance the algebra considered in Example 7.8 has only one component which contains a cycle and therefore is not preprojective. However, there are nice classes of algebras which do have a preprojective and a preinjective component. For this, it will be useful to systematize many examples which we already have considered.

An algebra A is called **hereditary** if each submodule of a projective A-module is projective again. Note that this happens if and only if each A-module has a projective presentation $0 \to Q_1 \to Q_0$, that is, if and only if $\mathrm{Ext}_A^2(M, N) = 0$ for any two A-modules M and N. Now, this last condition can also be stated in different terms: it means that each A-module has an injective presentation $J_0 \to J_1 \to 0$, in other words, that each quotient of an injective A-module is injective again. A different description is given in Exercise 7.5.1: a basic and finite-dimensional K-algebra A is hereditary if and only if it is isomorphic to the path algebra KQ of a quiver Q without cycle.

Proposition 7.10. *If A is a hereditary and finite-dimensional K-algebra, then in Γ_A each projective A-module is contained in a preprojective component and dually each injective A-module in a preinjective component. If A is connected, then there exists a unique preprojective component and a unique preinjective component. Moreover in that case, the preprojective component is preinjective if and only if A is of finite representation type.*

Proof. Note that without loss of generality, we may assume A to be basic, see Proposition 3.16. Hence by Exercise 7.5.1, we have $A = KQ$, for some quiver Q. Since A is finite-dimensional the quiver Q has no cycle, see Exercise 2.4.4.

Let $Q^{(0)}$ be the **full subquiver** of Γ_A obtained by the projectives, that is, the quiver with vertices the projective indecomposable A-modules in Γ_A together with *all* the arrows in Γ_A between them. More generally, a **subquiver** Q' of a quiver Q is a quiver given by two subsets, that is $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq Q_1$ such that $s_{Q'}(\alpha) = s_Q(\alpha)$ and $t_{Q'}(\alpha) = t_Q(\alpha)$ for each $\alpha \in Q_1$.

It follows from the Yoneda Lemma 4.24 that each homomorphism $f\colon P_x \to P_y$ is given by $f = P_a$ for some $a \in e_x A e_y$, where $P_a\colon P_x \to P_y, b \mapsto ba$. Hence it follows that the arrows in Γ_A between projectives can be chosen as P_α for an arrow α of the quiver of A. This shows that $Q^{(0)}$ is isomorphic to

Q^{op} and hence contains no cycle.

Therefore there exists $P_x \in Q^{(0)}$ which is a source, or equivalently $x \in Q$ is a sink. Note that

$$\sigma_x = \begin{bmatrix} \cdots & P_\alpha & \cdots \end{bmatrix}_\alpha : P_x \to \bigoplus_{\alpha\colon y \to x} P_y$$

is a source map for P_x by Proposition 7.4. Hence we obtain $\tau_A^- P_x$ as cokernel of σ_x:

$$\rho_x = \begin{bmatrix} \vdots \\ \rho_{x,\alpha} \\ \vdots \end{bmatrix} : \bigoplus_{\alpha\colon y \to x} P_y \to \tau_A^- P_x.$$

We may assume that the irreducible homomorphisms $\rho_{x,\alpha}$ are chosen as arrows in Γ_A and that these are all arrows ending in $\tau_A^- P_x$.

Denote by $Q^{(1)}$ the quiver obtained by deleting P_x from $Q^{(0)}$ and adding $\tau_A^- P_x$ together with all arrows $\rho_{x,\alpha}$. Note, that $Q^{(1)}$ is isomorphic to the quiver obtained from $Q^{(0)}$ by inverting the direction of all arrows ending in P_x. We may now repeat the arguments with every source in $Q^{(0)}$ to obtain $Q^{(t)}$. Let $P_y \in Q^{(t)}$ be a source. It follows from Proposition 7.4 that the source map for P_y looks like

$$\sigma_y = \begin{bmatrix} \cdots & P_\beta & \cdots & \rho_{x,\alpha} & \cdots \end{bmatrix}_{\beta,\alpha} : P_y \to \bigoplus_{\beta\colon z \to y} P_z \oplus \bigoplus_{\alpha\colon y \to x} \tau_A^- P_x.$$

Note that we may proceed that way until no vertex of the original quiver $Q^{(0)}$ is left. Finally, we get a quiver $Q^{(n)}$ which is isomorphic to $Q^{(0)}$ and consists of the vertices $\tau_A^- P_x$ for all $x \in Q_0$ together with the arrows in Γ_A between them. Proceeding this way, we either continue forever or at some step reach an injective A-module. In the latter case all τ_A-orbits are finite. Thus every connected component of Q yields a component C of Γ_A.

Clearly in each such component C there are only paths from $\tau^\ell P_x$ to $\tau^m P_y$ if $\ell \geq m$. This shows that there is no cycle in C and that C is a preprojective component. The statement about the components containing the injectives is proved dually.

If A is connected, then the projective indecomposable A-modules are connected by irreducible homomorphisms and therefore all projectives in Γ_A must belong to the same component. The statement about the preinjective component is proved dually. $\square$

Proposition 7.11. *If C is a preprojective or preinjective component of Γ_A, then for each indecomposable A-module $X \in C$ we have $\operatorname{End}_A(X) \simeq K$ and $\operatorname{Ext}^1_A(X, X) = 0$.*

Proof. Note that by definition, the arrows in C induce a partial order "$\leq$" in the vertices of C. Thus for $X, Y \in C$ we have $X \leq Y$ if and only if there exists a path from X to Y or $X = Y$. We show that $\operatorname{Hom}_A(X, Y) \neq 0$ implies $X \leq Y$. Indeed, in case C is preprojective, we consider the subquiver C' of C obtained by deleting all $Z \in C$ for which $Z \not\leq Y$. Note that C' is a finite quiver: first let $Y = \tau_A^{-\ell} P_x$ for some $x \in Q_0$ and some $\ell \geq 0$. Then the elements of the τ_A-orbit of Y which belong to C' are $\tau_A^{-i} P_x$ for $i = 0, \dots, \ell$. If $Z \in Y^{\leftarrow}$ then similarly only finitely many elements of the τ_A-orbit of Z belong to C'. Proceeding this way can consider all τ_A-orbits, each having only a finite number of elements in C'.

We have to show that $X \in C'$. Now, if $f\colon X \to Y$ is a non-zero homomorphism, we get that f factors through $Y^{\leftarrow} \subseteq C'$, say $f = \sum_{i=1}^{t} f_i g_i$ where $g_i\colon Z_i \to Y$ for some $Z_i \in Y^{\leftarrow}$ and $f_i\colon X \to Z_i$. Note that $Z_i \in C'$ for all i. If we suppose that $X \notin C'$, then we can continue factoring f through arbitrarily large number of irreducible homomorphisms, but since C' is finite there is a bound M for which $\operatorname{rad}^M(Z, Y) = 0$ for all $Z \in C'$, a contradiction. This shows that $X \in C'$ and therefore $X \leq Y$.

As a consequence we have $\operatorname{Hom}_A(X, E) = 0$ if E is the middle term of the Auslander-Reiten sequence ending in X and therefore $\operatorname{rad}_A(X, X) = 0$, which in turn implies $\operatorname{End}_A(X) \simeq K$. By Theorem 6.16, we have that $\dim_K \operatorname{Ext}^1_A(X, X) = \dim_K \overline{\operatorname{Hom}}_A(X, \tau_A X) \leq \dim_K \operatorname{Hom}_A(X, \tau_A X) = 0$ since there is no path in C from X to $\tau_A X$. $\square$

Example 7.12. The preprojective, resp. preinjective, resp. regular modules over the Kronecker algebra are of the form Q_n, resp. J_n, resp. $R_{n,\lambda}$, see Section 6.2. $\lozenge$

Exercises

7.5.1 Let $A = KQ/I$, where Q is an arbitrary quiver and I is an admissible ideal. Let $\rho_1, \dots, \rho_t$ be relations such that $I = \langle \rho_1, \dots, \rho_t \rangle$ and assume that t is minimal. Then we have that $\langle \rho_1, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_t \rangle \neq I$ for each $i = 1, \dots, t$. Show that $\operatorname{rad} P_{s(\rho_i)}$ is not projective. This implies that A is not hereditary. Prove conversely that KQ is hereditary for any quiver Q without cycle.

7.5.2 Knit the preprojective component of Γ_A for $A = KQ$, where Q is the following quiver.

$$
\begin{tikzcd}[row sep=small, column sep=small]
Q: & 1 \arrow[dr] & & & & \\
& & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \\
& 2 \arrow[ur] & & & &
\end{tikzcd}
$$

Chapter 8

Combinatorial invariants

The reduction from modules to dimension vectors can not only be seen as a loss of information but also as a gain in simplicity. Several useful tools are presented for working with dimension vectors. The first of them is based on homological algebra, the second linearizes the Auslander-Reiten translation as far it is possible and the last plays a crucial role for characterizing the indecomposable modules in the reduction process.

8.1 Grothendieck group

We have seen in Section 7.3 that dimension vectors are very useful. We now give a more theoretic approach to them. For this we recall some notions from ring theory.

Let R be a ring. An R-module F is said to be **free** if there exists a subset $S \subset F$ such that for each R-module M and each map $\varphi\colon S \to M$ there exists a unique homomorphism of R-modules $\hat{\varphi}\colon F \to M$ such that $\hat{\varphi}|_S = \varphi$. In that case S is called a **basis** of F. If the basis S is finite, then F is called **finitely generated free**. Note that this **universal property** defines the free R-module uniquely, up to isomorphism, see Exercise 8.1.1. If S is a set then $R^{(S)}$ denotes the R-module of functions $f\colon S \to R$ which are zero almost everywhere, that is, $\{s \mid f(s) \neq 0\}$ is a finite set. Commonly one identifies $s \in S$ with $\delta_s \in R^{(S)}$ where

$$\delta_s\colon S \to R, t \mapsto \begin{cases} s, & \text{if } s = t, \\ 0, & \text{else.} \end{cases}$$

Then $R^{(S)}$ is a free R-module with basis S, see Exercise 8.1.2. This shows that for each set S there exists a free R-module with basis S. This explicit construction shows also that in a free R-module F with basis S each element $x \in S$ can be written as (finite) linear combination of elements in S with coefficients in R. A particular interesting case is when $R = \mathbb{Z}$ the ring of integers. We then speak of free abelian groups.

Temporarily we denote the isomorphism class of an A-module X by $\mathrm{cl}(X)$. We know from Lemma 4.14 that the isomorphism classes of A-modules form a set

$$S = \{\mathrm{cl}(X) \mid X \in \operatorname{mod} A\}.$$

The **Grothendieck group** $\mathrm{K}_\circ(A)$ of an algebra A is the quotient

$$\mathrm{K}_\circ(A) = \mathbb{Z}^{(S)}/I$$

where I is the ideal generated by all linear combinations

$$\delta_{\mathrm{cl}(X')} - \delta_{\mathrm{cl}(X)} + \delta_{\mathrm{cl}(X'')}$$

whenever there exists a short exact sequence $Y' \to Y \to Y''$ of A-modules for some $Y' \in \mathrm{cl}(X')$, $Y \in \mathrm{cl}(X)$ and $Y'' \in \mathrm{cl}(X'')$. The coset of $\delta_{\mathrm{cl}(X)}$ in $\mathrm{K}_\circ(A)$ is called the **class** of an A-module X in $\mathrm{K}_\circ(A)$ and will be denoted by $[X]$.

Since $0 \to 0 \to 0$ is a short exact sequence we have $\delta_{\mathrm{cl}(0)} = \delta_{\mathrm{cl}(0)} - \delta_{\mathrm{cl}(0)} + \delta_{\mathrm{cl}(0)} \in I$ and therefore $[0]$ is the neutral element of $\mathrm{K}_\circ(A)$, denoted by 0. Furthermore, $[X] = [Y]$ holds if X and Y are isomorphic A-modules since any isomorphism $f\colon X \to Y$ yields a short exact sequence $X \xrightarrow{f} Y \to 0$ and therefore $[X] + [0] - [Y] = 0$ n $\mathrm{K}_\circ(A)$. Moreover $[X' \oplus X''] = [X'] + [X'']$ since the split sequence $X' \to X' \oplus X'' \to X''$ is a short exact sequence.

Remark 8.1. The Grothendiek group is defined as a quotient F/I of a free abelian group F modulo an ideal I generated by some set of relations R. We denote by $\pi\colon F \to F/I$ the canonical projection.

Now, every homomorphism $\varphi\colon F/I \to \mathbb{Z}$ yields a group homomorphism $\hat{\varphi} = \varphi\pi\colon F \to \mathbb{Z}$ which satisfies $\hat{\varphi}(r) = 0$ for each $r \in R$. Conversely, each homomorphism $\psi\colon F \to \mathbb{Z}$ which satisfies $\psi(r) = 0$ for all $r \in R$ induces a group homomorphism $\varphi\colon F/I \to \mathbb{Z}$ such that $\varphi\pi = \psi$.

Therefore, to give a $\mathbb{Z}$-linear map $f\colon \mathrm{K}_\circ(A) \to \mathbb{Z}$ it is necessary and sufficient to define the values $f(M)$ for each $M \in \operatorname{mod} A$ such that $f(M) = f(M') + f(M'')$ whenever there exists a short exact sequence $M' \to M \to M''$. $\Diamond$

Proposition 8.2. *If A is a finite-dimensional K-algebra with quiver Q, then*

$$\mathrm{K}_\circ(A) \to \mathbb{Z}^{Q_0}, [M] \mapsto \underline{\dim}\, M.$$

is a $\mathbb{Z}$-linear bijection, that is, an isomorphism of abelian groups.

Proof. Let S be the set of isomorphism classes in $\operatorname{mod} A$. Define

$$S \to \mathbb{Z}^{Q_0}, \mathrm{cl}(M) \mapsto \underline{\dim}\, M.$$

Now, if $M' \to M \to M''$ is a short exact sequence, then $\underline{\dim}\, M = \underline{\dim}\, M' + \underline{\dim}\, M''$, see Exercise 5.1.1. Thus, by Remark 8.1 the above map induces a linear function

$$\mathrm{K}_\circ(A) \to \mathbb{Z}^{Q_0}, [M] \mapsto \underline{\dim}\, M.$$

If M is not a simple module, then there exists a submodule M' and hence $[M] = [M'] + [M/M']$. This shows that for any A-module M the class $[M]$ is a linear combination of the classes of the simple A-modules $[S_1], \dots, [S_n]$, where the coefficient of $[S_i]$ is the number of times S_i appears as factor in a Jordan-Hölder filtration of M.

Hence $[M] = \sum_{i \in Q_0} n_i [S_i]$, where n_i is the dimension of M in the vertex i, viewed as representation of the quiver Q of A. This shows that $\mathrm{K}_\circ(A)$ is the free abelian group with basis $[S_i]$, $i \in Q_0$, that is $\mathrm{K}_\circ(A) \to \mathbb{Z}^{Q_0}, [M] \mapsto \underline{\dim}\, M$ is an isomorphism of abelian groups. □

In the forthcoming we will not distinguish between $[M]$ and $\underline{\dim}\, M$. Furthermore we denote the class $[S_i]$ by e_i, that is, the canonical basis vector of $\mathbb{Z}^{Q_0}$ which should not be confounded with the trivial path in i which is denoted by the same symbol.

Exercises

8.1.1 Let F (resp. F') be a free R-module with basis S (resp. S'). Show that if there exists a bijection $\psi\colon S \to S'$, then there exists an isomorphism $\hat{\psi}\colon F \to F'$ such that $\hat{\psi}|_S = \psi$. Hint: use that id_F is the unique homomorphism which satisfies $\mathrm{id}_F\, j = j$, where $j\colon S \to F$ denotes the inclusion.

8.1.2 Show for a ring R and a set S that $R^{(S)}$ is a free R-module with basis S.

8.1.3 Let

$$0 \xrightarrow{a_0} M_1 \xrightarrow{a_1} M_2 \xrightarrow{a_2} \dots \xrightarrow{a_{t-1}} M_t \xrightarrow{a_t} 0$$

be an exact sequence. Show that the alternating sum of the classes in the Grothendieck group vanishes:

$$\sum_{i=1}^{t}(-1)^i[M_i] = 0.$$

Hint: consider the short exact sequences $\operatorname{Ker} a_i \to M_i \to \operatorname{Im} a_i = \operatorname{Ker} a_{i+1}$.

8.2 The homological bilinear form

Suppose that the finite-dimensional K-algebra A is of finite global dimension. Then for any two A-modules X and Y we define

$$\langle X, Y\rangle = \sum_{i=0}^{\infty}(-1)^i \dim_K \operatorname{Ext}_A^i(X, Y). \tag{8.1}$$

Note that the sum on the right hand side is finite since $\operatorname{gldim} A < \infty$ and that $\operatorname{Ext}_A^0(X, Y) = \operatorname{Hom}_A(X, Y)$ by Exercise 5.3.3. The next result shows that the value of $\langle X, Y\rangle$ depends only on the classes $[X]$ and $[Y]$ in $\mathrm{K}_\circ(A)$.

Lemma 8.3. *Let A be a finite-dimensional K-algebra of finite global dimension. If X, X', Y and Y' are A-modules such that $[X] = [X']$ and $[Y] = [Y']$, then $\langle X, Y\rangle = \langle X', Y'\rangle$ holds. Moreover, the map*

$$\langle -, -\rangle\colon\ \mathrm{K}_\circ(A) \times \mathrm{K}_\circ(A) \longrightarrow \mathbb{Z}$$

induced by (8.1) *is bilinear.*

Proof. By Theorem 5.20 for each short exact sequence $M' \to M \to M''$ of A-modules and each A-module L we have a long exact sequence

$$\cdots \to \operatorname{Ext}^i(L, M') \to \operatorname{Ext}^i(L, M) \to \operatorname{Ext}^i(L, M'') \to \operatorname{Ext}^{i+1}(L, M') \to \cdots .$$

Note that in an exact sequence the alternating sum of the dimensions is zero, see Exercise 8.2.1. This shows that

$$\langle L, M'\rangle - \langle L, M\rangle + \langle L, M''\rangle = 0.$$

By Remark 8.1 the map $\langle L, -\rangle$ induces a $\mathbb{Z}$-linear map $\mathrm{K}_\circ(A) \to \mathbb{Z}$. Similarly, for every fixed N the map $\langle -, N\rangle$ induces a $\mathbb{Z}$-linear map $\mathrm{K}_\circ(A) \to \mathbb{Z}$. Thus the result follows. $\square$

The bilinear form $\mathrm{K}_\circ(A) \times \mathrm{K}_\circ(A) \to \mathbb{Z}$ defined in Lemma 8.3

$$\langle [X], [Y] \rangle = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}_A^i(X, Y). \tag{8.2}$$

is called **homological bilinear form** of A.

Example 8.4. Let A be a hereditary algebra, that is, $A = KQ$ for some quiver Q by Exercise 7.5.1. For any projective A-module P and any A-module M we have $\langle [P], [M] \rangle = \dim_K \operatorname{Hom}_A(P, M)$ since all extension groups are zero. Thus if P is indecomposable, say $P = P_i$, then $\langle [P_i], [M] \rangle = \dim_K M_i$. Since there is no cycle in Q we therefore have $\langle [P_i], [P_i] \rangle = 1$.

For simple A-modules S_i and S_j we have

$$\langle [S_i], [S_j] \rangle = \dim_K \operatorname{Hom}_A(S_i, S_j) - \dim_K \operatorname{Ext}_A^1(S_i, S_j).$$

Note that by Exercise 5.3.2 we know that $\dim_K \operatorname{Ext}_A^1(S_i, S_j)$ is the number of arrows $j \to i$ in the quiver Q and by Schur's Lemma 4.33 we have that $\operatorname{Hom}_A(S_i, S_j) = 0$ for $i \neq j$. $\Diamond$

Let A be a finite-dimensional K-algebra of finite global dimension and quiver Q with vertices $Q_0 = \{1, \ldots, n\}$. We denote by P_i projective indecomposable A-module associated to the vertex i and by $p_i = [P_i]$ its class in $\mathrm{K}_\circ(A)$. Then the **Cartan matrix** $C_A \in K^{n \times n}$ of A is defined by

$$(C_A)_{ij} = \dim_K \operatorname{Hom}_A(P_i, P_j) = \langle p_i, p_j \rangle.$$

Lemma 8.5. *The Cartan matrix satisfies the following identities*

$$\begin{aligned} C_A[S_j] &= [P_j], \\ C_A^\top[S_i] &= [I_i]. \end{aligned}$$

Furthermore, C_A is invertible if and only if the classes $p_1, \ldots p_n$ of the indecomposable projectives are linearly independent in $\mathrm{K}_\circ(A)$.

Proof. Note that $\langle p_i, p_j \rangle = \dim_K \operatorname{Hom}_A(P_i, P_j) = (p_j)_i$ the i-th entry of the vector $p_j = [P_j]$. Thus the j-th column of the matrix C_A is precisely p_j. Since $\operatorname{Hom}_A(P_i, P_j) \simeq \operatorname{Hom}_A(I_i, I_j) = [I_i]_j$ the i-th row of C_A is $[I_i]^\top$. This shows the first two equations.

If the vectors $p_1, \ldots, p_n$ are linearly independent, then C_A is certainly invertible. To see the converse we assume that C_A is invertible and denote by r

the rank of the subgroup of $\mathrm{K}_\circ(A)$ generated by $p_1, \dots, p_n$. Then $e_i = C_A^{-1} p_i$ implies $r \geq n$ since $C_A^{-1}p_1, \dots, C_A^{-1}p_n$ generate a subgroup of rank at most r. □

Lemma 8.6. *Let A be a finite-dimensional K-algebra. If A is of finite global dimension, then the classes of the indecomposable projective A-modules generate $\mathrm{K}_\circ(A)$ and are linearly independent.*

Proof. If A is of finite global dimension, then each A-module admits a projective resolution which has finite length. It follows thus from Exercise 8.1.3 that the class $[M]$ of an A-module M can be expressed as linear combination of the vectors $p_1, \dots, p_n$ with integer coefficients. Therefore the vectors $p_1, \dots, p_n$ generate $\mathrm{K}_\circ(A)$ and since $\mathrm{K}_\circ(A)$ is a free abelian group isomorphic to $\mathbb{Z}^n$ these vectors must also be linearly independent. □

Remarks 8.7 (a) The statement of Lemma 8.6 can be reformulated as follows: under the given hypothesis the Cartan matrix C_A of A is invertible.

(b) The converse of Lemma 8.6 is wrong: it is easy to construct algebras of infinite global dimension but with invertible Cartan matrix, see Exercise 8.2.2.

(c) A note of caution: $\mathrm{K}_\circ(A)$ is a free abelian group of rank n, that is, it is isomorphic to $\mathbb{Z}^n$ as abelian group. If $x_1, \dots, x_n$ generate $\mathbb{Z}^n$, then these vectors are linearly independent, but the converse is false. Consider the case where $x_i = 2e_i$ for all i. Then the vectors $x_1, \dots, x_n$ are linearly independent, but they do not generate $\mathbb{Z}^n$ since there exist vectors, for example e_i, which cannot be expressed as linear combination of $x_1, \dots, x_n$ with *integer* coefficients. We can reformulate these conditions using the matrix M whose i-th column is x_i. To say that $x_1, \dots, x_n$ are linearly independent is equivalent to $\det M \neq 0$, whereas $x_1, \dots, x_n$ generate $\mathbb{Z}^n$ if and only if $\det M = \pm 1$. ♢

Proposition 8.8. *Let A be a finite-dimensional K-algebra. If A is of finite global dimension, then*

$$\langle x, y \rangle = x^\top C_A^{-\top} y$$

holds for all $x, y \in \mathrm{K}_\circ(A)$, where $C_A^{-\top} = (C_A^{-1})^\top$.

Proof. Since $\operatorname{gldim} A < \infty$ it follows by Lemma 8.6 that C_A is invertible. Consequently $e_i = C_A^{-1}[P_i]$ and we see

$$\langle p_i, p_j \rangle = e_i^\top p_j = p_i^\top C_A^{-\top} p_j$$

holds for all i, j. Since $p_1, \ldots, p_n$ generate $\mathrm{K}_\circ(A)$ the equation $\langle x, y \rangle = x^\top C_A^{-\top} y$ follows for all $x, y \in \mathrm{K}_\circ(A)$. □

Exercises

8.2.1 Show the following generalization of Exercise 5.1.1: if $0 \to M_1 \to M_2 \to \ldots \to M_t \to 0$ is an exact sequence of A-modules, then the alternating sum of the dimension vanishes:

$$\sum_{i=1}^{t} (-1)^i \dim_K M_i = 0.$$

8.2.2 Let $A = KQ/I$, where the quiver Q is as shown in the following picture and the ideal I is generated by the relation $\beta\alpha\beta\alpha$.

$$Q: \quad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

Calculate the Cartan matrix C_A and verify that C_A is invertible. Calculate the projective resolution of the simple S_1 to see that $\operatorname{gldim} A = \infty$.

8.3 The Coxeter transformation

We suppose that the finite-dimensional K-algebra A is of finite global dimension. For each A-module M we get two $\mathbb{Z}$-linear maps $\mathrm{K}_\circ(A) \to \mathbb{Z}$, namely $\langle [M], - \rangle$ and $\langle -, [M] \rangle$. In this section we are looking for a $\mathbb{Z}$-linear map $\mathrm{K}_\circ(A) \to \mathrm{K}_\circ(A)$ which sends $[X]$ to $[\tau_A X]$. Note that one cannot expect $[X] \mapsto [\tau_A X]$ to be a well-defined map on $\mathrm{K}_\circ(A)$, see Exercise 8.3.1.

We define the **Coxeter transformation** as the unique linear map $\mathrm{K}_\circ(A) \to \mathrm{K}_\circ(A)$ which satisfies $\Phi_A[P_i] = -[I_i]$. The matrix representing Φ_A in the canonical basis is called **Coxeter matrix** and denoted also by Φ_A.

Lemma 8.9. *For a projective presentation $f\colon Q \to P$ of an indecomposable A-module M the following equation holds*

$$[\tau_A M] = \Phi_A[M] - \Phi_A[\operatorname{Ker} f] + [\nu_A M].$$

Proof. By definition we have $[\nu_A Q] = -\Phi_A[Q]$ and $[\nu_A P] = -\Phi_A[P]$. We apply Exercise 8.1.3 to the exact sequence

$$0 \to \operatorname{Ker} f \to Q \xrightarrow{f} P \to M \to 0$$

and get $[P] - [Q] = [M] - [\operatorname{Ker} f]$. Applying Exercise 8.1.3 to the exact sequence (6.23):

$$0 \to \tau_A M \to \nu_A Q \xrightarrow{\nu_A f} \nu_A P \to \nu_A M \to 0.$$

we can calculate the class of $\tau_A M$

$$\begin{aligned}[\tau_A M] &= [\nu_A Q] - [\nu_A P] + [\nu_A M] \\ &= \Phi_A[P] - \Phi_A[Q] + [\nu_A M] \\ &= \Phi_A[M] - \Phi_A[\operatorname{Ker} f] + [\nu_A M]\end{aligned}$$

which is what we wanted to prove. □

That the formula of Lemma 8.9, valid for all A-modules M, can take a particularly nice form in special cases is shown in the next result.

Proposition 8.10. *Let A be a finite-dimensional K-algebra of finite global dimension and M an indecomposable A-module. If $\operatorname{Hom}_A(M, A) = 0$ and $\operatorname{pdim} M \leq 1$, then $\Phi_A[M] = [\tau_A M]$.*

Proof. Since $\operatorname{pdim} M \leq 1$ there exists a projective presentation $f\colon Q \to P$ which is injective. Thus $\operatorname{Ker} f = 0$ and $\operatorname{Hom}_A(M, A) = 0$ implies that $\nu_A M = \operatorname{D}\operatorname{Hom}_A(M, A) = 0$. Hence $[\tau_A M] = \Phi_A[M]$ by Lemma 8.9. □

The next result shows how the dimension vectors of preprojective or preinjective modules over finite-dimensional hereditary K-algebras can be calculated.

Corollary 8.11. *If $A = KQ$ is a finite-dimensional hereditary K-algebra, then $[\tau_A^{-\ell} P_i] = \Phi^{-\ell}[P_i]$ holds for each $i \in Q_0$ and each $\ell > 0$ such that $\tau^{-\ell} P_i \neq 0$ and similarly $[\tau_A^{\ell} I_i] = \Phi^{\ell}[I_i]$ holds for each $i \in Q_0$ and each $\ell > 0$ such that $\tau^{\ell} I_i \neq 0$.*

Proof. Let M be an indecomposable A-module. If $\operatorname{Hom}_A(M, A) \neq 0$, then M is projective since A is hereditary. Therefore, if $M = \tau_A^{-\ell} P_i$ for $\ell > 0$, then $\operatorname{Hom}_A(M, A) = 0$. Hence by Proposition 8.10 we have $[\tau_A M] = \Phi_A[M]$. Inductively we get $[P_i] = [\tau_A^{\ell} M] = \Phi_A^{\ell}[M]$ thus $\Phi_A^{-\ell}[P_i] = [M] = [\tau_A^{-\ell} P_i]$. Similarly, if $M = \tau_A^{\ell} I_i$ is not projective, then $[\tau_A M] = \Phi_A[M]$ again by Proposition 8.10. Therefore $[\tau_A^{\ell} I_i] = \Phi_A^{\ell}[I_i]$ follows inductively. □

Lemma 8.12. *If the finite-dimensional K-algebra A has finite global dimension, then $\Phi_A = -C_A^\top C_A^{-1}$ and*

$$\langle x, y\rangle = -\langle y, \Phi_A x\rangle = \langle \Phi_A x, \Phi_A y\rangle \tag{8.3}$$

holds for all $x, y \in \mathrm{K}_\circ(A)$.

Proof. Since $p_i = C_A e_i$ and $[I_i] = C_A^\top e_i$ we have

$$\Phi_A p_i = -[I_i] = -C_A^\top e_i = -C_A^\top C_A^{-1} p_i$$

and therefore $\Phi_A = -C_A^\top C_A^{-1}$.

Now, by Proposition 8.10

$$-\langle y, \Phi x\rangle = y^\top C_A^{-\top} C_A^\top C_A^{-1} x = y^\top C_A^{-1} x = x^\top C_A^{-\top} y = \langle x, y\rangle$$

and thus the first equation of (8.3) follows. The second equation of (8.3) follows by applying twice the first. □

Example 8.13. Let A be the Kronecker algebra. The Auslander-Reiten quiver was determined in Section 6.2 but we were not able to determine the AR-translate $\tau_A M$ for those indecomposable modules M which lie neither in the preinjective nor in the preprojective component, that is $M = R_{n,\lambda}$ for some $n > 0$ and $\lambda \in K \cup \{\infty\}$, see (6.1). The dimension vector of M is $[M] = (n, n) \in \mathbb{Z}^2$ which is an eigenvector of Φ_A, see Exercise 8.3.2. Since $\mathrm{Hom}_A(R_{n,\lambda}, A) = 0$, see Section 6.2, we may apply Proposition 8.10 and get $[\tau_A R_{n,\lambda}] = \Phi_A[R_{n,\lambda}] = [R_{n,\lambda}]$. Because $\tau_A R_{n,\lambda}$ is indecomposable we must have $\tau_A R_{n,\lambda} = R_{n,\mu}$ for some $\mu \in K \cup \{\infty\}$. The Auslander-Reiten sequence ending in $R_{n,\lambda}$ yields non-zero morphisms $R_{m,\mu} \to E \to R_{n,\lambda}$ and therefore $\mu = \lambda$ follows from the description of the morphism spaces given in Section 6.2. This shows that

$$\tau_A R_{n,\lambda} = R_{n,\lambda}$$

for all $n > 0$ and $\lambda \in K \cup \{\infty\}$ completing the description of the Auslander-Reiten translate for the Kronecker algebra. ◊

Exercises

8.3.1 Show that $[X] \mapsto [\tau_A X]$ is not a well-defined map. For this let $A = K\overrightarrow{\mathbb{A}}_2$ and let $X = P_1$, $Y = S_1 \oplus S_2$. Show that $[X] = [Y]$ but $[\tau_A X] \neq [\tau_A Y]$.

8.3.2 Calculate the Coxeter transformation $\Phi_A = -C_A^\top C_A^{-1}$ for the Kronecker algebra A. Show that 1 is the unique eigenvalue of Φ_A and that the eigenvectors x with respect to this eigenvalue satisfy $x_1 = x_2$.

8.4 Quadratic forms

Let A be a finite-dimensional K-algebra. If A is of finite global dimension, then the quadratic form associated to the homological bilinear form

$$\chi_A\colon \mathrm{K}_\circ(A) \to \mathbb{Z}, x \mapsto \langle x, x\rangle \tag{8.4}$$

is called **Euler form** or sometimes **homological form**.

Note that χ_A can only be defined if A is of finite global dimension, since otherwise the right hand side of (8.2) is not a finite sum. If A is not of finite global dimension, we still can define a bilinear form $\langle -, -\rangle_T$ and from this then a quadratic form by truncating the Euler from in the following way: set

$$\begin{aligned}\langle [S_i], [S_j]\rangle_T = &\dim_K \mathrm{Hom}_A(S_i, S_j) - \dim_K \mathrm{Ext}^1_A(S_i, S_j)\\ &+ \dim_K \mathrm{Ext}^2_A(S_i, S_j)\end{aligned} \tag{8.5}$$

for simple A-modules S_i, S_j and extend it linearly to obtain a bilinear form on $\mathrm{K}_\circ(A)$. The associated quadratic form $q_A\colon \mathrm{K}_\circ(A) \to \mathbb{Z}, x \mapsto \langle x, x\rangle_T$ is given by

$$q_A(x) = \sum_{i\in Q_0} x_i^2 - \sum_{(i\to j)\in Q_1} x_i x_j + \sum_{i,j\in Q_0} x_i x_j \dim_K \mathrm{Ext}^2_A(S_i, S_j). \tag{8.6}$$

It is called **Tits form**, sometimes **geometric form** or **Ringel form**.

Remark 8.14. If $\operatorname{gldim} A = \infty$ then χ_A is undefined. If $\operatorname{gldim} A \leq 2$ we have $\chi_A = q_A$ since we have

$$\langle e_i, e_j\rangle = \sum_{i=0}^{\infty} \dim_K \mathrm{Ext}^i_A(S_i, S_j) = \sum_{i=0}^{2} \dim_K \mathrm{Ext}^i_A(S_i, S_j) = \langle e_i, e_j\rangle_T$$

for all $i, j \in Q_0$. If the global dimension of A is finite but larger than 2 then χ_A and q_A are both defined but do not have to be equal. $\Diamond$

Note that both quadratic forms χ_A and q_A have the following shape:

$$q\colon \mathbb{Z}^n \to \mathbb{Z}, q(x) = \sum\nolimits_{i=1}^{n} q_i x_i^2 + \sum\nolimits_{i<j} q_{ij} x_i x_j \tag{8.7}$$

for some integer coefficients q_i, q_{ij}. If $q_i = 1$ for all i then q is called **unit form**.

If $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is a unit form, we define its **graph** $\Delta(q)$ to have $1,\dots,n$ as vertices and edges as follows: between the vertices i and j there are $|q_{ij}|$ **full** edges if $q_{ij} < 0$ and $|q_{ij}|$ **dotted** edges if $q_{ij} > 0$. Sometimes multiple edges are indicated by labels written close to a single edge. Conversely, if G is a graph with full and dotted edges but without loops nor mixed edges, that is, a pair of vertices (i,j) such that there is a full and a dotted edge, then we associate to G the unit form $p = q(G)$ such that $G = \Delta(p)$.

If Q is a quiver then the **underlying graph** of Q is the graph having the same vertices as Q and $a_{ij} + a_{ji}$ full edges joining the vertices i and j, where a_{ji} is the number of arrows from i to j. It is customary to say that the underlying graph is obtained "by forgetting the orientation" of the arrows in Q.

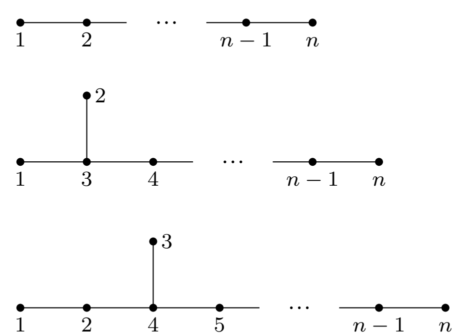
We start by investigating very special cases of unit forms.

Example 8.15. Let q be a unit form such that its graph $\Delta(q)$ is a **Dynkin diagram**, that is, one of the graphs of the following list:

$\mathbb{A}_n$ $(n \geq 1)$:

$\mathbb{D}_n$ $(n \geq 4)$:

$\mathbb{E}_n$ $(n = 6, 7, 8)$:



(8.8)

Note that in these examples there is no dotted edge. Moreover, there are two infinite families and three cases $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ which are called **exceptional**. These graphs play a fundamental role in many classification problems in modern algebra. Nowadays it is common to speak of Dynkin diagrams even though these structures first appeared at the end of the 19th century when Wilhelm Killing and Elie Cartan classified semisimple Lie algebras over the complex numbers. $\Diamond$

We observe that the linearly oriented quiver $\overrightarrow{\mathbb{A}}_n$ has $\mathbb{A}_n$ as underlying graph.

Now we compare two unit forms $p\colon \mathbb{Z}^m \to \mathbb{Z}$ and $q\colon \mathbb{Z}^n \to \mathbb{Z}$. For this we assume $T \in \mathbb{Z}^{m\times n}$ to be some integer matrix such that $p = q \circ T$. If $n = m$ and T is $\mathbb{Z}$-invertible, $T \in \mathrm{GL}_n(\mathbb{Z})$, then we say that p and q are **equivalent** and denote it by $p \sim q$. In the particular case where T is a permutation matrix, that is, $Te_i = e_{\sigma(i)}$ for some permutation $\sigma \in \mathrm{S}_n$, then we say that p and q are **isomorphic**.

There are also interesting cases for $p = q \circ T$ for $T \in \mathbb{Z}^{n \times m}$ not a square matrix: If $m \leq n$ and each column of T contains precisely one entry 1 and each row at most one entry 1 whereas all other entries are zero, then p is called a **restriction** of q. Note that this happens precisely when $\Delta(p)$ is obtained from $\Delta(q)$ by deleting some of its vertices (and relabeling the remaining ones).

We recall from linear algebra that by Sylvester's law of inertia a quadratic form $q \colon \mathbb{R}^n \to \mathbb{R}$ is, up to equivalence, uniquely defined by the **indices of inertia**: q is equivalent to a form $p \colon \mathbb{R}^n \to \mathbb{R}$ which is given by $p(x) = \sum_{i=1}^n p_i x_i^2$. The number n_+ (resp. n_-) of positive (resp. negative) coefficients p_i is called **positive** index of inertia, resp. **negative** index of inertia and both indices are uniquely determined by q. The difference $n - n_+ - n_-$ is called **corank** of q.

We can apply these concepts to a unit form $q \colon \mathbb{Z}^n \to \mathbb{Z}$ by defining $q_{\mathbb{R}} \colon \mathbb{R}^n \to \mathbb{R}$ by $q_{\mathbb{R}}(x) = q(x)$ for all $x \in \mathbb{Z}^n$. That $q_{\mathbb{R}}$ is uniquely defined by this follows from the fact that the coefficients q_i and q_{ij} in the expression (8.7) are uniquely defined by $q_i = q(e_i)$ and $q_{ij} = q(e_i + e_j) - q(e_i) - q(e_j)$. Since a unit form satisfies $q(e_i) = 1$ we conclude $n_+ > 0$. Hence there are three important cases to be considered: if $n_+ = n$, then q is said to be **positive definite**. If $n_- = 0$ but possibly $n > n_+$ then q is called **positive semidefinite**. The third case is when $n_- > 0$ and in that case q is called **indefinite**.

Lemma 8.16. *Let $q \colon \mathbb{Z}^n \to \mathbb{Z}$ be any quadratic form. Then the following characterizations hold.*

(a) *The form q is positive definite if and only if $q(x) > 0$ for all $x \in \mathbb{Z}^n \setminus \{0\}$ if and only if $q_{\mathbb{R}}(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*

(b) *The form q is positive semidefinite if and only if $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$ if and only if $q_{\mathbb{R}}(x) \geq 0$ for all $x \in \mathbb{R}^n$.*

Proof. First, we observe that q is positive semidefinite if and only if $q_{\mathbb{R}}(x) \geq 0$ for all $x \in \mathbb{R}^n$ and that this implies that $q(x) \geq 0$ holds for all $x \in \mathbb{Z}^n$. Similarly q is positive definite if and only if $q_{\mathbb{R}}(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and this implies that $q(x) > 0$ for all $x \in \mathbb{Z}^n \setminus \{0\}$.

Assume now that $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$ and denote by $q_{\mathbb{Q}} \colon \mathbb{Q}^n \to \mathbb{Q}$ the obvious extension of q. Since $q_{\mathbb{Q}}(\frac{a}{b}) = \frac{1}{b^2} q(a)$, we see that $q_{\mathbb{Q}}(x) \geq 0$ for all $x \in \mathbb{Q}^n$ and by continuity we have $q_{\mathbb{R}}(x) \geq 0$ for all $x \in \mathbb{R}^n$. This proves (b). To see (a) it remains to show that if there exists some non-zero $x \in \mathbb{R}^n$ with $q_{\mathbb{R}}(x) = 0$, then there exists some non-zero $z \in \mathbb{Z}^n$ with $q(z) = 0$.

Indeed, if $q_{\mathbb{R}}(x) = 0$ for some non-zero x, then $q_{\mathbb{R}}$ attains in x a global minimum. Therefore

$$\frac{\partial q_{\mathbb{R}}}{\partial x_i}(x) = 0 \tag{8.9}$$

for all $i = 1, \ldots, n$. But that means that x satisfies the system of linear equations (8.9) non-trivially. Since all coefficients in (8.9) are rational there exists also a non-trivial solution y in $\mathbb{Q}^n$. That is $q_{\mathbb{Q}}(y) = 0$. By multiplying y with the least common multiple m of the denominators we obtain a non-zero vector $z = my \in \mathbb{Z}^n$ with $q(z) = 0$. □

Lemma 8.17. *Each unit form q whose graph $\Delta(q)$ is a Dynkin diagram is positive definite.*

Proof. We start by doing this for the case $\Delta(q) = \mathbb{D}_n$. Then we have

$$\begin{aligned} q(x) =& x_1^2 - x_1x_3 + x_2^2 - x_2x_3 + x_3^2 - x_3x_4 + x_4^2 - \ldots \\ & - x_{i-1}x_i + x_i^2 - \ldots - x_{n-1}x_n + x_n^2 \\ =& (x_1 - \tfrac{1}{2}x_3)^2 + (x_2 - \tfrac{1}{2}x_3)^2 + \tfrac{1}{2}(x_3 - x_4)^2 + \ldots \\ & + \tfrac{1}{2}(x_i - x_{i+1})^2 + \ldots + \tfrac{1}{2}(x_{n-1} - x_n)^2 + \tfrac{1}{2}x_n^2. \end{aligned}$$

This shows that $q(x) > 0$ whenever $x \neq 0$ and hence q is positive definite if $\Delta(q) = \mathbb{D}_n$.

Clearly, the diagram $\mathbb{A}_n$ can be obtained as restriction of $\mathbb{D}_{n+1}$ and a restriction of a positive definite quadratic form is again positive definite. Therefore q is positive definite if $\Delta(q) = \mathbb{A}_n$.

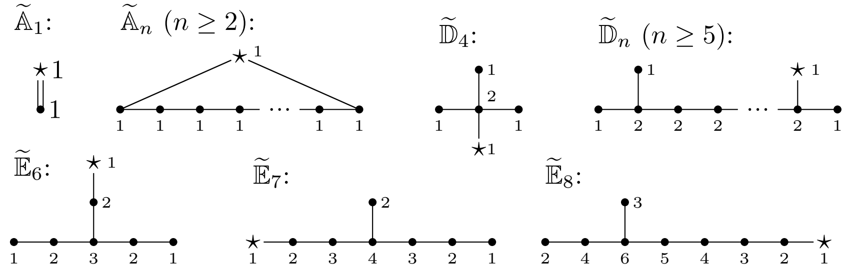
Finally, we study the case $\Delta(q) = \mathbb{E}_8$. Then we have

$$\begin{aligned} q(x) =& x_1^2 - x_1x_2 + x_2^2 - x_2x_4 + x_3^2 - x_3x_4 + x_4^2 - \ldots \\ & - x_{i-1}x_i + x_i^2 - \ldots + x_8^2 \\ =& (x_1 - \tfrac{1}{2}x_2)^2 + \tfrac{3}{4}(x_2 - \tfrac{2}{3}x_4)^2 + (x_3 - \tfrac{1}{2}x_4)^2 + \tfrac{5}{12}(x_4 - \tfrac{6}{5}x_5)^2 \\ & + \tfrac{2}{5}(x_5 - \tfrac{5}{4}x_6)^2 + \tfrac{3}{8}(x_6 - \tfrac{4}{3}x_7)^2 + \tfrac{1}{3}(x_7 - \tfrac{3}{2}x_8)^2 + \tfrac{1}{4}x_8^2 \end{aligned}$$

which shows that q is positive definite if $\Delta(q) = \mathbb{E}_8$. Since $\mathbb{E}_7$ and $\mathbb{E}_6$ are both restrictions of $\mathbb{E}_8$, we get the desired result. □

We also give examples of unit forms which are not positive.

Example 8.18. The graphs of the following list are called **extended Dynkin diagrams**, by some authors they are called **Euclidean diagrams**.



(8.10)

Let q be a unit form such that $\Delta(q)$ is an extended Dynkin diagram. Notice that the numbers on the vertices of the graphs in (8.10) do not indicate the vertices itself but they define a vector x for which $q(x) = 0$, as easily can be checked. Hence q is not positive definite. Moreover, if p is positive definite, then $\Delta(p)$ cannot contain an extended Dynkin diagram as subgraph. It is shown in Exercise 8.4.2 that q is positive semidefinite of corank 1 if $\Delta(q)$ is an extended Dynkin diagram. $\Diamond$

Exercises

8.4.1 Calculate the Euler form of the algebra given in Example 7.8.

8.4.2 Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a unit form whose graph is an extended Dynkin diagram, see list (8.10). Show that q is positive semidefinite of corank 1. Hint: denote by x the vector indicated in (8.10). Verify that x satisfies the following remarkable property.

$$2x_i + \sum_{j \neq i} q_{ij} x_j,$$

that is, twice the value x_i equals the sum of the values x_j over all neighbours j of i in the graph $\Delta(q)$. Use this formula to prove that

$$q(y + x) = q(y)$$

for all $y \in \mathbb{Z}^n$.

Now observe that $v_\star = 1$, where $\star$ is a vertex of the extended Dynkin diagram $\widetilde{\Delta}$, as indicated in (8.10), such that the restriction to all other vertices is the Dynkin diagram Δ. Now use that every vector $w \in \mathbb{Z}^n$ can be written as $w = w' + \alpha v$ with $w'_\star = 0$ and $\alpha \in \mathbb{Z}$ and conclude that $q(w) \geq 0$.

8.4.3 Let $A = KQ/I$, where $Q = \overrightarrow{\mathbb{A}}_5$ and I is the ideal generated by all compositions of two consecutive arrows in Q. Calculate the Cartan matrix C_A and the graph $\Delta(\chi_A)$.

8.5 Roots

Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a unit form. In general, a vector $x \in \mathbb{Z}^n$ is called a **root**, or 1-**root**, of q if $q(x) = 1$. A root $x \in \mathbb{Z}^n$ is called **positive** if $x_i \geq 0$ for all i. A root $x \in \mathbb{Z}^n$ is called **negative** if $-x$ is positive. We denote set of all roots by

$$R(q) = \{x \in \mathbb{Z}^n \mid q(x) = 1\}$$

and the set of all positive roots by

$$P(q) = \{x \in R(q) \mid x_i \geq 0 \text{ for all } i = 1, \ldots, n\}.$$

By Example 8.4 the canonical basis vectors e_i and the dimension vectors of the indecomposable projective A-modules are roots of $\chi_A = q_A$ if A is a finite-dimensional K-algebra. This shows that there are interesting indecomposable A-modules whose classes in $\mathrm{K}_\circ(A)$ are roots. But these are no exceptions: by Lemma 8.12

$$\chi_A(x) = \chi_A(\Phi_A x).$$

and therefore $[\tau_A^{-\ell} P_x]$ is a root for each x and $\ell \geq 0$. Similarly, preinjective indecomposables define roots. However not for every indecomposable A-module M the class $[M]$ is a root of χ_A, see Exercise 8.5.1.

Proposition 8.19. *Let $q\colon \mathbb{Z}^n \to Z$ be a positive definite quadratic form. Then for any $r \in \mathbb{Z}_{\geq 0}$ the fibre $q^{-1}(r)$ is finite.*

Proof. We know from Lemma 8.16 (a) that $q_{\mathbb{R}}(x) > 0$ holds for each $x \in \mathbb{Z}^n \setminus \{0\}$ and therefore $\frac{1}{q_{\mathbb{R}}}$ is well defined on $\mathbb{R}^n \setminus \{0\}$. Moreover, since $\mathrm{S}^1 = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is compact, $q_{\mathbb{R}}$ attains there a global minimum m. Hence we have for all x with $q_{\mathbb{R}}(x) = r$ that $\frac{r}{\|x\|^2} = \frac{q_{\mathbb{R}}(x)}{\|x\|^2} = q_{\mathbb{R}}(\frac{x}{\|x\|}) \geq m$ and hence $\|x\| \leq \sqrt{\frac{r}{m}}$. This shows that there can be only finitely many vectors in $q_{\mathbb{R}}^{-1}(r) \cap \mathbb{Z}^n = q^{-1}(r)$. □

Corollary 8.20. *If q is a unit form such that its graph $\Delta(q)$ is a Dynkin diagram, then there are only finitely many roots of q.*

Proof. This is an immediate consequence of Proposition 8.19 and Lemma 8.17. ☐

Proposition 8.21. *Let q be a unit form such that its graph $\Delta(q)$ is a Dynkin diagram. Then any root of q is either positive or negative.*

Proof. Let v be any root of q. Write $x = x^+ + x^-$ with $x_i^+ \geq 0$, $x_i^- \leq 0$ but at most one of them non-zero, for each i. Then $1 = q(x) = q(x^+) + q(x^-) + \sum_{i,j} q_{ij} x_j^+ x_i^-$, where the sum is taken over all (i,j) such that $x_i^+ \neq 0$ and $x_j^- \neq 0$. Thus the right hand side consists of three non-negative summands, the latter since $q_{ij} \leq 0$ and $x_i^+ x_j^- \leq 0$ for any such (i,j). Hence $q(x^+) = 0$ or $q(x^-) = 0$ which implies $x^+ = 0$ or $x^- = 0$, respectively. ☐

If $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is a quadratic form, we define the symmetric bilinear form $(-|-)_q\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ associated to q by

$$(x|y)_q = q(x+y) - q(x) - q(y). \tag{8.11}$$

The next rather technical result is very useful in order to determine the roots in practice.

Lemma 8.22. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a unit form such that $\Delta(q)$ is a Dynkin diagram. Then for any positive root x of q with $\|x\| > 1$ there exists some vertex i such that $x - e_i$ is again a root.*

Proof. It follows from $\|x\| > 1$ that $x - e_i \neq 0$ and therefore $q(x - e_i) > 0$ for all indices i. We assume that for all i the vector $x - e_i$ is not a root, that is $q(x - e_i) > 1$ for $i = 1, \ldots, n$.

Since x is a root we have $(x|x)_q = 2q(x) = 2$. On the other hand, it follows from

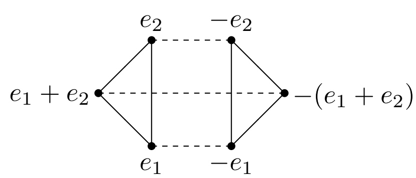
$$2 \leq q(x - e_i) = q(x) + q(e_i) - (x|e_i)_q = 2 - (x|e_i)_q,$$

that $(x|e_i)_q \leq 0$. Therefore $2 = (x|x)_q = \sum_i x_i (x|e_i)_q \leq 0$, a contradiction. ☐

Remark 8.23. The previous lemma is really very practical to determine all positive roots. If x is a positive root, then either $\|x\| = 1$ or $\|x\| > 1$. In the first case we have $x = e_i$ for some i. In the second case there exists some i for which $x - e_i$ is again a root, and therefore again positive, by Proposition 8.21. Hence we can compute all positive roots in the following way: start from the canonical ones $e_1, \ldots, e_n$ and add e_i for $i = 1, \ldots, n$ to each of them and discard those which are not roots. Proceed this way and you will necessary get all positive roots. $\diamondsuit$

For a given unit form q we define the **root diagram** $\Psi(q)$ as the graph whose vertices are the roots of q, that is, $\Psi(q)_0 = R(q)$, whereas the edges are assigned as follows: there is a full edge joining x and y if $x+y$ or $x-y$ is again a root and a dotted edge if $x=-y$.

Example 8.24. The following picture shows the root diagram $\Psi(q)$ if $\Delta(q) = \mathbb{A}_2$.



$\diamondsuit$

Proposition 8.25. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a unit form such that $\Delta(q)$ is a Dynkin diagram. Then the root diagram $\Psi(q)$ is connected. Furthermore the number of roots $|R(q)|$ for the various Dynkin types is as indicated in the following list.*

$\Delta(q)$	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\lvert R(q)\rvert$	$n(n+1)$	$2(n-1)n$	72	126	240

As a consequence the number $|R(q)|$ together with n determines $\Delta(q)$ uniquely.

Proof. Clearly, if x is a root, then also $-x$ and x is connected to $-x$ by a dotted edge. Hence, it is sufficient to show that the positive roots are connected among themselves. First, if i and j are two adjacent vertices in the graph $\Delta(q)$, then clearly $e_i + e_j$ is a root and hence the roots e_i and e_j are connected. This shows that the roots $e_1, \dots, e_n$ are connected. By Lemma 8.22 and induction over $\|x\|$ the connectedness of $R(q)$ follows.

The positive roots of q if $\Delta(q) = \mathbb{A}_n$ are of the form $0\cdots01\cdots10\cdots0$ where the first 1 stands in place ℓ and the last in place $h \geq \ell$. Clearly, any such vector is a root and furthermore for any such root x the vector $x + e_i$ is either again in the list or not a root as direct inspection shows. Hence this must be all positive roots and there are $\frac{n(n+1)}{2}$ of them.

A similar argument shows that the positive roots of q for $\Delta(q) = \mathbb{D}_n$ are of the following form.

$\begin{matrix}0\\ 00\cdots01\cdots10\cdots0\end{matrix}$	$\frac{(n-2)(n-1)}{2}$	$\begin{matrix}1\\ 12\cdots\cdots\cdots21\cdots10\cdots0\end{matrix}$	$\frac{(n-3)(n-2)}{2}$
$\begin{matrix}0\\ 11\cdots\cdots\cdots10\cdots0\end{matrix}$	$n-1$	$\begin{matrix}1\\ 01\cdots\cdots\cdots10\cdots0\end{matrix}$	$n-1$
$\begin{matrix}1\\ 11\cdots\cdots\cdots10\cdots0\end{matrix}$	$n-2$		

We indicated the number of such roots to the right and by adding them up we get that there are $n^2 - n$ positive roots and another $n^2 - n$ negative roots and therefore $2(n-1)n$ roots altogether by Proposition 8.21. The roots for $\mathbb{E}_n$ with $n = 6, 7, 8$ can be obtained following Remark 8.23, see Exercise 8.5.3. This shows that $|R(q)|$ has the size as indicated in the statement.

It remains to verify that $\Delta(q)$ is uniquely determined by the two numbers n and $|R(q)|$. But there is only one case when $|R(q)| = |R(p)|$ being $\Delta(q)$ and $\Delta(p)$ two different Dynkin diagrams, namely if $\Delta(q) = \mathbb{A}_8$ and $\Delta(p) = \mathbb{E}_6$, but the two unit forms have different number of variables. $\square$

Theorem 8.26. *In each equivalence class of positive definite unit forms there exists one and only one unit form q with the property that each connected component of $\Delta(q)$ is a Dynkin diagram.*

Proof. Let q be a positive definite unit form. By Proposition 8.19 the set of positive roots $P(q)$ is finite. Furthermore, it follows from

$$0 \leq q(e_i \pm e_j) = q(e_i) + q(e_j) \pm q_{ij} = 2 \pm q_{ij}$$

that $-1 \leq q_{ij} \leq 1$ for all $i \neq j$. If there exists a coefficient $q_{ij} = 1$, then let $T = \mathbf{1}_n - \mathbf{E}^{ij}$. Then $q' = qT$ is again a unit form since $q'(e_a) = q(e_a) = 1$ for $a \neq j$ and $q'(e_j) = q(e_j - e_i) = 2 - q_{ij} = 1$. For any $x \in \mathbb{Z}^n$ denote $x' = T^{-1}x$. Then $q'(x') = q(x)$ and we obtain an injective map $P(q) \to P(q'), x \mapsto x'$, since $x'_h = e_h^\top T^{-1} x = e_h^\top x = x_h$ for $h \neq i$ and $x'_i = e_i^\top T^{-1} x = (e_i + e_j)^\top x = x_i + x_j \geq 0$. However, this inclusion is proper since $e_i \in N(q')$ but $Te_i \notin P(q)$. So $|P(q)| < |P(q')| \leq |R(q')| = |R(q)|$. Thus after finitely many steps we must end up with a unit form $q^{(m)}$ such that $q^{(m)}_{ij} \leq 0$ for all $i \neq j$.

This shows that each positive definite unit form q is equivalent to a unit form p whose graph contains only simple full edges, that is, it does not contain a dotted edge nor multiple full edges.

Let Δ be a component of $\Delta(p)$. We want to show that Δ is contained in the list (8.8). First of all Δ cannot contain a cycle because otherwise some $\widetilde{\mathbb{A}}_n$ would be contained. Next, if we look at the **degree** of a vertex, that is, the number of vertices joined by an edge to it, we see that for each vertex the degree is 3 or less since otherwise $\widetilde{\mathbb{D}}_4$ is contained in Δ. Furthermore, Δ has at most one vertex with degree 3 because otherwise Δ would contain $\widetilde{\mathbb{D}}_n$ for some $n \geq 5$.

If there is no vertex of degree 3, then we have the case $\mathbb{A}_n$ and if there is one such vertex i, then the lengths $l_1 \leq l_2 \leq l_3$ of the arms (counting the vertices including i) are bounded: using that Δ does not contain $\widetilde{\mathbb{E}}_6$, we have $l_1 = 2$. Using $\widetilde{\mathbb{E}}_7 \not\subseteq \Delta$, we have $l_2 = 2, 3$ and in case $l_2 = 3$ we use $\widetilde{\mathbb{E}}_8 \not\subseteq \Delta$ to see that $l_3 \leq 5$.

It remains to see that in each equivalence class there is at most one such "normal form" p. But the root diagram $\Psi(p)$ is invariant under any equivalence and hence the connected components of $\Psi(p)$ indicate the connected components of $\Delta(p)$ and the number of variables involved for a fixed component is given by the rank of the subgroup generated by the roots of that component. Hence by Proposition 8.25 the corresponding Dynkin diagram of each component is uniquely determined. $\square$

Remark 8.27. The connection between indecomposables and roots can be remarkably strong, see Exercise 8.5.4. That's why the whole last chapter will be devoted to it. $\diamondsuit$

Exercises

8.5.1 Let A be the Kronecker algebra. Use Exercise 8.3.2 to show that $\chi_A([R_{n,\lambda}]) = 0$ holds for each $n \geq 1$ and each $\lambda \in K \cup \{\infty\}$.

8.5.2 Calculate $\chi_A([M])$ for each indecomposable A-module M for A the algebra considered in Example 7.8.

8.5.3 Use Remark 8.23 to determine the positive roots of q, where $\Delta(q) = \mathbb{E}_8$. Use the list obtained this way to determine the roots for $\Delta(q) = \mathbb{E}_6$, resp. $\Delta(q) = \mathbb{E}_7$: show that these are the vectors x in the list which satisfy $x_7 = x_8 = 0$, resp. $x_8 = 0$).

8.5.4 There are three quivers whose underlying graph is $\mathbb{A}_3$. Knit the preprojective component in each case and verify directly that always the same dimension vectors occur. Show that these vectors are precisely the positive roots of the unit form $q(\mathbb{A}_3)$.

Chapter 9

Indecomposables and dimensions

Some of the deepest results in the theory of representations of algebras is presented within this chapter. All of them deal with the representation type, in other words, with the amount of non-isomorphic indecomposable modules and their dimension vectors.

9.1 The Brauer-Thrall conjectures

In this last chapter we gathered some results concerning the representation type and in particular dimensions and dimension vectors of indecomposable modules. We will not be able to proof all of them, for many we have to refer to the literature. We start with two early statements, brought into life by Brauer around 1940 as exercises for his students and which remained unproved for a long time. Nowadays they are known as the **first** and **second Brauer-Thrall conjecture**. Both statements concern the representation type of an algebra. We prove the first Brauer-Thrall conjecture and refer to the literature for the second since its proof is well beyond the scope of this book.

For a fixed, finite-dimensional K-algebra A with quiver Q we denote by $\kappa(m) \in \mathbb{N} \cup \{\infty\}$ the number of isomorphism classes of indecomposable A-modules with dimension m. The algebra A is called **of bounded representation type** if there exists some number $M > 0$ such that $\kappa(m) = 0$ for all $m \geq M$. If A is not of bounded representation type, it is called **of**

unbounded representation type. The algebra A is called **of strongly unbounded representation type** if there exists an infinite sequence of dimensions $m_0 < m_1 < m_2 < \dots$ such that $\kappa(m_i) = \infty$ for all $i \in \mathbb{N}$.

For the proof of the first Brauer-Thrall conjecture we need the following technical result.

Lemma 9.1. *Let A is a finite-dimensional algebra. Furthermore let $M_0, \dots, M_t$ be indecomposable A-modules and*

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \dots \to M_{t-1} \xrightarrow{f_t} M_t$$

be radical homomorphisms. If $t \geq 2^d - 1$ and $\dim M_i \leq d$ for all $0 \leq i \leq t$, then the composition $f_t \cdots f_1$ is zero.

Proof. The crucial part in the proof is to show the following claim: If g and h are homomorphisms

$$L \xrightarrow{g} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{h} N$$

such that $\dim_K \operatorname{Im}(g) \leq \ell$ and $\dim_K \operatorname{Im}(h) \leq \ell$ for some $\ell > 0$, then it follows that $\dim_K \operatorname{Im}(hf_ig) \leq \ell - 1$. Assume the contrary. Then, since $\operatorname{Im}(hf_ig) \subseteq \operatorname{Im}(hf_i) \subseteq \operatorname{Im}(h)$ we have

$$\ell \leq \dim_K \operatorname{Im}(hf_ig) \leq \dim_K \operatorname{Im}(hf_i) \leq \dim_K \operatorname{Im}(h) \leq \ell$$

and therefore all these dimension must be equal to ℓ. This shows that $\operatorname{Ker}(hf_i) \cap \operatorname{Im}(g) = 0$ and therefore $\operatorname{Ker}(hf_i) \oplus \operatorname{Im}(g)$ is a submodule of M_i. Since $\operatorname{Im}(hf_i) \simeq M_i / \operatorname{Ker}(hf_i)$ we get that $\dim_K \operatorname{Ker}(hf_i) = \dim M_i - \dim_K \operatorname{Im}(hf_i) = \dim M_i - \ell$. Hence

$$\dim_K (\operatorname{Ker}(hf_i) \oplus \operatorname{Im}(g)) = \dim_K M_i - \ell + \ell = \dim_K M_i$$

showing that $M_i = \operatorname{Ker}(hf_i) \oplus \operatorname{Im}(g)$. Since $\operatorname{Im}(g) \neq 0$ and M_i is indecomposable we conclude that $\operatorname{Ker}(hf_i) = 0$. This shows that hf_i and hence f_i is injective. Similarly, we conclude from

$$\ell \leq \dim_K \operatorname{Im}(hf_ig) \leq \dim_K \operatorname{Im}(f_ig) \leq \dim_K \operatorname{Im}(g) \leq \ell,$$

that all these dimensions equal ℓ and therefore $\operatorname{Ker}(h) \cap \operatorname{Im}(f_ig) = 0$. Since $\operatorname{Im}(h) \simeq M_{i+1} / \operatorname{Ker}(h)$ we get $\dim_K \operatorname{Ker}(h) = \dim_K M_{i+1} - \ell$ and therefore $\dim_K (\operatorname{Ker}(h) \oplus \operatorname{Im}(f_ig)) = \dim M_{i+1}$. This shows that $M_{i+1} =$

$\operatorname{Ker}(h) \oplus \operatorname{Im}(f_i g)$ and since $f_i g \neq 0$ and M_{i+1} is indecomposable we conclude that $\operatorname{Ker}(h) = 0$ and $M_{i+1} = \operatorname{Im}(f_i g)$. Consequently $f_i g$ and hence f_i is surjective. Hence f_i is bijective in contradiction to the assumptions. This shows the claim.

Now, if we take $g = \mathrm{id}_{M_i}$ and $h = \mathrm{id}_{M_{i+1}}$, then $\dim_K \operatorname{Im}(g) \leq d$ and $\dim_K \operatorname{Im}(h) \leq d$. Thus, we conclude in the first step that $\dim_K \operatorname{Im}(f_i) \leq d-1$ for all i. In the second step we set $g = f_{i-1}$ and $h = f_{i-1}$ and conclude that the image $f_{i+1} f_i f_{i-1}$ has dimension at most $d-2$ for all i. Inductively we get that

$$\dim_K \operatorname{Im}(f_{i+m} \cdots f_i) \leq d - p$$

if $m \geq 2^p - 1$. The result follows then by setting $p = d$. $\square$

Theorem 9.2 (First Brauer-Thrall conjecture)**.** *An algebra of bounded representation type is of finite representation type.*

Proof. Suppose A is of bounded representation type. Then there exists some number $d > 0$ such that $\kappa(m) = 0$ for all $m > 0$. In other words, $\dim_K M \leq d$ for each indecomposable A-module M. We now will define a series of finite subsets of the Auslander-Reiten quiver Γ_A of A. The first set consists of all indecomposable projective A-modules $S_0 = \{P-1, \ldots, P_N\} \subseteq \Gamma_A$. Inductively if S_i is defined, we define

$$S_{i+1} = \{Y \in \Gamma_A \mid \text{there exists an arrow } X \to Y \text{ in } \Gamma_A \text{ with } X \in S_i\}$$

which, since S_i is finite, is again finite by Remark 7.3. Note that by construction, each radical morphism $X \to Y$ with $X \in S_i$ factors over a direct sum of A-modules in S_i.

Thus, if M is an indecomposable A-module then there exists a non-zero morphism $f\colon P \to M$ for some indecomposable projective A-module $P \in S_0$, the either f is an isomorphism and M belongs to S_0 or the factorization over S_1 yields two homomorphisms $f_1\colon P \to M_1$ and $g_1\colon M_1 \to M$ with $g_1 f_1 \neq 0$ with f_1 radical and $M_1 \in S_1$. In the next step we see that if g_1 is not an isomorphism, the factorization over S_2 yields a radical homomorphism $f_2\colon M_1 \to M_2 \in S_2$ and $g_2\colon M_2 \to M$ with $g_2 f_2 f_1 \neq 0$. Inductively we obtain that either M belongs to $\bigcup_{m=0}^{2^d-1} S_m$ or there exist homomorphisms

$$P \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \ldots \to M_{2^d-2} \xrightarrow{f_{2^d-2}} M_{2^d-1} \xrightarrow{g_{2^d-1}} M$$

with $g_{2^d-1} f_{2^d-1} \cdots f_2 f_1 \neq 0$. Since the latter is impossible by Lemma 9.1, we obtain that each indecomposable A-module belongs to the finite set $\bigcup_{m=0}^{2^d-1} S_m$. $\square$

The first Brauer-Thrall conjecture states that it is not possible to have $\kappa(m) = \infty$ for some small dimensions m but $\kappa(m) = 0$ for all large m. In other words: a finite-dimensional K-algebra of infinite representation type has indecomposable modules of arbitrarily large dimension. The first Brauer-Thrall conjecture was first proved by Roiter in [23] for algebraically closed fields and later generalized by Auslander in [1].

Theorem 9.3 (Second Brauer-Thrall conjecture)**.** *An algebra of unbounded representation type is of strongly unbounded representation type.*

This result states that if an algebra is of infinite representation type, then there exists infinitely many dimensions for which there are infinite families of indecomposable modules. The first complete proof of the second Brauer-Thrall conjecture was given in [3] by Bautista. For arbitrary fields, the problem remains open, see also Ringel's report [21] for a detailed discussion about the developments in the proofs of the Brauer-Thrall conjectures.

9.2 Gabriel's Theorem

Gabriel's Theorem is the first classification result in the representation theory of finite-dimensional algebras, which successfully links the indecomposable representations to the roots of the Tits form (8.5) for a special class of algebras. It classifies those hereditary algebras which are of finite representation type. First we prove an auxiliary result concerning short exact sequences.

Lemma 9.4. *Let A be a finite-dimensional K-algebra and let*

$$\varepsilon\colon 0 \to M \xrightarrow{a} E \xrightarrow{b} N \to 0$$

be a short exact sequence of A-modules. If the sequence ε is not split, then the following inequality holds:

$$\dim_K \operatorname{End}_A(E) < \dim_K \operatorname{End}_A(M \oplus N).$$

Proof. Applying the functor $\operatorname{Hom}_A(-, E)$ to the exact sequence ε we get

$$0 \to \operatorname{Hom}_A(N, E) \xrightarrow{\operatorname{Hom}_A(b,E)} \operatorname{End}_A(E) \xrightarrow{\operatorname{Hom}_A(a,E)} \operatorname{Hom}_A(M, E)$$

and therefore

$$\begin{aligned} \dim_K \operatorname{End}_A(E) &= \dim_K \operatorname{Hom}_A(N, E) + \dim_K(\operatorname{Im} \operatorname{Hom}_A(a, E)) \\ &\leq \dim_K \operatorname{Hom}_A(N, E) + \dim_K \operatorname{Hom}_A(M, E). \end{aligned} \tag{9.1}$$

With a similar reasoning we get by applying $\operatorname{Hom}_A(M,-)$ to ε the following inequality

$$\begin{aligned}\dim_K \operatorname{Hom}_A(M,E) &\leq \dim_K \operatorname{Hom}_A(M,M) + \dim_K \operatorname{Hom}_A(M,N)\\ &= \dim_K \operatorname{Hom}_A(M, M\oplus N).\end{aligned} \tag{9.2}$$

Finally, by applying $\operatorname{Hom}_A(E,-)$ to ε, we get the exact sequence

$$0 \to \operatorname{Hom}_A(N,M) \to \operatorname{Hom}_A(N,E) \to \operatorname{Hom}_A(N,N) \xrightarrow{\delta\neq 0} \operatorname{Ext}^1_A(N,M),$$

where δ is the connecting homomorphism, which is non-zero since $\delta(\mathrm{id}_N) = \varepsilon \neq 0$. Therefore we get a strict inequality

$$\begin{aligned}\dim_K \operatorname{Hom}_A(N,E) &< \dim_K \operatorname{Hom}_A(N,M) + \dim_K \operatorname{Hom}_A(N,N)\\ &= \dim_K \operatorname{Hom}_A(N, M\oplus N).\end{aligned} \tag{9.3}$$

Substituting (9.2) and (9.3) in (9.1) yields

$$\begin{aligned}\dim_K \operatorname{End}_A(E) &< \dim_K \operatorname{Hom}_A(M, M\oplus N) + \dim_K \operatorname{Hom}_A(N, M\oplus N)\\ &= \dim_K \operatorname{End}_A(M\oplus N)\end{aligned}$$

which is what we wanted to prove. □

Theorem 9.5 (Gabriel)**.** *Let $A = KQ$ be the path algebra of a quiver Q without cycle. Denote by q_A the Euler form of A. Then A is of finite representation type if and only if q_A is positive definite or equivalently if and only if Q is the disjoint union of connected quivers whose underlying graphs are Dynkin diagrams. Moreover, in that case, the function*

$$\Psi\colon (\Gamma_A)_0 \to P(q_A), X \mapsto \underline{\dim}\, X$$

is bijective, where $P(q_A)$ denotes the set of positive roots of q_A.

Proof. By Remark 8.14 the Euler form and the Tits form coincide, $\chi_A = q_A$, since A is hereditary. Note that it is enough to prove the result for connected A. Otherwise $A = A_1 \times A_2$ and $q_A = q_{A_1} \times q_{A_2}$ holds and then A is of finite representation type if and only if A_1 and A_2 both are so and similarly q_A is positive definite if and only if q_{A_1} and q_{A_2} are both positive definite.

Next we observe that by Proposition 7.10 there exists a unique preprojective component C in Γ_A since A is hereditary and connected. Hence we conclude from Theorem 7.7 that A is of finite representation type if and only if C is finite and that this happens if and only if $C = \Gamma_A$. Furthermore, it follows from Proposition 7.11 that each A-module $M \in C$ satisfies $\dim_K \operatorname{End}_A(M) = 1$ and $\operatorname{Ext}^1_A(M, M) = 0$. Therefore $\underline{\dim}\, M$ is a root of q_A for each $M \in C$.

Now suppose that q_A is positive definite. Then it follows from Proposition 8.19 that there are only finitely many roots of q_A. Therefore, if we can show that for each root d there exists *at most one* indecomposable A-module $M \in C$ such that $\underline{\dim}\, M = d$, then we have shown that C is finite. So, assume that $M, N \in C$ are two indecomposable A-modules with the same dimension vector d. Then

$$1 = q_A(d) = \langle \underline{\dim}\, M, \underline{\dim}\, N\rangle = \dim_K \operatorname{Hom}_A(M, N) - \dim_K \operatorname{Ext}^1_A(M, N)$$

shows that $\operatorname{Hom}_A(M, N) \neq 0$. Similarly, we get $\operatorname{Hom}_A(N, M) \neq 0$ and conclude that $M = N$ since there is no cycle in C.

Conversely assume now that C is finite. Note that, if we can show that for each positive root d of q_A there exists *at least one* indecomposable A-module M with $\underline{\dim}\, M = d$, then there exist only finitely many roots since $\Gamma_A = C$. So, assume that d is a positive root of q_A. Since d is positive it is possible to find A-modules with $\underline{\dim}\, M = d$, for example $M = \bigoplus_{i \in Q_0} S_i^{d_i}$ has dimension vector d. Choose such an A-module M such that $\dim_K \operatorname{End}_A(M)$ is minimal. Decompose M into indecomposables $M = \bigoplus_{i=1}^t M_i$. By Proposition 7.11 we have $\operatorname{Ext}^1_A(M_i, M_i) = 0$ and it follows from Lemma 9.4 that $\operatorname{Ext}^1_A(M_i, M_i') = 0$, where $M_i' = \bigoplus_{j \neq i} M_j$, since otherwise there would exist some non-split short exact sequence $M_i' \to E \to M_i$ and consequently

$$\dim_K \operatorname{End}_A(E) < \dim_K \operatorname{End}_A(M_i \oplus M_i') = \dim_K \operatorname{End}_A(M),$$

a contradiction to the minimality of $\dim_K \operatorname{End}(M)$ since $\underline{\dim}\, E = \underline{\dim}\, M_i + \underline{\dim}\, M_i' = \underline{\dim}\, M$. This shows that $\operatorname{Ext}^1_A(M, M) = 0$. Then it follows from

$$1 = q_A(\underline{\dim}\, M) = \dim_K \operatorname{End}_A(M) \geq \sum_{i=1}^t \dim_K \operatorname{End}_A(M_i) = t$$

that $t = 1$, in other words, that M is indecomposable.

Observe that we also proved the claimed bijection between the indecomposable A-modules and the positive roots of q_A. $\square$

Theorem 9.5 was first proved by Gabriel in [10]. Soon a more conceptual proof was given by Bernstein, Gelfand and Ponomarev in [4]. The presented approach is taken from Ringel [22].

Exercises

9.2.1 Generalize Theorem 9.5 to finite-imensional K-algebras A which have a preprojective component C in the Auslander-Reiten quiver Γ_A and which satisfy $\operatorname{gldim} A \leq 2$ by adjusting the proof slightly to the more general setting.

9.3 Reflection functors

Observe that Gabriel's Theorem 9.5 implies that the dimension vectors of the indecomposables do not depend on the orientation of the quiver, an unexpected fact, which already could be observed in Exercise 8.5.4. In this and the next section we focus more closely on this peculiarity of hereditary algebras.

Let Q be a finite quiver without cycle. Note that this implies that Q always has a source and a sink, see Exercise 9.3.1.

If the x is a source or a sink of Q, we denote by $\sigma_x Q$ the quiver which is obtained from Q by reversing the direction of all arrows which start or end in x. Thus if x is a source of Q, then it will be a sink in $\sigma_x Q$ and vice versa. We now define functors $\sigma_x \colon \operatorname{rep} Q \to \operatorname{rep} \sigma_x Q$ in the following way. Let first x be a source of Q. Then for each representation $V \in \operatorname{rep} Q$ we have a linear map

$$V_{x\to\bullet} \colon V_x \xrightarrow{\begin{bmatrix} \vdots \\ V_\alpha \\ \vdots \end{bmatrix}} \bigoplus_{\alpha\in Q_1 \colon s(\alpha)=x} V_{t(\alpha)}$$

and denote its cokernel by

$$\bigoplus_{\alpha\in Q_1 \colon s(\alpha)=x} V_{t(\alpha)} \xrightarrow{\begin{bmatrix} \cdots & V'_\alpha & \cdots \end{bmatrix}} V'_x.$$

Hence we get a representation $\sigma_x V$ of $\sigma_x Q$ which is defined by

$$(\sigma_x V)_y = \begin{cases} V_y, & \text{if } y \neq x, \\ V'_x, & \text{if } y = x, \end{cases} \qquad (\sigma_x V)_\beta = \begin{cases} V_\beta, & \text{if } t(\beta) \neq x, \\ V'_\beta, & \text{if } t(\beta) = x. \end{cases}$$

Now if $f \colon V \to W$ is a morphism of representations of Q, then we define $f'_x \colon V'_x \to W'_x$ as the map induced by $\bigoplus_\alpha f_\alpha \colon \bigoplus_\alpha V_{t(\alpha)} \to \bigoplus_\alpha W_{t(\alpha)}$. This defines the functor σ_x if x is a source of Q and the inverse construction works if x is a sink of Q. These functors were called **Coxeter reflection functors** by Bernstein, Gelfand and Ponomarev in [4] and are nowadays also called **BGP-reflection functors**.

Proposition 9.6. *Let Q be a finite quiver without cycle and x a source or a sink of Q. Then $\sigma_x S_x = 0$ and if V is an indecomposable representation of Q satisfying $V \not\simeq S_x$, then $\sigma_x V$ is an indecomposable representation of $\sigma_x Q$ satisfying $\sigma_x V \not\simeq S_x$. In particular σ_x induces a bijection between the isomorphism classes of the indecomposable representations $V \not\simeq S_x$ of Q and the isomorphism classes of the indecomposable representations $W \not\simeq S_x$ of $\sigma_x Q$.*

Proof. Notice first that for all representations V we have $V \simeq S_x^d \oplus W$ where d is the dimension of $\operatorname{Ker} V_{x\to\bullet}$. Hence, if V is indecomposable, then either $V = S_x$ or $V_{x\to\bullet}$ is injective. It follows directly from the definition that $\sigma_x S_x = 0$ and that S_x cannot be a direct summand of $\sigma_x V$ for any representation V of Q. If $V_{x\to\bullet}$ is injective then $\sigma_x\sigma_x V \simeq V$. Now, if $V = U \oplus W$, then $\sigma_x V = \sigma_x U \oplus \sigma_x W$. Thus if V is indecomposable and $V \not\simeq S_x$, then $\sigma_x V$ cannot be decomposable since otherwise $V = \sigma_x\sigma_x V$ would also be decomposable. This finishes the proof. $\square$

Exercise 9.3.2 gives an even stronger version of Proposition 9.6. We now study how the reflection functors change the dimension vectors.

Lemma 9.7. *Let Q be a finite quiver without cycle, x a source of Q and q the quadratic form of the path algebra KQ. Then for any representation V of Q with injective map $V_{x\to\bullet}$ the dimension vector of $\sigma_x V$ is given by*

$$\underline{\dim}\, \sigma_x V = \underline{\dim}\, V - (\underline{\dim}\, V|e_x)_q e_x,$$

where $(x|y)_q = q(x+y)-q(x)-q(y)$ is the symmetric bilinear form associated to q.

Proof. Since $V_{x\to\bullet}$ is injective we have a short exact sequence

$$V_x \to \bigoplus_{\alpha\in Q_1 :\, s(\alpha)=x} V_{t(\alpha)} \to V'_x$$

and therefore we can calculate $\underline{\dim}\, \sigma_x V$ as follows:

$$\underline{\dim}\, \sigma_x V = \underline{\dim}\, V + \delta e_x,$$

where $\delta = \sum_{\alpha\in Q_1 :\, s(\alpha)=x} \dim_K V_{t(\alpha)} - 2\dim_K V_x$. It follows from the description of the Tits form given in (8.6) that for $v = \underline{\dim}\, V$ we have

$$\begin{aligned}(v|e_x)_q &= q(v+e_x) - q(v) - q(e_x)\\ &= \sum_{i\in Q_0} (v+e_x)_i^2 - \sum_{(i\to j)\in Q_1} (v+e_x)_i(v+e_x)_j\\ &\qquad - \sum_{i\in Q_0} v_i^2 + \sum_{(i\to j)\in Q_1} v_i v_j - 1.\end{aligned}$$

After cancelling all possible terms we are left with $(v|e_x)_q = 2v_x - \sum_{i\to x} v_i - \sum_{x\to j} v_i$ and since x is a source we get $\sum_{i\to x} v_i = 0$ and therefore conclude that $(v|e_x)_q = \delta$. □

We define the linear maps

$$s_x\colon\ \mathrm{K}_\circ(KQ) \to \mathrm{K}_\circ(KQ), \quad s_x(v) = v - (v|e_x)_q e_x, \tag{9.4}$$

and get that the dimension vectors of representations change under σ_x according to s_x in the sense that

$$\underline{\dim}\, \sigma_x V = s_x(\underline{\dim}\, V), \tag{9.5}$$

if the representation V does not have any direct summands isomorphic to S_x. In the following we assume that the vertices of Q are $1,\dots,n$ and therefore $\mathrm{K}_\circ(KQ) \simeq \mathbb{Z}^n$. We extend the symmetric bilinear form $(-|-)_q$ to $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the linear maps s_x to $\mathbb{R}^n \to \mathbb{R}^n$ in the obvious way.

Lemma 9.8. *With the above definitions the following properties hold.*

(a) s_x *is an idempotent, that is,* $s_x^2 = \mathrm{id}_{\mathbb{R}^n}$,

(b) s_x *is orthogonal, that is,* $(s_x v|s_x w)_q = (v|w)_q$,

(c) s_x *reflects* e_x*, that is,* $s_x(e_x) = -e_x$,

(d) $\mathbb{R}^n = \mathbb{R}e_x \oplus e_x^\perp$ *where* $e_x^\perp = \{v \in \mathbb{R}^n \mid (v|e_x) = 0\}$.

Proof. Note that $(e_x|e_x)_q = q(2e_x) - 2q(e_x) = 2$ and hence (c) follows immediatly. All other properties follow from these two facts. Since

$$s_x^2(v) = s_x(v) - (v|e_x)_q s_x(e_x) = v - (v|e_x)_q e_x + (v|e_x)_q e_x = v.$$

property (a) follows. Also (b) is straightforward:

$$\begin{aligned}(s_xv|s_xw)_q &= (v-(v|e_x)_qe_x \mid w-(w|e_x)_qe_x)\\ &= (v|w)_q-(v|e_x)_q(e_x|w)_q\\ &\qquad -(w|e_x)_q(v|e_x)_q+(v|e_x)_q(w|e_x)_q(e_x|e_x)_q\\ &= (v|w)_q\end{aligned}$$

To see (d) write any $v\in\mathbb{R}^n$ as

$$v=\frac{1}{2}(v|e_x)_qe_x+\left(v-\frac{1}{2}(v|e_x)_qe_x\right)$$

and observe that the second summand lies in $e_x^\perp$ since $(v-\frac{1}{2}(v|e_x)_qe_x \mid e_x)=(v|e_x)_q-\frac{1}{2}(v|e_x)_q(e_x|e_x)_q=0$. This shows that $\mathbb{R}e_x+e_x^\perp=\mathbb{R}^n$. Since $(e_x|e_x)_q\neq 0$ we also have $\mathbb{R}e_x\cap e_x^\perp=0$ showing the result. □

Remarks 9.9 (a) It is really formula (9.5) which gave the idea of naming the functors σ_x Coxeter reflections since this formula is well known from the study of reflection groups and the generalization to Coxeter groups, see [12] for an elementary introduction. It also justifies to call the maps s_x **reflections** even when the bilinear form $(-|-)_q$ is not positive definite.

(b) Bernstein, Gelfand and Ponomarev used in [4] this reflection functors to give a proof of the bijection statement in Gabriel's Theorem. It is possible to reach any root of a positive definite unit form by starting from the roots $e_1,\dots,e_n$ and using these reflections, see Exercise 9.3.3.

We would like to use this for representations: It is clear that for each j there exists, up to isomorphism, a unique indecomposable representation with dimension vector e_j, namely the simple representation S_j. So let $v=s_{i_t}\cdots s_{i_2}s_{i_1}(e_j)$ be a positive root obtained by reflections. Now, if i_1 is a source or a sink of Q and iteratively i_h is a source or a sink of the quiver $\sigma_{i_{h-1}}\cdots\sigma_{i_1}Q$ for $1<h\leq t$, then we see that there exists a unique indecomposable representation with dimension vector v, namely $\sigma_{i_t}\cdots\sigma_{i_1}S_j$. Unfortunately if for some h the vertex i_h is not a source or a sink of the quiver $\sigma_{i_{h-1}}\cdots\sigma_{i_1}Q$ then this construction does not work. For the case of Dynkin diagrams this can be fixed, see [4]. ◊

The next section shows the most general result concerning the dimension vectors of indecomposable representations obtained so far.

Exercises

9.3.1 Prove that a finite quiver Q without cycle always has at least one source and one sink. Show that both hypothesis are necessary by giving two examples of quivers without source nor sink: one which is finite but has some cycle and another which is infinite but without cycle.

9.3.2 Let x be a source or a sink of a finite quiver Q. Prove that the Coxeter reflection functors induce equivalences between the categories $\operatorname{rep} Q/I_x$ and $\operatorname{rep} \sigma_x Q/J_x$ where I_x (resp. J_x) is the ideal of $\operatorname{rep} Q$ (resp. of $\operatorname{rep} \sigma_x Q$) consisting of all morphisms which factor over S_x^d for some $d > 0$.

9.3.3 Let Q be a **Dynkin quiver**, that is, a quiver whose underlying graph is a Dynkin diagram. Then the associated unit form $q = q_{KQ}$ is poisitive definite and by Lemma 8.22 for each root v with $\|v\| > 1$ there exists an index i such that $v - e_i$ is again a root. Show that in this case $v - e_i = s_x(v)$. This shows that in the case of Dynkin diagrams it is possible to reach any root using the reflections $s_1, \ldots, s_n$ starting from the roots $e_1, \ldots, e_n$.

9.4 Kac's Theorem

Let Q be a finite quiver with vertices $1, \ldots, n$ and let q be the associated quadratic form. The group W generated by the reflections $s_1, \ldots, s_n$ is called the **Weyl group**. The Weyl group acts on $\mathbb{Z}^n$ in the obvious way.

In case Q is a Dynkin quiver, we know from Theorem 9.5 that the positive roots of q correspond to the indecomposable representations of Q. In view of Exercsie 9.3.3 we can say that the positive roots of q are precisely those positive vectors which can be obtained by a sequence of reflections s_x as defined in (9.4). In other words, the set $P(q)$ of positive roots equals $\mathbb{N}^n \cap W\{e_1, \ldots, e_n\}$.

It is remarkable that this relationship extends to all finite quivers without cycle if the notion of roots is enhanced the right way. To do so, we define the **support** $\operatorname{supp} v$ of a vector $v \in \mathbb{Z}^n$ as the full subquiver of Q given by those vertices i for which $v_i \neq 0$. Furthermore, the set

$$F = \{v \in \mathbb{N}^n \mid \operatorname{supp} v \text{ is connected and } (v|e_j)_q \leq 0 \text{ for } 1 \leq j \leq n\} \qquad (9.6)$$

is called the **fundamental region**, where $(-|-)_q$ is the symmetric bilinear form (8.11) associated to q.

A vector $v \in \mathbb{Z}^n$ is called a **Schur root** if it belongs to the W-orbit WS, where $S = \{e_1, \ldots, e_n\} \cup F \cup (-F)$. The elements of the form we_j for some

$j = 1, \dots, n$ and some $w \in W$ are called **real** whereas the elements of the form $w(\pm f)$ for some $f \in F$ and some $w \in W$ are called **imaginary**. Note that $q(v) = 1$ for any real Schur root v since by Lemma 9.8(b) the Weyl group preserves the bilinear form $(-|-)$. Similarly one can see that $q(v) \leq 0$ holds for any imaginary Schur root, see Exercise 9.4.1. Moreover, if v is a real Schur root, then $\pm v$ are the unique multiples of v which are Schur roots, whereas if v is imaginary, then every non-zero multiple of v is a Schur root, see Exercise 9.4.2.

Theorem 9.10 (Kac)**.** *Let K be any field and Q a finite quiver without cycle. If V is an indecomposable representation of Q, then $\underline{\dim}\, V$ is a Schur root. Conversely we have the following characterization.*

(a) *If v is a positive real Schur root, then up to isomorphism there exists a unique indecomposable representation V of Q with $\underline{\dim}\, V = v$.*

(b) *If v is a positive imaginary Schur root, then there exist infinitely many pairwise non-isomorphic indecomposable representations V of Q with $\underline{\dim}\, V = v$.*

The proof of Kac's Theorem is too difficult to give in this notes. It was proved by Kac in [13] and later generalized by Kac in [14] to quivers with loops. For an approach to Kac's Theorem using algebraic geometry (as briefly outlined in Section 9.6) we refer to Kraft and Riedtmann [17].

Remark 9.11. Note that it is a premature idea to think that by Kac's Theorem we already gain access to all finite-dimensional algebras using the fact that $\operatorname{mod} A$ is equivalent to a full subcategory of $\operatorname{rep} Q$, where Q is the quiver of A. If A is Morita equivalent to KQ/I for some admissible ideal I, then Kac's Theorem yields those dimension vectors for which indecomposable A-modules may exist, but since we have no control of how many of those representations satisfy the ideal I we cannot deduce more than the first statement: if M is an indecomposable A-module, then $\underline{\dim}\, M$ is a Schur root. $\Diamond$

Exercises

9.4.1 Let q be the quadratic form associated to a quiver Q. Prove that $q(f) \leq 0$ for any f in the fundamental region F given in (9.6).

9.4.2 A vector $v \in \mathbb{Z}^n$ is called **indivisible** if $v = \lambda w$ for $w \in \mathbb{Z}^n$ and $\lambda \in \mathbb{Z}$ implies that $\lambda = \pm 1$. Show that if v is indivisible, then also $s_x(v)$ is indivisible and conclude from this that each real Schur root is indivisible. Prove further that if $v \in \mathbb{Z}^n$ is an imaginary Schur root then λv is also an imaginary Schur root for each $\lambda \in \mathbb{Z} \setminus \{0\}$.

9.5 Tame and wild

At the end of Chapter 1 we found three very distinct phenomena in classifying indecomposable elements of matrix problems. We later translated these problems into the language of representations of a quiver and modules over an algebra. We now come back to these three types of complexity and study them in more detail. We assume throughout the rest of this chapter that K is algebraically closed.

To compare the complexity of the classification problems for two different finite-dimensional K-algebras A and B, we introduce the following language. A functor $F\colon \operatorname{mod} A \to \operatorname{mod} B$ **preserves indecomposability** if it maps indecomposable A-modules to indecomposable B-modules and F **reflects isomorphisms** if $FM \simeq FN$ implies $M \simeq N$ for any A-modules M and N. Thus, if we have a functor $F\colon \operatorname{mod} A \to \operatorname{mod} B$ which preserves indecomposability and reflects isomorphisms, then a classification of indecomposable B-modules would yield a classification of indecomposable A-modules. However, note that such a functor does not necessarily transport a lot of structure from $\operatorname{mod} A$ to $\operatorname{mod} B$, see Exercsie 9.5.1.

We start by studying the wild behaviour in some detail. Let $Q^{(t)}$ be the quiver with one vertex 0 and t loops $\alpha_1, \dots, \alpha_t$. A representation of $Q^{(t)}$ is a vector space together with t (not necessarily commuting) endomorphisms. Such a representation it is given by t square matrices $(V_{\alpha_1}, \dots, V_{\alpha_t})$. Two such tuples $(V_{\alpha_1}, \dots, V_{\alpha_t})$ and $(W_{\alpha_1}, \dots, W_{\alpha_t})$ represent isomorphic representations if and only if they are simultaneously conjugate, that is, if there exists an invertible square matrix U such that $W_{\alpha_i} = UV_{\alpha_i}U^{-1}$ for all $i = 1, \dots, t$.

Lemma 9.12. *There exists a functor $F\colon \operatorname{rep} Q^{(t)} \to \operatorname{rep} Q^{(2)}$ which preserves indecomposability and reflects isomorphisms.*

Proof. Given a representation $V \in \operatorname{rep} Q^{(t)}$ we define FV by setting $(FV)_0 =$

$(V_0)^{t+1}$ and

$$(FV)_{\alpha_1} = \begin{bmatrix} 0 & \mathrm{id} & 0 & \cdots & 0 \\ 0 & 0 & \mathrm{id} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{id} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (FV)_{\alpha_2} = \begin{bmatrix} 0 & V_{\alpha_1} & 0 & \cdots & 0 \\ 0 & 0 & V_{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & V_{\alpha_t} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

If V and W are representations of $Q^{(t)}$ then a morphism $g\colon V \to W$ is given by single linear map $g = g_0\colon V_0 \to W_0$ such that $W_{\alpha_i} g = gV_{\alpha_i}$ for $i = 1, \ldots, t$. To define $Fg\colon FV \to FW$ we observe first that any morphism $h\colon FV \to FW$ is given by a matrix of size $(t+1) \times (t+1)$ whose entries h_{ij} are morphisms $V_0 \to W_0$ satisfying $(FW)_{\alpha_\ell} h = h(FV)_{\alpha_\ell}$ for $\ell = 1, 2$. Thus we see that

$$Fg = \begin{bmatrix} g & 0 & 0 & \cdots & 0 \\ 0 & g & 0 & \cdots & 0 \\ 0 & 0 & g & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & g \end{bmatrix}$$

defines a morphism $FG\colon FV \to FW$ of representations of $Q^{(2)}$. By definition F is functorial and **faithful**, that is, $Fg \neq 0$ for $g \neq 0$. Now suppose that $h\colon FV \to FW$ is a morphism. Then it follows from $(FW)_{\alpha_1} h = h(FV)_{\alpha_1}$ that $h_{11} = h_{22} = \ldots = h_{tt} = h_{t+1,t+1}$ and $h_{ij} = 0$ for $i > j$. Using this fact we conclude from $(FW)_{\alpha_2} h = h(FV)_{\alpha_2}$ that $W_{\alpha_i} h_{11} = h_{11} V_{\alpha_i}$ for all $i = 1, \ldots, t$, that is, h_{11} is a morphism $V \to W$. Thus, if h is an isomorphism, then so must be h_{11} since $h = (h_{ij})$ is upper triangular. This shows that F reflects isomorphisms.

Furthermore, if FV decomposes into two non-zero summands, then there exist two non-zero idempotents $h, h' \in \mathrm{End}_{Q^{(2)}}$ such that $h + h' = \mathrm{id}_{FV}$ and $hh' = h'h = 0$. So h_{11} and h'_{11} are two non-zero idempotents in $\mathrm{End}_{Q^{(t)}}$ with $h_{11} + h'_{11} = \mathrm{id}_V$ and $h_{11} h'_{11} = h'_{11} h_{11} = 0$ showing that V is decomposable. Thus F preserves indecomposability. □

A functor $F\colon \mathscr{C} \to \mathscr{D}$ is called **full** if for every pair of objects $x, y \in \mathscr{C}$ we have $F\mathscr{C}(x, y) = \mathscr{D}(Fx, Fy)$ and the functor is called an **embedding** if it is full and faithful.

Lemma 9.13. *Let A and B be two finite-dimensional K-algebras. Any embedding* $\operatorname{mod} A \to \operatorname{mod} B$ *preserves indecomposability and reflects isomorphisms.*

Proof. Let $F\colon \operatorname{mod} A \to \operatorname{mod} B$ be a full functor and M, N be two A-modules. If $\varphi\colon FM \to FN$ is an isomorphism, then $\varphi = F\psi$ for some $\psi\colon M \to N$ and also $\varphi^{-1} = F\eta$ for some $\eta\colon N \to M$. Now, $F(\mathrm{id}_M) = \mathrm{id}_{FM} = \varphi^{-1}\varphi = F\eta F\psi = F(\eta\psi)$ implies $\mathrm{id}_M = \eta\psi$. Similarly, we get $\mathrm{id}_N = \psi\eta$ and therefore $M \simeq N$. This implies that F reflects isomorphisms.

As a consequence a non-zero A-module cannot be mapped to zero. Now, if FMdecomposes, say $FM \simeq Q' \oplus Q''$ with $Q' \neq 0$ and $Q'' \neq 0$, then there exist non-zero, orthogonal idempotents $\varphi', \varphi''\colon FM \to FM$ with $\varphi' + \varphi'' = \mathrm{id}_{FM}$. Since F is full there exist ψ', ψ'' such that $\varphi' = F\psi'$ and $\varphi'' = F\psi''$. Observe that ψ', ψ'' are non-zero, orthogonal idempotents satisfying $\psi' + \psi'' = \mathrm{id}_M$. This implies that $M \simeq \operatorname{Im}\psi' \oplus \operatorname{Im}\psi''$ with $\operatorname{Im}\psi' \neq 0$ and $\operatorname{Im}\psi'' \neq 0$. Therefore F preserves indecomposability. $\square$

The next result resumes that solving the classification problem for $Q^{(2)}$ would solve the classification problem for all finite-dimensional K-algebras at once.

Proposition 9.14. *For each finite-dimensional algebra A there exists a functor $\operatorname{mod} A \to \operatorname{rep} Q^{(2)}$ which preserves indecomposability and reflects isomorphisms.*

Proof. Since K is algebraically closed Gabriel's Theorem 3.28 implies that there exists a quiver Q and an admissible ideal I of KQ such that $\operatorname{mod} A$ is equivalent to $\operatorname{rep}_I Q$. Now, the inclusion $J\colon \operatorname{rep}_I Q \to \operatorname{rep} Q$ is an embedding and therefore preserves indecomposability and reflects isomorphisms by Lemma 9.13.

Let t be the number of arrows of Q, say $Q_1 = \{\beta_1, \dots, \beta_t\}$. Then we define a functor $G\colon \operatorname{rep} Q \to \operatorname{rep} Q^{(t)}$ in the following way. For a representation V of Q we define the representation GV of $Q^{(t)}$ by $(GV)_0 = \bigoplus_{x\in Q_0} V_x$ and $(GV)_{\alpha_h} = \iota^V_{t(\beta_h)} V_{\beta_h} \pi^V_{s(\beta_h)}$, that is, the composition of the following maps

$$\bigoplus_{x\in Q_0} V_x \xrightarrow{\pi^V_{s(\beta_h)}} V_{s(\beta_h)} \xrightarrow{V_{\beta_h}} V_{t(\beta_h)} \xrightarrow{\iota^V_{t(\beta_h)}} \bigoplus_{x\in Q_0} V_x.$$

If $\psi\colon V \to W$ is a morphism of representations of Q then we define the morphism $G\psi\colon GV \to GW$ by $G\psi = \bigoplus_{x\in Q_0} \psi_x$. Thus by definition G is faithful.

To see that G is full, we assume $f\colon GV \to GW$ to be a morphism of representations of $Q^{(t)}$. Then for each $x \in Q_0$ we define $\varphi_x = \pi^W_x f \iota^V_x \colon V_x \to W_x$.

We want to show that the family $\varphi = (\varphi_x)_{x \in Q_0}$ is a morphism of representations $V \to W$. For this assume that $\beta = \beta_h \colon x \to y$ is an arrow. Then we get that $f(GV)_{\alpha_h} = (GW)_{\alpha_h} f$, that is,

$$f \iota_y^V V_\beta \pi_x^V = \iota_y^W W_\beta \pi_x^W f. \tag{9.7}$$

Applying $\pi_y^W \circ ? \circ \iota_x^V$ to equation (9.7) we get

$$\pi_y^W f \iota_y^V V_\beta = W_\beta \pi_x^W f \iota_x^V$$

hence $\varphi_y V_\beta = W_\beta \varphi_x$. This shows that φ is indeed a morphism of representations and $f = G\varphi$. Thus G is a full functor and therefore by Lemma 9.13 preserves indecomposability and reflects isomorphisms.

By Lemma 9.12 there exists a functor $F \colon \operatorname{rep} Q^{(t)} \to \operatorname{rep} Q^{(2)}$ which preserves indecomposability and reflects isomorphisms, so the composition

$$\operatorname{mod} A \simeq \operatorname{rep}_I Q \xrightarrow{J} \operatorname{rep} Q \xrightarrow{G} \operatorname{rep} Q^{(t)} \xrightarrow{F} \operatorname{rep} Q^{(2)}$$

preserves indecomposability and reflects isomorphisms. This finishes the proof. $\square$

A finite-dimensional K-algebra A is called **wild** if there exists a functor $F \colon \operatorname{rep} Q^{(2)} \to \operatorname{mod} A$ which preserves indecomposability and reflects isomorphisms. Thus a full classification of all modules of some wild algebra would yield the classification for all finite-dimensional K-algebras at once. This is considered a “hopeless situation”.

Now let us look at the phenomenon of tameness. Since the formal definition of tameness is more involved than that of wildness we will approach the definition gradually and start with an example of a tame algebra, where we have the full knowledge about its indecomposables.

Example 9.15. Let A be the Kronecker algebra. We know that the indecomposables occur in every dimension. For odd $m \in \mathbb{N}$ there exist precisely two indecomposable A-modules, namely Q_ℓ and J_ℓ where $2\ell + 1 = m$. For even $m \in \mathbb{N}$, say $m = 2\ell$, there exist infinitely many indecomposables, namely $R_{\ell,\lambda}$ where $\lambda \in K \cup \{\infty\}$.

Thus for every even dimension $m = 2\ell$ there exists a one-paremeter family. Rephrasing it differently, we can obtain each of these indecomposables, except $R_{n,\infty}$, by specializing the indeterminate X to λ in the following infinite-dimensional representation of the Kronecker quiver:

$$K[\mathrm{X}]^\ell \quad \overset{\mathbf{1}_\ell}{\underset{J(\ell,\mathrm{X})}{\rightrightarrows}} \quad K[\mathrm{X}]^\ell,$$

where $J(\ell,\mathrm{X})$ is the matrix in $K[X]^{\ell\times\ell}$ with entries

$$J(\ell,\mathrm{X})_{ij} = \begin{cases} \mathrm{X}, & \text{if } i=j \\ \mathbf{1}_K, & \text{if } i=j-1 \\ 0, & \text{else.} \end{cases}$$

If we specialize X to λ then any polynomial in X becomes an element of K and thus the polynomial algebra $K[\mathrm{X}]$ "specializes" to K. In the following, we will see how to formalize this properly. $\Diamond$

Fix a finite-dimensional K-algebra A. Of course a "one-paremeter family of A-modules" $(M_\lambda)_\lambda$ should not mean any arbitrary assignment $\lambda \mapsto M_\lambda$ since we do not seek merely to index the modules in the family. First of all, each considered modules should have the same dimension vector $d \in \mathbb{N}^{Q_0}$. One way to visualize something which comes close to a one-paremeter family is a fixed family $(V_x)_{x\in Q_0}$ of vector spaces in the vertices of the quiver Q of A and a family of matrices $(V_{\alpha,\lambda})_{\alpha\in Q_1}$ such that for each α the family of matrices $V_{\alpha,\lambda}$ *depends polynomially* on λ.

Instead of thinking $V_{\alpha,\lambda}$ as depending polynomially on λ we can think of a single matrix $\tilde{V}_\alpha$ whose entries belong to the polynomial algebra $K[\mathrm{X}]$, that is, a matrix in $K[\mathrm{X}]^{d_y\times d_x}$, if $\alpha\colon x\to y$ and $d\in\mathbb{N}^{Q_0}$ denotes the dimension vector of the family. To recover a single element of the family we only have to specialize $\mathrm{X}\to\lambda$. Thus we can resume: to give a one-parameter family of representations of the quiver Q of A we give a representation in the ring $K[\mathrm{X}]$. Recalling the equivalence between $\operatorname{rep}_I Q$ and $\operatorname{mod} KQ/I$ we see that such a family corresponds to an A-module whose underlying vector space is $\tilde{V} = \bigoplus_{x\in Q_0} K[\mathrm{X}]^{d_x}$ and whose multiplication with A consists of multiplications with matrices in $K[\mathrm{X}]^{m\times m}$, where $m=\sum_{x\in Q_0} d_x$.

Hence $\tilde{V}$ becomes a $K[\mathrm{X}]$-module which is finitely generated free. Since the multiplication of A is compatible with the multiplication of $K[\mathrm{X}]$ we get an A-$K[\mathrm{X}]$-bimodule $\tilde{V}$ which is finitely generated free as right $K[\mathrm{X}]$-module. We have reached the final definition: a **one-parameter family** of A-modules is an A-$K[\mathrm{X}]$-bimodule which is finitely generated free as right $K[\mathrm{X}]$-module.

The recovering of the A-module M_λ from $\tilde{V}$ formally is done by tensoring with $K[\mathrm{X}]/(\mathrm{X}-\lambda)$, that is

$$M_\lambda = \tilde{V} \otimes_{K[\mathrm{X}]} K[\mathrm{X}]/(\mathrm{X}-\lambda).$$

We are now able to give a formal definition of tameness. A finite-dimensional K-algebra is called **tame** if for each dimension vector d there exists a finite family $F_{i=1}^{L_d}$ of one-paremeter families such that the following two properties hold.

(i) Up to isomorphism, there exist only finitely many indecomposable A-modules with dimension vector d which are not isomorphic to $F_{i,\lambda} = F_i \otimes_{K[\mathrm{X}]} K[\mathrm{X}]/(\mathrm{X}-\lambda)$, for some i and some λ.

(ii) Up to finitely many exceptions, the A-modules $F_{i,\lambda}$ are indecomposable and pairwise non-isomorphic.

Example 9.16. Each algebra of finite representation type is tame: just specify the empty family for each dimension vector. Then (ii) is empty and (i) holds since we can allow finitely many exceptions. $\Diamond$

The crucial result concerning the classes of tame and wild algebras is the following.

Theorem 9.17 (Tame-Wild Dichotomy)**.** *Each finite-dimensional algebra over an algebraically closed field is either tame or wild.*

The proof is too involved and we hence have to refer to the literature: The first proof of Theorem 9.17 was given in [9] by Drozd in 1980. Crawley-Boevey rewrote and cleaned it in [6]. In 1993 Gabriel, Nazarova, Roiter, Sergeijuk and Vossieck gave an alternative proof, see [11], improving the result as mentioned above.

Let $A = KQ/I$ be a finite-dimensional K-algebra given by a quiver Q and an admissible ideal I of KQ. We assume that A admits a "two-parameter family of indecomposables", that is, there exists a dimension vector $d \in \mathbb{N}^{Q_0}$ such that in $\mathrm{rep}_I(Q,d)$ there exists a two-parameter family of modules which are (except finitely many exceptions) indecomposable and pairwise non-isomorphic. We observe that then A can not be tame and hence must be wild by the Tame-Wild Dichotomy Theorem 9.17. In that case, for any $t \geq 1$ there exists a dimension vector d such that $\mathrm{rep}_I(Q,d)$ contains a t-parameter family of indecomposable pairwise non-isomorphic A-modules.

Exercises

9.5.1 Let A be a finite-dimensional K-alegbra. Show that the canonical projection functor $\operatorname{mod} A \to \operatorname{mod} A/\operatorname{rad} A$ preserves indecomposablity and reflects isomorphisms.

9.5.2 Show that there exists a functor $F\colon \operatorname{rep} Q^{(2)} \to K[\mathrm{X}, \mathrm{Y}]$ which preserves indecomposablity and reflects isomorphisms. Hint: define the ideal $I = \langle \alpha_1\alpha_2 - \alpha_2\alpha_1 \rangle$ of the infinite dimensional K-algebra $KQ^{(2)}$ and show that $K[\mathrm{X}, \mathrm{Y}] \simeq KQ^{(2)}/I$. For a representation V of $Q^{(2)}$ define the representation FV of $Q^{(2)}$ by setting $(FV)_0 = V_0^3$ and

$$(FV)_{\alpha_\ell} = \begin{bmatrix} 0 & \mathrm{id}_{V_0} & V_{\alpha_\ell} \\ 0 & 0 & \mathrm{id}_{V_0} \\ 0 & 0 & 0 \end{bmatrix}$$

for $\ell = 1, 2$. Show that FV satisfies the ideal I and define F on morphisms to get a functor $F\colon \operatorname{rep} Q^{(t)} \to \operatorname{rep}_I Q^{(t)}$ and then verify that F satisfies the two claimed properties.

9.6 Module varieties

In this section we treat a geometric approach to the question about the dimension vectors of indecomposables and the number of indecomposables of fixed dimension vector. We give here only a quick overview and refer to the expository texts [16] by Kraft and [17] by Kraft and Riedtmann.

Let $A = KQ/I$ be a finite-dimensional algebra and recall that A-modules can be viewed as representations of Q. Moreover, each representation $V = ((V_i)_{i \in Q_0}, (V_\alpha)_{\alpha \in Q_1})$ is isomorphic to a representation of the form $M = \big((K^{d_i})_{i \in Q_0}, (M_\alpha)_{\alpha \in Q_1}\big)$, where for each arrow $\alpha\colon i \to j$ the matrix $M_\alpha \in K^{d_j \times d_i}$ defines a linear map $K^{d_i} \to K^{d_j}$.

For the sake of simplicity, we assume that the vertices of Q are $1, \ldots, n$. If we fix a dimension vector $d \in \mathbb{N}^n$, then each A-module with dimension vector d is isomorphic to a point in the space

$$\operatorname{rep}(Q, d) = \prod\nolimits_{(i \to j) \in Q_1} K^{d_j \times d_i}. \tag{9.8}$$

If $I \neq 0$, then not every choice of matrices $(M_\alpha)_{\alpha \in Q_1} \in \operatorname{rep}(Q, d)$ defines an A-module, but only those which satisfy the ideal I, see Section 3.8.

Example 9.18. Let $Q = \overrightarrow{\mathbb{A}}_3$ and $I = \langle \alpha_2\alpha_1 \rangle$. Furthermore, let $d \in \mathbb{N}^3$ be the dimension vector $d_1 = d_2 = d_3 = 2$. Let $M \in \operatorname{rep}(Q, d)$ be any pair of matrices $R = M_{\alpha_1}, S = M_{\alpha_2} \in K^{2\times 2}$. Then (R,S) defines an A-module if and only if $SR = 0$, equivalently if and only if the following four equations are satisfied.

$$\begin{aligned} S_{11}R_{11} + S_{12}R_{21} &= 0, & S_{11}R_{12} + S_{12}R_{22} &= 0, \\ S_{21}R_{11} + S_{22}R_{21} &= 0, & S_{21}R_{12} + S_{22}R_{22} &= 0. \end{aligned}$$

Note that each of these equation is polynomial. $\Diamond$

In general we denote by $\operatorname{rep}_I(Q,d)$ the subset of $\operatorname{rep}(Q,d)$ given by all families of matrices $M = (M_\alpha)_{\alpha\in Q_1}$ which satisfy the ideal I. The set $\operatorname{rep}_I(Q,d)$ is an **algebraic variety**: it is the set of all solutions of polynomial equations in some affine space K^t. This is why $\operatorname{rep}_I(Q,d)$ is called **module variety**.

Two points $M, N \in \operatorname{rep}(Q,d)$ define isomorphic representations if and only if there exist linear invertible maps $g_i\colon K^{d_i} \to K^{d_i}$, one for each vertex i, such that for any arrow $\alpha : i \to j$ in Q we have $N_\alpha \circ g_i = g_j \circ M_\alpha$. We can state this differently: We define the group

$$\operatorname{GL}(Q,d) = \prod\nolimits_{i\in Q_0} \operatorname{GL}(d_i) \tag{9.9}$$

and the group action of $\operatorname{GL}(Q,d)$ on $\operatorname{rep}(Q,d)$ by

$$g \cdot M = (g_j M_\alpha g_i^{-1})_{(\alpha\colon i\to j)\in Q_1}. \tag{9.10}$$

The **orbit** of $M \in \operatorname{rep}_I(Q,d)$ under the action of the group $\operatorname{GL}(Q,d)$ is denoted by $\mathcal{O}(M)$. Then M and N are isomorphic if and only if they lie in the same orbit under the action of $\operatorname{GL}(Q,d)$.

Example 9.19. Let $A = KQ$, where Q is the three-Kronecker quiver and $d \in \mathbb{N}^2$ given by $d_1 = d_2 = 1$. Then $\operatorname{rep}(Q,d) \simeq K^3$. Two points $M, N \in \operatorname{rep}(Q,d)$ define isomorphic representations if and only if there exists some $\lambda \in K \setminus \{0\}$ such that $M = \lambda N$. Hence the orbits in $\operatorname{rep}(Q,d)/\sim$ under the $\operatorname{GL}(Q,d)$-action are given by $\mathbb{P}^2(K) \cup \{0\}$, that is, the projective space of dimension 2, where each of its point represents an isomorphism class of indecomposable, together with a single point 0 representing the semisimple $S_1 \oplus S_2$. $\Diamond$

We now consider the dimensions of the space $\operatorname{rep}(Q,d)$ and of the group $\operatorname{GL}(Q,d)$ presented in the previous section.

The dimension of $\operatorname{rep}(Q,d)$ is calculated easily:

$$\dim \operatorname{rep}(Q,d) = \sum\nolimits_{(i\to j)\in Q_1} d_i d_j.$$

On the other hand $\mathrm{GL}(Q,d)$ is not a vector space, but an open set in $G = \prod_{i\in Q_0} K^{d_i\times d_i}$ defined by a single polynomial inequality

$$(\det g_1)(\det g_2)\cdots(\det g_n) \neq 0.$$

and therefore $\mathrm{GL}(Q,d)$ has the same dimension than G, that is,

$$\dim \mathrm{GL}(Q,d) = \sum\nolimits_{i\in Q_0} d_i^2.$$

The next result shows that these are exactly the ingredients of the quadratic form q_{KQ}.

Proposition 9.20. *Let Q be a finite quiver without cycle and $A = KQ$. Then for all dimension vectors $d \in \mathbb{N}^{Q_0}$ we have*

$$q_A(d) = \dim \mathrm{GL}(Q,d) - \dim \mathrm{rep}(Q,d) \tag{9.11}$$

Proof. The geometric form q_A was defined in (8.6). Since A is hereditary $\mathrm{Ext}^2_A(S_i,S_j) = 0$ and therefore

$$q_A(d) = \sum\nolimits_{i\in Q_0} d_i^2 - \sum\nolimits_{(i\to j)\in Q_1} d_i d_j.$$

and hence the result follows. □

Equation (9.11) is the first crucial equation, showing that the quadratic form expresses some geometric information. This also justifies the name geometric form for the Tits form. The second crucial equation is stated in Proposition 9.21, which we will not prove. It concerns for a fixed representation $M \in \mathrm{rep}(Q,d)$ the **stabilizer**

$$\mathrm{GL}(Q,d)_M = \{g \in \mathrm{GL}(Q,d) \mid g\cdot M = M\},$$

which is a subgroup of $\mathrm{GL}(Q,d)$.

Proposition 9.21. *Let Q be a finite quiver without cycle, $d \in \mathbb{N}^{Q_0}$ and $M \in \mathrm{rep}(Q,d)$. Then the following equation holds*

$$\dim \mathcal{O}(M) = \dim \mathrm{GL}(Q,d) - \dim \mathrm{GL}(Q,d)_M. \tag{9.12}$$

Note that the equality 9.12 is intuitively clear: The acting group changes M in some directions, whereas in others M remains unchanged, hence the dimension of the orbit is the dimension of the group minus the number of directions in which M remains fixed. We notice that $\mathrm{GL}(Q,d)_M$ always contains the one-dimensional subgroup consisting of the scalar multiples of the identity. Hence $\dim \mathrm{GL}(Q,d)_M \geq 1$.

Remark 9.22. We can use these dimension formulas to give an alternative argument for Gabriel's Thereom 9.5 as exposed in Section 9.2 which is also known as the **Tits argument**. It shows that if $A = KQ$ is of finite representation type, then q_A is positive definite.

Namely, assume that q_A is not positive definite. Then there exists a non-zero vector d such that $q_A(d) \leq 0$. Write $d = d^+ - d^-$ with $d_i^{\pm} \geq 0$ and $d_i^+ d_i^- = 0$ for any i. Then $0 \geq q_A(d) = q_A(d^+) + q_A(d^-) - \sum q_{ij} d_i^+ d_j^-$, where the sum runs over all i, j such that $d_i^+ > 0$ and $d_j^- > 0$. Since $q_{ij} \leq 0$ we conclude that $q_A(d) + \sum q_{ij} d_i^+ d_j^- \leq 0$ and consequently $q_A(d^+) \leq 0$ with $d^+ \neq 0$ or $q_A(d^-) \leq 0$ with $d^- \neq 0$. In any case, there exists a positive vector d such that $q_A(d) \leq 0$.

Hence by (9.11) and (9.12) we have

$$\begin{aligned}\dim \operatorname{rep}(Q,d) &\geq \dim \mathrm{GL}(Q,d) \\ &= \dim \mathcal{O}(M) + \dim \mathrm{GL}(Q,d)_M \\ &> \dim \mathcal{O}(M)\end{aligned}$$

and hence there can be no orbit in $\operatorname{rep}(Q,d)$ with the same dimension than $\operatorname{rep}(Q,d)$ itself and consequently there must exist infinitely many orbits. This shows that A is not of finite representation type.

Note that in this argument it does not matter if these orbits considered in the end correspond to indecomposable modules or not, since if A would be of finite representation type there were only finitely many orbits for any dimension vector. Unfortunately, this argument does not yield part (c) of Kac's Theorem 9.10 as we cannot deduce from the existence of infinitely many modules with dimension vector d that there are infinitely many indecomposables among them. $\lozenge$

Although it is considered "hopeless" to classify all indecomposables of a wild algebra A it still makes sense to try to find some geometric "space" which classifies all A-modules of a fixed dimension vector (or a fixed dimension). This conducts to the notion of **moduli**, as defined by King in [15], see also the overview [20] of Reineke. The notion of moduli attempts to view $\operatorname{rep}(Q,d)/G(Q,d)$ as a geometrical quotient.

Exercises

9.6.1 Let A be the algebra considered in Example 9.18 and $d \in \mathbb{N}^3$ be given by $d_1 = d_2 = d_3 = 1$. Calculate the orbits under the $\mathrm{GL}(Q,d)$-action.

9.6.2 Let $A = KQ/I$, where $Q = \overrightarrow{\mathbb{A}}_n$ and I is the ideal generated by all possible compositions of two consecutive arrows. Let $d \in \mathbb{N}^n$ be given by $d_i = m$ for all i. Calculate the dimension of $\operatorname{rep}(Q, d)$ as vector space. Furthermore determine how many polynomial equations are needed to get $\operatorname{rep}_I(Q, d)$.

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