Non-reduced automorphism schemes

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Abstract

We give a criterion for a group scheme to be not reduced involving infinitesimal multiplicative subgroups. As a consequence the scheme of outer automorphisms of a finite-dimensional algebra A over an algebraically closed field of characteristic p is not reduced if A admits a $\mathbb{Z}_p$-grading which can not be lifted to a $\mathbb{Z}$-grading. This applies in particular when A is the group-algebra of a p-group.

1 Introduction

Let A be a finite-dimensional associative algebra over an algebraically closed field k. We write $\mathfrak{Aut}(A)$ for the affine group scheme of automorphisms of A and $\mathfrak{Inn}(A)$ for the smooth characteristic subgroup of inner automorphisms. Then $\mathfrak{Out}(A) := \mathfrak{Aut}(A) \tilde{/} \mathfrak{Inn}(A)$ is the affine group-scheme [1, III,§3,5.6] of outer automorphisms. The tangent space $T_{\mathfrak{Out}(A),1}$ is naturally isomorphic to the Hochschild cohomology group $H^1(A)$.

[5] proves that the identity component $\mathfrak{Out}^o(A)$ of the outer automorphisms is invariant under derived equivalences, see also [4]. Thus it should be interesting to inquire if $\mathfrak{Out}(A)$ is reduced or not, one of the most basic properties of an algebraic group.

Now, by the theorem of Cartier [1, II,§6,1.1] $\mathfrak{Aut}(A)$ is always reduced if $\mathrm{char}(\mathrm{k}) = 0$, but for example $\mathfrak{Aut}(\mathrm{k}[x]/(x^2))$ is not reduced if $\mathrm{char}(\mathrm{k}) = 2$. On the other hand in case $\mathrm{char}(\mathrm{k}) = p > 0$ it is known that $\mathfrak{Aut}(A)$ is reduced if and only if each derivation $D\colon A \longrightarrow A$ is integrable i.e. if there exist linear maps $D^{(m)}\colon A \longrightarrow A$ with $D^{(0)} = \mathbb{1}, D^{(1)} = D$ such that for all $m \in \mathbb{N}_0$ we have

$$D^{(m)}(ab) = \sum_{i+j=m} D^{(i)}(a) D^{(j)}(b) \quad a, b \in A.$$

See for example [2]. As discussed there, in some cases it is easy to produce a k-basis of integrable derivations. However, to show that a derivation is actually not integrable might involve some tricky calculations, see for example [2, section 3].

We present here a criterion which allows to show for particular algebras A efficiently that $\mathfrak{Aut}(A)$ and thus $\mathfrak{Out}^o(A)$ is not reduced. It is based on the following result about algebraic groups which we will prove in 3.1.

1.1 Theorem *Let $\mathfrak{G}$ be a smooth algebraic group over an algebraically closed field* k *of characteristic $p > 0$. If $\mathfrak{M}$ is an infinitesimal multiplicative subgroup of $\mathfrak{G}$ then there exists a maximal torus $\mathfrak{T}$ of $\mathfrak{G}$ with $\mathfrak{M} \subset \mathfrak{T}$.*

Let $\pi \colon \mathbb{Z} \longrightarrow \mathbb{Z}_p$ be the natural projection. We say that a $\mathbb{Z}_p$-grading $A = \oplus_{d \in \mathbb{Z}_p} A_d$ is *critical* if there is no $\mathbb{Z}$-grading $A = \oplus_{i \in \mathbb{Z}} A'_i$ with $A_d = \oplus_{i \in \pi^{-1}(d)} A'_i$ for all $d \in \mathbb{Z}_p$.

Corollary *Let A be a finite-dimensional algebra over an algebraically closed field* k *of characteristic $p > 0$. Suppose A admits a critical $\mathbb{Z}_p$ grading, then the group schemes $\mathfrak{Aut}(A)$ and $\mathfrak{Out}(A)$ are not reduced.*

The proof will be given in 3.2. Note however, that each k^+-grading of A yields a *diagonalizable* derivation $a \mapsto \sum_{t \in \mathrm{k}^+} t \cdot a_t$. Thus, our corollary says that if $\mathfrak{Aut}A$ is smooth, each diagonalizable derivation is integrable “in a diagonalizable way”, see [2, section 2].

2 Examples

2.1 Let G be a finite p-group, then the commutator subgroup G' is a proper normal subgroup of G, thus we can find a surjective homomorphism of groups $G \longrightarrow \mathbb{Z}_p$. This gives rise to a critical $\mathbb{Z}_p$-grading of the group-algebra $\mathrm{k}G =: A = \oplus_{j \in \mathbb{Z}_p} A_j$. In fact, if we have any $\mathbb{Z}$-grading $A = \oplus_{i \in \mathbb{Z}} A'_i$ of this local finite-dimensional algebra then the elements in A'_i are nilpotent for $i \neq 0$. On the other hand the non-invertible elements in A_j form a linear subspace of codimension ≥ 1 since A is local and A_j contains invertible elements. Thus $\mathfrak{Aut}(\mathrm{k}G)$ is not reduced if $\mathrm{char}(k) = p$, see also [2, 2] for an alternative proof of this result.

2.2 Let A be the following k-algebra which we give by its quiver with relations:

$$e_3 \underset{\alpha_1}{\overset{\alpha_0}{\leftleftarrows}} e_2 \underset{\beta_1}{\overset{\beta_0}{\leftleftarrows}} e_1 \qquad \alpha_0\beta_0 - \alpha_1\beta_1, \alpha_1\beta_0 - \alpha_0\beta_1$$

Then it is easy to see that

$$\begin{aligned} A_0 &= \mathrm{k}e_1 \oplus \mathrm{k}e_2 \oplus \mathrm{k}e_3 \oplus \mathrm{k}\alpha_0 \oplus \mathrm{k}\beta_0 \oplus \mathrm{k}(\alpha_0\beta_0) \\ A_1 &= \mathrm{k}\alpha_1 \oplus \mathrm{k}\beta_1 \oplus \mathrm{k}(\alpha_0\beta_1) \end{aligned}$$

gives a critical $\mathbb{Z}_2$-grading of A. Thus $\mathfrak{Aut}(A)$ is not reduced if $\mathrm{char}(\mathrm{k}) = 2$. We illustrate the phenomenon by a direct calculation of the automorphism group of the corresponding locally bounded category. The affine coordinate ring is given by

$$\mathrm{k}[Y, Y_{0,0}, Y_{0,1}, Y_{1,0}, Y_{1,1}, X, X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}]/I$$

where I is generated by the 6 elements

$$\begin{array}{r} Y(Y_{0,0}Y_{1,1} - Y_{0,1}Y_{1,0}) - 1, \\ X(X_{0,0}X_{1,1} - X_{0,1}X_{1,0}) - 1, \\ \begin{pmatrix} Y_{0,0} & Y_{1,0} & -Y_{0,1} & -Y_{1,1} \\ Y_{1,0} & Y_{0,0} & -Y_{1,1} & -Y_{0,1} \\ Y_{0,1} & Y_{1,1} & -Y_{0,0} & -Y_{1,0} \\ Y_{1,1} & Y_{0,1} & -Y_{1,0} & -Y_{0,0} \end{pmatrix} \cdot \begin{pmatrix} X_{0,0} \\ X_{1,0} \\ X_{0,1} \\ X_{1,1} \end{pmatrix} \end{array}$$

A straightforward calculation shows that the tangent space at the identity is 6-dimensional if $\mathrm{char}(\mathrm{k}) = 2$ while it is 4-dimensional otherwise. Since the determinant of the above matrix is

$$\begin{aligned} &(Y_{0,0} + Y_{1,0} + Y_{0,1} + Y_{1,1}) \cdot (Y_{0,0} + Y_{1,0} - Y_{0,1} - Y_{1,1}) \\ &\qquad \cdot (Y_{0,0} - Y_{1,0} + Y_{0,1} - Y_{1,1}) \cdot (Y_{0,0} - Y_{1,0} - Y_{0,1} + Y_{1,1}) \end{aligned}$$

we get in any characteristic the k-rational points as

$$\begin{aligned} &\{((\begin{smallmatrix} c_1 & c_2 \\ c_2 & c_1 \end{smallmatrix}), (\begin{smallmatrix} a_1 & a_2 \\ a_2 & a_1 \end{smallmatrix})) \text{ with } (c_1 - c_2)(c_1 + c_2) \neq 0 \neq (a_1 - a_2)(a_1 + a_2)\} \\ \cup &\{((\begin{smallmatrix} c_1 & -c_2 \\ c_2 & -c_1 \end{smallmatrix}), (\begin{smallmatrix} a_1 & -a_2 \\ a_2 & -a_1 \end{smallmatrix})) \text{ with } (c_1 - c_2)(c_1 + c_2) \neq 0 \neq (a_1 - a_2)(a_1 + a_2)\}. \end{aligned}$$

We conclude, that the automorphism group is isomorphic to

$$\mathrm{k}^\times \times \mathrm{k}^\times \times \mathrm{k}^\times \times \mathrm{k}^\times \times \mathbb{Z}_2$$

if $\mathrm{char}\,\mathrm{k} \neq 2$, while it is a connected non-reduced 4-dimensional group if $\mathrm{char}\,\mathrm{k} = 2$. Finally we would like to remark that A is wild if $\mathrm{char}(\mathrm{k}) = 2$ and tame otherwise [3].

3 Proofs

3.1 (Proof of theorem) We write $_{\mathfrak{F}^n}\mathfrak{H}$ for the n-th Frobenius kernel of an algebraic group $\mathfrak{H}$ over k. We have $\mathfrak{M} \subset {}_{\mathfrak{F}^n}\mathfrak{G}$ for some $n \in \mathbb{N}$, and take $\mathfrak{M} \subset \mathfrak{M}_n \subset {}_{\mathfrak{F}^n}\mathfrak{G}$ a maximal multiplicative subgroup of $_{\mathfrak{F}^n}\mathfrak{G}$. Next we construct inductively an ascending chain of subgroups

$$\mathfrak{M}_n \subset \mathfrak{M}_{n+1} \subset \cdots \subset \mathfrak{M}_{n+k} \subset \cdots \subset \mathfrak{G}$$

such that $\mathfrak{M}_{n+k}$ is a maximal multiplicative subgroup of $_{\mathfrak{F}^{n+k}}\mathfrak{G}$. Consider now the descending chain of closed subgroups of

$$\mathfrak{G} \supset \mathfrak{G}_0 \supset \cdots \supset \mathfrak{G}_k \supset \cdots$$

given by $\mathfrak{G}_k = \mathfrak{Cent}_{\mathfrak{G}}(\mathfrak{M}_{n+k})$. Clearly, this becomes stationary, i.e. we have $\mathfrak{M}_i \subset \mathfrak{Cent}(\mathfrak{G}_l^o)$ for some $l \in \mathbb{N}$ and all $i \geq n$. Here $\mathfrak{G}_l^o$ denotes the identity component of $\mathfrak{G}_l$.

We claim that $_{\mathfrak{F}^i}\mathfrak{G}_l^o$ is nilpotent with $\mathfrak{M}_i$ as largest multiplicative subgroup. In fact, we have by construction $\mathfrak{M}_i \subset {}_{\mathfrak{F}^i}\mathfrak{G}_l^o$ as maximal multiplicative subgroup since $\mathfrak{M}_i \subset \mathfrak{G}_l^o \subset \mathfrak{G}$, moreover $\mathfrak{M}_i \subset \mathfrak{Cent}({}_{\mathfrak{F}^i}\mathfrak{G}_l^o)$. Now the quotient $_{\mathfrak{F}^i}\mathfrak{G}_l^o/\mathfrak{M}_i$ contains no nontrivial multiplicative subgroup, otherwise $\mathfrak{M}_i$ would not be maximal multiplicative in $_{\mathfrak{F}^i}\mathfrak{G}_l^o$ by [1, IV,§1, 4.5]. Thus $_{\mathfrak{F}^i}\mathfrak{G}_l^o/\mathfrak{M}_i$ is unipotent by [7, 2.62] and [1, IV, §3, 3.6]. We conclude that $_{\mathfrak{F}^i}\mathfrak{G}_l^o$ is nilpotent by [1, IV,§4, 1.2] and $\mathfrak{M}_i$ is its largest multiplicative subgroup [1, IV,§4, 1.11]. (Note that by definition a maximal multiplicative subgroup is not properly contained in any other multiplicative subgroup, while the largest multiplicative subgroup, if it exists, contains all multiplicative subgroups.)

Next we claim that each maximal torus $\mathfrak{T} \subset \mathfrak{G}_l^o$ lies in the center of $\mathfrak{G}_l^o$. In fact, since the Frobenius kernel $_{\mathfrak{F}^i}\mathfrak{T}$ is a multiplicative subgroup of $_{\mathfrak{F}^i}\mathfrak{G}_l^o$ it lies by our first claim in $\mathfrak{M}_i$, thus in the center of $\mathfrak{G}_l^o$. Since $\mathfrak{T}$ is connected our claim follows.

It follows from the construction of $\mathfrak{G}_l^o$ that this is a *smooth* algebraic group by [1, II,§3, 4.3.] and [1, II,§5, 2.8], and we may apply the classical theory of Borel subgroups. Let $\mathfrak{B} \subset \mathfrak{G}_l^o$ be a Borel subgroup that contains a maximal torus $\mathfrak{T} \subset \mathfrak{G}_l^o$. By our second claim $\mathfrak{B} = \mathfrak{U} \times \mathfrak{T}$ for some unipotent group $\mathfrak{U}$. Thus $\mathfrak{B}$ is nilpotent and we conclude $\mathfrak{G}_l^o = \mathfrak{U} \times \mathfrak{T}$, see for example [6, 6.2.10]. As a consequence $\mathfrak{T}$ is the largest multiplicative subgroup of $\mathfrak{G}_l^o$ and our theorem follows.

3.2 (Proof of Corollary) Since k is algebraically closed the algebraic multiplicative groups are by definition of the form $\mathfrak{Sp}(\mathrm{k}H)$ for some finitely generated abelian group H. In particular we are interested in the torus $k^\times = \mathfrak{Sp}(\mathrm{k}\mathbb{Z})$ and its infinitesimal multiplicative subgroups ${}_{p^n}\mu = \mathfrak{Sp}(\mathrm{k}\mathbb{Z}_{p^n})$.

Now it follows from [1, IV,§1, 1.6] that the set of subgroups of $\mathfrak{Aut}(A)$ isomorphic to $\mathfrak{Sp}(\mathrm{k}H)$ corresponds naturally to the set of essential H-gradings on A (we say that an H-grading $A = \oplus_{h\in H} A_h$ is essential if H is generated by $\{h \in H \mid A_h \neq 0\}$). For example, a $\mathbb{Z}$-grading corresponds to the map $\mathrm{k}^\times \to \mathfrak{Aut}(A), t \mapsto (a \mapsto \sum_i a_i t^i)$. We conclude that a critical $\mathbb{Z}_p$ grading gives an infinitesimal multiplicative subgroup of $\mathfrak{Aut}(A)$ that is not contained in a torus.

Our claim for $\mathfrak{Aut}(A)$ follows now directly from the theorem above since k is perfect, and thus $\mathfrak{Aut}(A)$ is reduced if and only if it is smooth [1, II,§5, 2.1]. Now the structure morphism $\mathfrak{Aut}(A) \longrightarrow \mathfrak{Out}(A)$ is faithfully flat [1, III,§3, 2.5] with smooth fibres (isomorphic to $\mathfrak{Inn}(A)$) thus it is smooth by definition, see for example [1, I,§4,4.1]. We conclude that $\mathfrak{Out}(A)$ is smooth if and only if $\mathfrak{Aut}(A)$ is smooth.

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