Gentle algebras are Gorenstein

Ch. Geiß and I. Reiten

1 Introduction

The aim of this note is to show that that gentle algebras are Gorenstein. These are both interesting classes of (finite dimensional) algebras. The first class was introduced in [2] as appropriate context for the investigation of algebras derived equivalent to hereditary algebras of type $\widetilde{\mathbb{A}}_n$. The gentle algebras which are trees are precisely the algebras derived equivalent to hereditary algebras of type $\mathbb{A}_n$, see [1]. It is also known from [12] that the algebras with derived discrete category which are not piecewise hereditary are gentle algebras with one cycle violating the clock condition, see [4] for their derived classification. It is interesting to notice that the class of gentle algebras is closed under derived equivalence [11]. See also [10] for further interesting properties.

On the other hand the concept of a Gorenstein algebra Λ, where by definition Λ has finite injective dimension both as a left and a right Λ-module, is inspired from commutative ring theory. This class of algebras contains both the selfinjective algebras and the algebras of finite global dimension, and were investigated in [3],[7].

In the bounded derived category $D^b(\Lambda)$ consider the subcategories $K^b(\mathcal{P})$ of bounded complexes of projectives and $K^b(\mathcal{I})$ of bounded complexes of injectives. Then Λ is Gorenstein exactly when the last two coincide, and it is easy to see that the property of being Gorenstein is preserved under derived equivalence, see [7]. Moreover the AR-translation $\tau : K^b(\mathcal{P}) \longrightarrow K^b(\mathcal{I})$ shows that we have AR-triangles in $K^b(\mathcal{P})$ in this case, see [6, 1.4].

The property of beeing Gorenstein is also preserved under the skew group ring construction with a finite group whose order is invertible in Λ, see [9],[3]. Thus we may conclude that also the skewed-gentle algebras considered in [5] are Gorenstein, at least if the field is not of characteristic 2. Skewed gentle algebras form a class of derived tame algebras which is not closed under derived equivalence.

2 Definitions and setup

2.1 (Gentle algebras) Let k be a field, and note that we compose arrows in a quiver Q from left to right. We call a finite dimensional k-algebra Λ *special biserial* if it is Morita equivalent to an algebra $\mathrm{k}Q/I$ where Q is a quiver and $I \subset \mathrm{k}Q$ an admissible ideal subject to the following conditions:

(1) at each vertex of Q at most 2 arrows start;

(1') at each vertex of Q at most 2 arrows stop;

(2) for each arrow $\beta \in Q_1$ there is at most one arrow $\gamma \in Q_1$ with $\beta\gamma$ a path not contained in I;

(2') for each arrow $\beta \in Q_1$ there is at most one arrow $\alpha \in Q_1$ with $\alpha\beta$ a path not contained in I;

If moreover the following conditions are fulfilled we call A *gentle*

(3) I is generated by paths of length 2;

(4) for each arrow $\beta \in Q_1$ there is at most one arrow $\gamma' \in Q_1$ with $\beta\gamma'$ a path contained in I;

(4') for each arrow $\beta \in Q_1$ there is at most one arrow $\alpha' \in Q_1$ with $\alpha'\beta$ a path contained in I;

It is known [8] (see also [10]) that Λ is gentle if and only if the trivial extension $T(\Lambda)$ (or equivalently the repetitive algebra $\widehat{\Lambda}$) is special biserial.

2.2 (σ and τ) It will be convenient to give a description of gentle algebras in terms of the existence of two functions $\sigma, \tau \colon Q_1 \longrightarrow \{-1, +1\}$. Denote by s and t the functions from Q_1 to Q_0 which assign to each arrow α its start and end point respectively, so that we have $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. Then we have the following:

Lemma *Let $\Lambda = \mathrm{k}Q/I$, where I is generated by paths of length 2. Then the following are equivalent:*

(i) Λ is gentle.

(ii) There exist two functions $\sigma, \tau : Q_0 \longrightarrow \{-1, +1\}$ with the following properties:

(a) If $s(\alpha) = s(\beta)$ and $\sigma(\alpha) = \sigma(\beta)$ then $\alpha = \beta$.

(a') If $t(\alpha) = t(\beta)$ and $\tau(\alpha) = \tau(\beta)$ then $\alpha = \beta$.

(b) If $t(\alpha) = \sigma(\beta)$, then $\tau(\alpha) = \sigma(\beta)$ if and only if $\alpha\beta \in I$.

Proof: (i)$\Longrightarrow$ (ii). We may concentrate on one vertex $q \in Q_1$. Since at most two arrows start at q we may define $\sigma(\beta_1) = 1$ if β_1 is one such arrow, and $\sigma(\beta_2) = -1$ if there is an other one, β_2. Next define τ for the arrows ending at q. If no arrow starts at q we may do this similar as for σ. Otherwise let β be an arrow starting at q. Then we set

$$\tau(\alpha) = \begin{cases} \sigma(\beta) & \text{if } \alpha\beta \in I \\ -\sigma(\beta) & \text{otherwise} \end{cases}$$

for each arrow α ending at q. It follows easily from the definition of gentle that this is well defined and that the pair (σ, τ) has the desired properties. The other implication follows directly from the definitions. $\square$.

2.3 (Words) In order to state our findings efficiently we need some more notation. We introduce formally a set of inverse arrows $Q_1^{-1} = \{\alpha^{-1} \mid \alpha \in Q_1\}$ and extend our definitions by: $s(\alpha^{-1}) = t(\alpha)$, $\sigma(\alpha^{-1}) = \tau(\alpha)$, $t(\alpha^{-1}) = s(\alpha)$ and $\tau(\alpha^{-1}) = \sigma(\alpha)$.

A *walk* of length n in Q is a sequence $\mathfrak{w} = w_1 \cdots w_n$ of elements $w_i \in Q_1 \cup Q_1^{-1}$ with $t(w_i) = s(w_{i+1})$ for $1 \le i < n$. The *inverse* walk $\mathfrak{w}^{-1}$ is $w_n^{-1} \cdots w_1^{-1}$ where we formally set $(\alpha^{-1})^{-1} = \alpha$. We call a walk *direct* resp. *inverse* if all its elements are from Q_1 resp. Q_1^{-1}. We extend the functions s, σ, t, τ in the obvious way to walks.

A *word* is a walk $\mathfrak{w} = w_1 \cdots w_n$ with $\tau(w_i) = -\sigma(w_{i+1})$ for $1 \le i < n$. We need moreover a *trivial* word 1_q and its formal inverse 1_q^{-1} for each vertex q of Q, with $s(1_q^\varepsilon) = q = t(1_q^\varepsilon)$ and $\sigma(1_q^\varepsilon) = \varepsilon = -\tau(1_q^\varepsilon)$ for $\varepsilon \in \{-1, +1\}$. We agree to consider 1_q^ε as both a direct and inverse word of length 0.

If $\mathfrak{v}$ and $\mathfrak{w}$ are words with $t(\mathfrak{v}) = s(\mathfrak{v})$ and $\tau(\mathfrak{v}) = -\sigma(\mathfrak{w})$ then the concatenation $\mathfrak{v}\mathfrak{w}$ is again a word, and if $\mathfrak{w}1_q^\varepsilon$ is a word we write $\mathfrak{w}1_q^\varepsilon = \mathfrak{w}$.

In order to describe the indecomposable injective and projective modules we need the following words: For $(q, \varepsilon) \in Q_0 \times \{-1, +1\}$ denote by $\mathfrak{i}(1_q^\varepsilon)$ the unique direct word $\mathfrak{w}$ of maximal length with $\mathfrak{w} = \mathfrak{w}1_q^\varepsilon$. Similarly we define $\mathfrak{p}(1_q^\varepsilon)$ to be the unique direct word $\mathfrak{v}$ of maximal length with $1_q^\varepsilon\mathfrak{v} = \mathfrak{v}$. Some of these words may be trivial. We finally write $\mathfrak{i}_q = \mathfrak{i}(1_q^{-1})\mathfrak{i}(1_q)^{-1}$ and $\mathfrak{p}_q = \mathfrak{p}(1_q)^{-1}\mathfrak{p}(1_q^{-1})$.

3 Main results

In order to prove our results it will be useful to obtain a convenient description of syzygies of injective modules. First we assign to each word $\mathfrak{w}$ a right $\mathrm{k}Q/I$-module $M(\mathfrak{w})$ in the obvious way. In particular for each vertex $q \in Q_0$ the module $M(\mathfrak{i}_q)$ resp. $M(\mathfrak{p}_q)$ is the associated indecomposable injective resp. projective.

3.1 We call an arrow β *gentle* if there exists no direct walk $\alpha\beta$ with $\tau(\alpha) = \sigma(\beta)$, moreover we call a direct walk $\mathfrak{w} = \alpha_1 \cdots \alpha_n$ *critical* if $\tau(\alpha_i) = \sigma(\alpha_{i+1})$ for $1 \leq i < n$. For such a critical walk there exists at most one arrow α_0 such that $\alpha_0\alpha_1 \cdots \alpha_n$ is critical, and there is at most one arrow α_{n+1} such that $\alpha_1 \cdots \alpha_n\alpha_{n+1}$ is critical.

Lemma *For a gentle algebra $\Lambda = \mathrm{k}Q/I$ there exists a bound $n(\Lambda) \leq |Q_1|$ for the maximal length of critical walks starting with a gentle arrow.*

Proof: Assume $\alpha_1 \cdots \alpha_{n+1}$ is critical with $\alpha_1, \ldots, \alpha_n$ pairwise different and $\alpha_i = \alpha_{n+1}$ for some $i \leq n$. Since α_1 is gentle we have $i \neq 1$. Then we find $\alpha_{i-1}\alpha_i \in I$ and $\alpha_n\alpha_i \in I$, a contradiction. □

Remark: If Λ has no gentle arrows at all, we set $n(\Lambda) = 0$.

3.2 For a word $\mathfrak{w}$ it is convenient to consider the sets $L(\mathfrak{w}) = \{\alpha \in Q_1 \mid \alpha^{-1}\mathfrak{w} \text{ a word}\}$ and $R(\mathfrak{w}) = \{\beta \in Q_1 \mid \mathfrak{w}\beta \text{ a word}\}$, each of them having at most one element.

Lemma *For a gentle algebra $\Lambda = \mathrm{k}Q/I$ the arrows in $L(\mathfrak{i}_q) \cup R(\mathfrak{i}_q)$ are gentle for any vertex q.*

Proof: Assume $L(\mathfrak{i}_q) = \{\beta\}$. If β is not gentle let α be the arrow with $t(\alpha) = s(\beta)$ and $\tau(\alpha) = \sigma(\beta)$. But then $\alpha\mathfrak{i}(1_q^{-1})$ would be a direct word, a contradiction. □

3.3 For an arrow $\alpha = 1_q^{\varepsilon}\alpha$ we denote by $\mathfrak{p}(\alpha)$ the unique direct word with $\mathfrak{p}(1_q^{\varepsilon}) = \alpha\mathfrak{p}(\alpha)$.

Proposition *Let $\Lambda = \mathrm{k}Q/I$ be a gentle algebra, and $M(\mathfrak{i}_q)$ an indecomposable injective module. For $j \geq 1$, each indecomposable non-projective summand of $\Omega^j M(\mathfrak{i}_q)$ is of the form $M(\mathfrak{p}(\alpha_j))$ for a critical path $\alpha_1\alpha_2 \cdots \alpha_j$ with α_1 gentle.*

Proof: It is easy to see that

$$\Omega^1(M(\mathfrak{i}_q)) \cong \oplus_{\alpha\in L(\mathfrak{i}_q)\cup R(\mathfrak{i}_q)} M(\mathfrak{p}(\alpha)) \oplus P,$$

where $P = 0$ if $M(\mathfrak{i}_q)$ is uniserial and $P = M(\mathfrak{p}_q)$ otherwise. Since by the above lemma α_1 is gentle, this proves our claim for $j = 1$. Suppose now that our claim is true for $j \geq 1$. $M(\mathfrak{p}(\alpha_j))$ is projective if and only the critical path $\alpha_1 \cdots \alpha_j$ was right maximal, and there is nothing to show. Otherwise we find α_{j+1} such that $\alpha_1 \cdots \alpha_j\alpha_{j+1}$ is critical, and it is easy to see that $\Omega^1 M(\mathfrak{p}(\alpha_j)) \cong M(\mathfrak{p}(\alpha_{j+1}))$. □

3.4 Theorem *Let $\Lambda = \mathrm{k}Q/I$ be a gentle algebra with $n(\Lambda)$ the maximum length of critical walks starting with a gentle arrow. Then $\mathrm{id}_\Lambda \Lambda = n(\Lambda) = \mathrm{pd}_\Lambda D(\Lambda^{op})$ if $n(\Lambda) > 0$ and $\mathrm{id}_\Lambda \Lambda = \mathrm{pd}_\Lambda D(\Lambda^{op}) \leq 1$ if $n(\Lambda) = 0$. In particular Λ is Gorenstein.*

Proof: $\mathrm{pd}_\Lambda D(\Lambda^{\mathrm{op}}) \leq n(\Lambda) + \delta_{0,n(\Lambda)}$ follows directly from the above proposition.

Assume $n(\Lambda) > 0$ and let $\alpha_1 \cdots \alpha_n$ be a critical path with α_1 gentle. If there is an arrow $\beta \neq \alpha_1$ with $s(\beta) = s(\alpha_1)$ it is easy to see that $\mathrm{pd}\, M(\mathfrak{i}_{t(\beta)}) \geq n$. Otherwise $\mathrm{pd}\, M(\mathfrak{i}_{s(\alpha_1)}) \geq n$.

On the other hand Λ^{op} is also gentle and it is not hard to see that $n(\Lambda^{\mathrm{op}}) = n(\Lambda)$. Now $\mathrm{id}_\Lambda \Lambda = \mathrm{pd}_{\Lambda^{\mathrm{op}}} D(\Lambda) = n(\Lambda^{\mathrm{op}})$. □

The above theorem can not be extended to special biserial algebras.

3.5 Example Let Q be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and $I = \langle \alpha\beta\alpha \rangle$. Then $\mathrm{k}Q/I$ is special biserial but not Gorenstein.

References

[1] I. Assem and D. Happel, *Generalized tilted algebras of type A_n*, Comm. Algebra **9** (1981), no. 20, 2101–2125.

[2] I. Assem and A. Skowroński, *Iterated tilted algebras of type $\tilde{\mathbf{A}}_n$*, Math. Z. **195** (1987), no. 2, 269–290.

[3] M. Auslander and I. Reiten, *Cohen-Macaulay and Gorenstein Artin algebras*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991) (Basel), Birkhäuser, Basel, 1991, pp. 221–245.

[4] G. Bobiński, Ch. Geiß, and A. Skowroński, *Classification of discrete derived categories*, Preprint (2001), www.matem.unam.mx/~christof/preprints/Discrete.ps.

[5] Ch. Geiß and J.A. de la Peña, *Auslander-Reiten components for clans*, Bol. Soc. Mexicana (3) **5** (1999), 307–326.

[6] D. Happel, *Auslander-Reiten triangles in derived categories of finite-dimensional algebras*, Proc. Amer. Math. Soc. **112** (1991), no. 3, 641–648.

[7] ———, *On Gorenstein algebras*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), Birkhäuser, Basel, 1991, pp. 389–404.

[8] Z. Pogorzały and A. Skowroński, *Self-injective biserial standard algebras*, J. Algebra **138** (1991), no. 2, 491–504.

[9] I. Reiten and Ch. Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra **92** (1985), no. 1, 224–282.

[10] C.M. Ringel, *The repetitive algebra of a gentle algebra*, Bol. Soc. Mat. Mexicana (3) **3** (1997), no. 2, 235–253.

[11] J. Schröer and A. Zimmermann, *Stable endomorphism algebras of modules over special biserial algebras*, Preprint (2001), www.maths.leeds.ac.uk/~jschroer/preprints/dergen.ps.

[12] D. Vossieck, *The algebras with discrete derived category*, J. Algebra **243** (2001), no. 1, 168–176.

Instituto de Matemáticas, UNAM, 04510 México DF, MEXICO

e-mail: `christof@math.unam.mx`
`www.matem.unam.mx/~christof`

Institutt for Matemtiske fag, NTNU, 7491 Trondheim, NORWAY

e-mail: `idunr@math.ntnu.no`
`www.math.ntnu.no/~idunr`