ON COMPONENTS OF TYPE $\mathbb{Z}\mathbb{A}_\infty^\infty$ FOR STRING ALGEBRAS

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1. Introduction

Let $\Lambda = \mathrm{k}Q/\langle P\rangle$ be a string algebra, see 3.1. From [BuR] we have an quite mechanical recipe for the calculation of Auslander-Reiten sequences, and moreover we know from there that the components of the Auslander-Reiten quiver Γ_Λ of Λ containing string modules are, besides a finite number of exceptions, of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$. The exceptional components are easy to find, since they contain particular Auslander-Reiten sequences parameterized by the arrows of the quiver Q of Λ.

In section 2 we obtain by rather simple combinatoric considerations, that each $\mathbb{Z}\mathbb{A}_\infty^\infty$-Component of a string algebra contains a unique module of minimal dimension.

In section 3 we review the relevant definitions and results from [BuR]. By the way we find that the set of strings has a natural structure of a poset, where edges in the Hasse diagram correspond to irreducible maps between string-modules; this observation should be useful for related problems like clans.

In section 4 we provide more precise information: The structure of $\mathbb{Z}\mathbb{A}_\infty^\infty$-components is determined by certain infinite sections in Γ_Λ passing through simple modules (Proposition 3), moreover these sections turn out to be periodic in some sense (Proposition 2). Thus we obtain also a practical method to determine all the strings which correspond to modules of minimal dimension in a $\mathbb{Z}\mathbb{A}_\infty^\infty$-component.

In section 5 we study a particular case in detail to illustrate the quite technical arguments of the foregoing sections.

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2. Words

Let $\mathcal{L}$ be a set of "letters" and $\mathcal{L}^{-1} := \{l^{-1} \mid l \in \mathcal{L}\}$ the set of formal inverse letters . We denote then by $\mathcal{W}'$ the set of finite sequences in $\mathcal{L} \cup \mathcal{L}^{-1}$ – the "words" and $\mathcal{W} := \mathcal{W}' \cup \{1_{\mathcal{W}}\}$. The concatenation "$\cdot$" gives $\mathcal{W}$ the structure of a semi-group with neutral element $1_{\mathcal{W}}$. For $W = w_1 w_2 \cdots w_n \in \mathcal{W}'$ we define $l(W) := n$ the *length* of W and $l(1_{\mathcal{W}}) := 0$. If $W = W' \cdot V \in \mathcal{W}$ we define $W \setminus V := W'$.

Finally, we say that a sequence $(W[i])_{i\in\mathbb{Z}}$ in $\mathcal{W}$ *has property*

(S) if for every $i \in \mathbb{Z}$ we have either $W[i+1] = W[i] \cdot H_i$ for some $H_i = h_i \cdot H_i'$ with $h_i \in \mathcal{L}$, or else $W[i+1] := W[i] \setminus C_i$ for some $C_i = c_i \cdot C_i'$ with $c_i \in \mathcal{L}^{-1}$.

Lemma 1. *Let* $(W[i])_{i\in\mathbb{Z}}$ *be a sequence in* $\mathcal{W}$ *with property* (S)*, then there exists* $i_0 \in \mathbb{Z}$ *such that* $l(W[i_0]) < l(W[i])$ *for all* $i \in \mathbb{Z} \setminus \{i_0\}$.

Proof. Without loss of generality we can assume $l(W[0]) \leq l(W[i])$ for all $i \in \mathbb{Z}$; then $W[1] = W[0] \cdot H_0$ with $H_0 = h_1 \cdots h_m \in \mathcal{H}$. We show by induction:

$$W[i] = W[0] \cdot h_1 \cdot S_i \text{ with } S_i \in \mathcal{W} \text{ for all } i \in \mathbb{N}$$

For $\underline{i = 1}$ nothing is to show. For the step $\underline{i \Longrightarrow i+1}$ we have two cases:
– If $W[i+1] = W[i] \cdot H_i = W[0] \cdot h_1 \cdot S_i \cdot H_i$ obviously $S_{i+1} = S_i \cdot H_i$.
– Else, $W[i+1] = W[i] \setminus C_i = (W[0] \cdot h_1 \cdot S_i) \setminus C_i$. In this case $l(W[i+1]) \geq l(W[i])$ implies $l(h_1 \cdot S_i) \geq l(C_i)$; moreover $h_1 \in \mathcal{L}$ while the first letter of C_i is in $\mathcal{L}^{-1}$, thus $S_{i+1} = S_i \setminus C_i$ is defined.

Dually we show $l(C[-i]) > l(C[0])$ for all $i \in \mathbb{N}$. □

Proposition 1. *Let* $(W[i,j])_{i,j\in\mathbb{Z}}$ *be a (double) sequence in* $\mathcal{W}$ *with the following three properties:*

(S1) *The sequence* $(W[i_0,j])_{j\in\mathbb{Z}}$ *has property* (S)*for all* $i_0 \in \mathbb{Z}$.
(S2) *The sequence* $((W[i,j_0])^{-1})_{i\in\mathbb{Z}}$ *has property* (S)*for all* $j_0 \in \mathbb{Z}$.
(E) $l(W[i,j]) + l(W[i+1,j+1]) = l(W[i+1,j]) + l(W[i,j+1])$ *for all* $i,j \in \mathbb{Z}$.

Then there exists $(i_0,j_0) \in \mathbb{Z} \times \mathbb{Z}$ *such that* $l(W[i_0,j_0]) < l(W[i,j])$ *for all* $(i,j) \in \mathbb{Z} \setminus \{(i_0,j_0)\}$.

Note: (S1), (S2) and (E) are fulfilled, if $(W[i,j])_{i,j\in\mathbb{Z}}$ is a family of strings parameterizing the indecomposable modules in a $\mathbb{Z}\mathbb{A}_\infty^\infty$-component of Γ_Λ, with Λ a string algebra, see section 3.

Proof. We can suppose $l(W[0,0]) \leq l(W[i,j])$ for all $i,j \in \mathbb{Z}$. By the Lemma we obtain then, using (S1) and (S2):
(1) $l(W[0,0]) < l(W(0,j))$ for all $j \in \mathbb{Z} \setminus \{0\}$ and
(2) $l(W[0,0]) < l(W(i,0))$ for all $i \in \mathbb{Z} \setminus \{0\}$.
Moreover we have for $(i,j) \neq (0,0)$:

(1) $$l(W[i,j]) + l(W[0,0]) \overset{(E)}{=} l(W[i,0]) + l(W[0,j]) \overset{(1),(2)}{>} 2l(W[0,0])$$ □

3. String algebras

In this section we repeat for convenience several important definitions and results concerning string algebras and give some additional setup. This review is based on [BuR], see however the references in [BuR] for previous work on string algebras.

3.1. Let k be a field. Let $Q = (Q_0, Q_1, s, e)$ be a *finite* quiver, i.e. Q_0 is the set of vertices, Q_1 is the set of arrows and the functions $s, e\colon Q_1 \to Q_0$ determine the start resp. endpoint of an arrow. If P is a set of paths in Q, the (monomial) algebra $\mathrm{k}Q/\langle P\rangle$ is called a *string algebra* in case the following three conditions hold:

(1) For all $p \in Q_0$ we have $|\{\alpha \in Q_1 \mid s(\alpha) = p\}| \leq 2$ and $|\{\alpha \in Q_1 \mid e(\alpha) = p\}| \leq 2$
(2) For all $\beta \in Q_1$ we have $|\{\alpha \in Q_1 \mid s(\alpha) = e(\beta) \text{ and } \alpha\beta \notin P\}| \leq 1$ and $|\{\gamma \in Q_1 \mid e(\gamma) = s(\beta) \text{ and } \beta\gamma \notin P\}| \leq 1$
(3) $\langle P\rangle$ is an admissible ideal of $\mathrm{k}Q$.

3.2. For every arrow $\alpha \in Q_1$ we introduce a formal inverse arrow α^{-1} with $s(\alpha^{-1}) = e(\alpha)$, $e(\alpha^{-1}) = s(\alpha)$ and $(\alpha^{-1})^{-1} = \alpha$; write $Q_1^{-1} := \{\alpha^{-1} \mid \alpha \in Q_1\}$. A *string* is a finite sequence $C = c_1c_2\cdots c_n$ with $c_i \in Q_1 \cup Q_1^{-1}$ for $i = 1, 2, \dots, n$ and $s(c_i) = e(c_{i+1})$ for $i = 1, \dots, n-1$ and such that C and $C^{-1} := c_n^{-1}c_{n-1}^{-1}\cdots c_1^{-1}$ do not contain a sub-path from P or of the form $\alpha^{-1}\alpha$. Moreover we introduce for every vertex $p \in Q_0$ *two* trivial strings $1_{(p,t)}$ for $t \in \{-1, 1\}$ with $1_{(p,t)}^{-1} := 1_{(p,-t)}$. The set of all strings will be denoted by $\mathcal{S}$ and we extend the functions s, e in the obvious way to $\mathcal{S}$. By definition a string $C = c_1\cdots c_n$ has *length* $l(C) := n$ and $l(1_{(p,t)}) := 0$.

Next, recall that we can define two functions $\sigma, \varepsilon\colon Q_1 \cup Q_1^{-1} \to \{-1, 1\}$ such, that for all $c_1, c_2 \in Q_1 \cup Q_1^{-1}$ with $s(c_1) = e(c_2)$ we have $c_1c_2 \in \mathcal{S}$ iff $\sigma(c_1) = -\varepsilon(c_2)$. We extend these functions to $\mathcal{S}$ in the following way: If $C = c_1\cdots c_n$ is a non trivial string, $\sigma(C) := \sigma(c_n)$ and $\varepsilon(C) := \varepsilon(c_1)$; moreover $\sigma(1_{(p,t)}) := -t$ and $\varepsilon(1_{(p,t)}) := t$. Finally, for $(p, t) \in Q_0 \times \{-1, 1\}$ let $\mathcal{S}_{(p,t)} := \{C \in \mathcal{S} \mid e(C) = p, \varepsilon(C) = t\}$.

For two non-trivial strings $C = c_1\cdots c_n$ and $D = d_1\cdots d_m$ we say, that $C \cdot D$ is *defined* if the concatenation $c_1\cdots c_nd_1\cdots d_m$ is a string. In this case $C \cdot D := c_1\cdots c_nd_1\cdots d_m$, and we have $s(C) = e(D)$, $\sigma(C) = -\varepsilon(D)$. Similarly $1_{(p,t)} \cdot C$ is *defined* iff $e(C) = p$ and $-t = \sigma(1_{(p,t)}) = -\varepsilon(C)$, in this case $1_{(p,t)} \cdot C := C$. Also $C \cdot 1_{(q,u)}$ is defined iff $s(C) = q$ and $\sigma(C) = -\varepsilon(1_{(q,u)}) = -u$, in this case $C \cdot 1_{(q,u)} := C$.

3.3. Let $U_{(p,t)}$ be the unique string in $\mathcal{S}_{(p,t)}$ of maximal length which consists only of direct arrows; similarly $V_{(p,t)}$ is the unique string in $\mathcal{S}_{(p,t)}$ of maximal length which consists only of inverse arrows (recall, that $\mathrm{k}Q/\langle P\rangle$ is finite dimensional). For $C \in \mathcal{S}_{(p,t)} \setminus \{U_{(p,t)}\}$ we define

$$C[1] := \begin{cases} C \cdot \alpha \cdot V_{(s(\alpha),-\sigma(\alpha))} & \text{if } \exists \alpha \in Q_1 : \ C \cdot \alpha \text{ defined} \\ C \setminus \beta^{-1} \cdot U_{(s(\beta^{-1}),-\sigma(\beta^{-1}))} & \text{else} \end{cases}$$

Dually we define $C[-1]$ for $C \in \mathcal{S}_{(p,u)} \setminus \{V_{(p,u)}\}$; finally we define inductively $C[i]$ for all $i \in \mathbb{Z}$ where this makes sense, else we say that $C[i]$ is not defined. Finally, for technical reasons we also introduce the notations $[i]C := (C^{-1}[1])^{-1}$ where this makes sense, and $I_p := U^{-1}_{(p,-1)} \cdot U_{(p,1)}$.

The set $\mathcal{S}_{(p,u)}$ is linearly ordered by

$$V < W \iff \begin{cases} \text{either} & W = V \cdot W' \text{ with } w'_1 \in Q_1 \\ \text{or} & V = W \cdot V' \text{ with } v'_1 \in Q_1^{-1} \\ \text{or} & V = CV',\ W = CW' \text{ with } w'_1 \in Q_1, v'_1 \in Q_1^{-1} \end{cases}$$

We leave the proof of the following as an exercise.

Lemma 2. *With the above notations we have:*

$$V_{(p,u)} = \min \mathcal{S}_{(p,u)} \qquad C[1] = \min\{C' \in \mathcal{S}_{(p,u)} \mid C' > C\}$$
$$U_{(p,u)} = \max \mathcal{S}_{(p,u)} \qquad C[-1] = \max\{C' \in \mathcal{S}_{(p,u)} \mid C' < C\}$$

Remarks . (1) In the notation of [BuR] we have $C[1] = C_h$ if C starts not on a peak, otherwise $C = C[1]_c$.

(2) By the above lemma it is clear that that $C[i] = C[j] \iff i = j$; by Theorem 2 below we can interpret this as a special case of [BaS].

3.4. For $C \in \mathcal{S}$ we can define naturally a Λ-module $M(C)$, see [BuR] for details but note that $M(C) \cong M(C^{-1})$, and for $C_1 < C_2$ in $\mathcal{S}_{(p,u)}$ there is a canonical morphism $M(C_1) \to M(C_2)$; moreover observe that $M(I_p)$ is an injective indecomposable module.

Define on $\mathcal{S}$ an equivalence relation $\sim$ by $C \sim D \iff D \in \{C, C^{-1}\}$ and let $\mathcal{S}'$ be a set of representatives of the corresponding equivalence classes.

We also need the set $\mathcal{B}$ of *bands*, by definition

$$\mathcal{B} := \{C \in \mathcal{S} \mid C^n \in \mathcal{S} \text{ for all } n \in \mathbb{N} \text{ and } C \neq B^m \text{ for } B \in \mathcal{S}, m \geq 2\}$$

For each $C \in \mathcal{B}$ we have a Λ-$\mathrm{k}[T, T^{-1}]$ bimodule $N(C)$; loosely speaking $N(C) = M(\cdots C \cdot C \cdot C \cdots)$. For $C = c_1 c_2 \cdots c_{l(C)} \in \mathcal{B}$ the *rotation* $C_{[r]} := c_{r+1} c_{r+2} \cdots c_{l(C)} c_1 \cdots c_r$ is also a band. Define on $\mathcal{B}$ an equivalence relation ρ by $C \rho D \iff C^t_{[r]} = D$ for some $r \in \{0, 1, \ldots, l(C)-1\}, t \in \{-1, 1\}$, and let $\mathcal{B}'$ be a set of representatives of the corresponding equivalence classes.

Finally, let Ψ be a set of representatives of the isomorphism classes of indecomposable $\mathrm{k}[T, T^{-1}]$-modules of finite length.

Theorem 1 ([BuR])**.** *Let $\Lambda = \mathrm{k}[Q]/\langle P \rangle$ be a string algebra and $\mathcal{S}', \mathcal{B}', \Psi$ the corresponding sets as above. Then the modules $M(C)$, $C \in \mathcal{S}'$ and $N(B) \otimes_{\mathrm{k}[T,T^{-1}]} S$, $B \in \mathcal{B}'$, $S \in \Psi$ form a (irredundant) set of representatives of the isoclasses of indecomposable Λ-modules of finite length.*

Theorem 2 ([BuR]). *Under the same hypothesis as above we have:*

(a) *The functors* $N(B)\otimes_{k[T,T^{-1}]}- : k[T,T^{-1}]\text{-mod} \longrightarrow \Lambda\text{-mod}$ *preserve Auslander-Reiten sequences.*

(b) *If for* $(p,t) \in Q_0 \times \{-1,1\}$ *the string* $U_{(p,t)} \neq I_p^t$ *(i.e.* $M(U_{(p,t)})$ *is not injective), then the canonical sequence*

$$0 \longrightarrow M(U_{(p,t)}) \longrightarrow M([1]U_{(p,t)}) \longrightarrow M(([1]U_{(p,t)})[1]) \longrightarrow 0$$

is an Auslander-Reiten sequence, otherwise

$$M(U_{(p,t)}) \longrightarrow M([1]U_{(p,t)}) \longrightarrow 0$$

is a source map.

(c) *If for* $C \in \mathcal{S} \setminus \{U^u_{(p,t)} \mid t,u \in \{-1,1\}, p \in Q_0\}$ *we have* $C \not\sim I_p$ *for all* $p \in Q_0$ *(i.e.* $M(C)$ *is not injective), then the canonical sequence*

$$0 \longrightarrow M(C) \longrightarrow M(C[1]) \oplus M([1]C) \longrightarrow M([1](C[1])) \longrightarrow 0$$

is an Auslander-Reiten sequence and $[1](C[1]) = ([1]C)[1]$*, otherwise*

$$0 \longrightarrow M(C) \longrightarrow M(C[1]) \oplus M([1]C)$$

is a source map.

Remark . If in situation (b) of the theorem $U_{(p,t)} \neq I_p$, there exists a unique $\alpha \in Q_1$ with $\alpha^{-1} \cdot U_{(p,t)}$ defined. In this case $([1]U_{(p,t)})[1] = V^{-1}_{(s(\alpha),-\sigma(\alpha))}$.

4. $\mathbb{Z}\mathbb{A}_\infty^\infty$-components

A (non-trivial) consequence of the above theorem also shown in [BuR] is, that the stable components of Γ_Λ containing string modules are of the form $\mathbb{Z}\mathbb{A}^\infty/\langle\tau^n\rangle$ or $\mathbb{Z}\mathbb{A}_\infty^\infty$. The (finitely many) components of the first type are easy to determine since they contain AR-sequences of type (b) in the theorem above. Here we describe the other type of stable components. The proofs of the corresponding Propositions 2 and 3, based on the lemmas in 4.1, are given at the end of the section.

Definition. Let $\Lambda = kQ/\langle P\rangle$ be a string algebra with $\varepsilon, \sigma\colon \mathcal{S} \longrightarrow \{-1,1\}$ as in 3.2.

(1) For a string $C \in \mathcal{S}$ let $R^+(C)$ be the maximal (left) substring of $U_{(s(C),-\sigma(C))}$ such that $C \cdot R^+(C)$ is defined. Similarly, let $L^+(C)$ be the maximal (right) substring of $V^{-1}_{(e(C),-\varepsilon(C))}$ such that $L^+(C)\cdot C$ is defined.

(2) We say, that $C = c_1c_2\cdots c_n$ is *critical*, if for any $1 \le i < j \le n$ we have, that $c_i = c_j \in Q_1$ implies $l(R^+(c_1\cdots c_i)) > l(R^+(c_1\cdots c_j))$.

(3) $\mathcal{S}^+_{(p,t)} := \{C \in \mathcal{S}_{(p,t)} \mid C \text{ is critical}\}$, this is a finite (!) set which contains $U_{(p,t)}$.

Proposition 2. *Let* $\Lambda = kQ/\langle P\rangle$ *be a string algebra with* $\varepsilon, \sigma\colon \mathcal{S} \longrightarrow \{-1,1\}$ *as in* 3.2*. Let* $(p,u) \in Q_0 \times \{-1,1\}$*. Then we are in exactly one of the following situations.*

(I) $1_{(p,u)}[i] \notin \mathcal{S}^+_{(p,u)}$ *for some* $i \in \mathbb{N}$. *In this case* $1_{(p,u)}[i]$ *is defined for all* $i \in \mathbb{N}$ *and there are* $P^+_{(p,u)}, Q^+_{(p,u)} \in \mathbb{N}_0$ *with* $M^+_{(p,u)} := P^+_{(p,u)} + Q^+_{(p,u)} \leq |\mathcal{S}^+_{(p,u)}|$ *such that*

$$1_{(p,u)}[Q^+_{(p,u)} + nP^+_{(p,u)} + r] = 1_{(p,u)}[Q^+_{(p,u)}] \cdot (1_{(p',t')}[P^+_{(p,u)}])^n \cdot 1_{(p',u')}[r]$$

for all $n, r \in \mathbb{N}_0$ *(with* $p' = s(1_{(p,u)}[Q^+_{(p,u)}])$, $u' = -\sigma(1_{(p,u)}[Q^+_{(p,u)}])$*). In this case for* $C \in \mathcal{S}$ *the composition* $C \cdot 1_{(p,u)}[i]$ *is defined for all* $i \in \mathbb{N}$ *iff* $C \cdot 1_{(p,u)}[M^+_{(p,u)}]$ *is defined.*

(F) $1_{(p,u)}[N^+_{(p,u)}] = U_{(p,u)}$ *for some* $0 \leq N^+_{(p,u)} \leq |\mathcal{S}^+_{(p,u)}|$. *In this case* $1_{(p,u)}[i] \in \mathcal{S}^+_{(p,u)}$ *for all* $i \in \mathbb{N}$ *where* $1_{(p,u)}[i]$ *is defined.*

There is an obvious "negative version" of this proposition dealing with the strings $1_{(p,u)}[-i]$, for $i \in \mathbb{N}$.

Proposition 3. *Under the same hypothesis as Proposition 2 let* $C \in \mathcal{S}$ *be a string. The following two conditions are equivalent:*

(i) $M(C)$ *is a module of minimal dimension in a component of type* $\mathbb{Z}\mathbb{A}^\infty_\infty$ *in the AR-quiver* Γ_Λ *of* Λ.

(ii) *For all* $i, j \in \mathbb{Z}$ *the expression*

$$C(i,j) := [i]1_{(e(C),\varepsilon(C))} \cdot C \cdot 1_{(s(C),-\sigma(C))}[j]$$

is defined.

In this case the modules $M([i](C[j])) = M(C(i,j))$ *for* $i, j \in \mathbb{Z}$ *are just the vertices of the component of* $M(C)$*, and moreover* $M(C)$ *is of strictly minimal dimension in that component.*

Remarks . (1) If $M(C)$ lies in a $\mathbb{Z}\mathbb{A}^\infty_\infty$-component of Γ_Λ the string $([i]C)[j] = [i](C[j])$ is defined for all $i, j \in \mathbb{Z}$, but this does not mean automatically that $C(i,j)$ is always defined.

(2) Let $\mathcal{C} := \{C \in \mathcal{S} \mid C(i,j) \text{ defined for all } i,j \in \mathbb{Z}\}$. By Proposition 2 we get for C a string: $C \in \mathcal{C}$ iff $C(M^-_{(e(C),-\varepsilon(C))}, M^+_{(s(C),-\sigma(C))})$ and $C(M^+_{(e(C),-\varepsilon(C))}, M^-_{(s(C),-\sigma(C))})$ are defined; thus condition (ii) above is easy to check.

(3) Of course, every $\mathbb{Z}\mathbb{A}^\infty_\infty$-component of Γ_Λ contains some module of minimal dimension; this module is by the proposition unique.

Corollary. *The set* $\mathcal{C}/\sim$ *parameterizes the* $\mathbb{Z}\mathbb{A}^\infty_\infty$*-components of* Γ_Λ*; the elements of* $\mathcal{C}/\sim$ *describe the (unique) modules of minimal dimension for each* $\mathbb{Z}\mathbb{A}^\infty_\infty$*-component.*

4.1. For the proof of the propositions we will need the following lemmas:

Lemma 3. *Let* $C \in \mathcal{S}$ *be a string and* $n \in \mathbb{N}_0$. *Then the following are equivalent:*

(i) $l(C[j]) \geq l(C)$ *for* $j = 0, 1, \ldots, n$.

(ii) $C \cdot 1_{(s(C),-\sigma(C))}[j]$ *is defined for* $j = 0, 1, \ldots, n$.
(iii) $C[j] = C \cdot 1_{(s(C),-\sigma(C))}[j]$ *for* $j = 0, 1, \ldots, n$.

For the proof of the non-obvious implications (i)$\Longrightarrow$(iii) and (ii)$\Longrightarrow$(iii) one easily adapts the argument of Lemma 1 (section 2).

Lemma 4. *Suppose* $1_{(p,t)}[n] = C \cdot \beta \cdot D$ *with* C, D *strings and* $\beta \in Q_1$*, then:*
(a) $C = 1_{(p,t)}[k]$ *for some natural number* $k < n$.
(b) $C \cdot 1_{(s(C),-\sigma(C))}[i]$ *is defined for* $i = 1, 2, \ldots, n-k$.

Proof. a) We have only to observe that $1_{(p,t)} < C < C \cdot \beta \cdot D = 1_{(p,t)}[n]$, and that by Lemma 2 the interval $[1_{(p,t)}, 1_{(p,t)}[n]]$ consists just of the elements $\{1_{(p,t)}[i] \mid i = 0, 1, \ldots n\}$.

b) By lemma 3 we would get otherwise $C[i_0]$ a (left) substring of C for some $i_0 \in \{2, 3, \ldots, n-k-1\}$, but this is impossible because $C < C[i_0] < C[n-k] = C \cdot \beta \cdot D$.

□

4.2. Proof of Proposition 2. If there is no $t \in \mathbb{N}$ with $1_{(p,u)}[t] \notin \mathcal{S}^+_{(p,u)}$ then, by remark (2) in 3.3 and the fact that $\mathcal{S}^+_{(p,u)}$ is a finite set, there must be some natural number $N^+_{(p,u)} \leq |\mathcal{S}^+_{(p,u)}|$ with $1_{(p,u)}[N^+_{(p,u)}] = U_{(p,u)}$.

Otherwise, let $t := \min\{i \in \mathbb{N}_0 \mid 1_{(p,u)}[i+1] \notin \mathcal{S}^+_{(p,u)}\}$. This means that $1_{(p,u)}[t+1]$ can be written as

$$1_{(p,u)}[t+1] = \underbrace{c_1 \cdots c_{j-1} \alpha c_{j+1} \cdots c_n}_{1_{(p,u)}[t]} \cdot \alpha \cdot V_{(s(\alpha),-\sigma(\alpha))}$$

with $l(R^+(c_1 \cdots c_{j-1}\alpha)) \leq l(R^+(c_1 \cdots c_n \alpha))$ $(*)$. By Lemma 4 we obtain for some $Q \in \mathbb{N}$:

$$1_{(p,u)}[Q+i] = c_1 \cdots c_{j-1} \cdot 1_{(s(c_{j-1}),-\sigma(c_{j-1}))}[i] \text{ for } i = 0, 1, \ldots t-Q := P.$$

As a consequence $1_{(p,u)}[Q+P] \cdot 1_{(s(c_{j-1}),-\sigma(c_{j-1}))}[i] = 1_{(p,u)}[Q+P+i]$ for $i = 0, 1, \ldots, P$, here the left hand side is defined by $(*)$ and thus is equal to the right side by Lemma 3. Finally observe, that $l(L^+(1_{(p,u)}[i]))$ is a monotonous decreasing function $\mathbb{N}_0 \to \mathbb{N}_0$. By the above it reaches its minimum at $Q+P$.

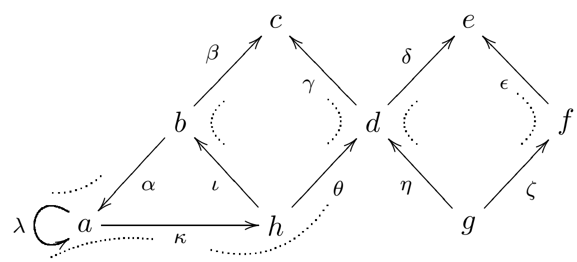
4.3. Proof of Proposition 3. a) $\Longrightarrow$ b) By Lemma 3 and its dual $C(0,k)$ and $C(k,0)$ are defined for all $k \in \mathbb{Z}$. If $C(i_0, j_0)$ is not defined for some $(i_0, j_0) \in \mathbb{Z} \times \mathbb{Z}$, then we may suppose that C consists only of direct arrows and that $i_0 < 0 < j_0$. Moreover we can suppose that $C(i,j)$ is defined for all $j \in \mathbb{Z}$ if $i > i_0$, and that $C(i_0, j)$ is defined for $j < j_0$. This means, that the module $M(C(i_0, j_0 - 1))$ is injective (!), but on the other hand it lies in the component of $M(C)$ - a contradiction.

b) $\Longrightarrow$ a) Applying twice Lemma 3 we get $C(i,j) = ([j]C)[i] = [j](C[i])$ for all $i, j \in \mathbb{Z}$, therefore (see Theorem 2) we have irreducible maps $M(C(i,j)) \to M(C(i,j{+}1))$ and $M(C(i,j)) \to M(C(i{+}1,j))$ for all $i, j \in \mathbb{Z}$.

Thus, by Theorem 2 the $M(C(i,j))$ are exactly the vertices of the component of Γ_Λ containing $M(C)$, and moreover this is a stable component, containing *no* Auslander-Reiten sequence with indecomposable middle term. Following [BuR, p. 175] we conclude that this component is of type $\mathbb{Z}\mathbb{A}_\infty^\infty$ containing $M(C)$ as a module of strictly minimal dimension.

5. An example

Consider the following quiver Q, with relations P' indicated by dotted lines and additional relations $P'' = \{\lambda^{m+1}, (\kappa\alpha\iota)^n\kappa\}$; thus $\Lambda_{m,n} := \mathrm{k}Q/\langle P' \cup P''\rangle$ is a string algebra for all $m, n \in \mathbb{N}$.



We want to determine the $\mathbb{Z}\mathbb{A}_\infty^\infty$ components of the Auslander-Reiten quiver of $\Lambda_{m,n}$.

First we determine the functions σ and ε by giving their values on Q_1.

	α	β	γ	δ	ϵ	ζ	η	θ	ι	κ	λ
σ	1	-1	1	-1	1	1	-1	-1	1	-1	1
ε	1	-1	-1	-1	1	1	-1	1	-1	-1	-1

Next we have to find the pairs $(p, u) \in Q_0 \times \{-1, 1\}$ where $1_{(p,u)}[i]$ is defined for all $i \in \mathbb{Z}$. The following simple calculations rule out most cases:

p/u	-1	1
a	$*$	$*$
b	$1_{(b,-1)}[-6] = \beta^{-1} = V_{(b,-1)}$	$1_{(b,1)} = U_{(b,1)}$
c	$1_{(c,-1)} = V_{(c,-1)}$	$1_{(c,1)} = V_{(c,1)}$
d	$1_{(d,-1)} = \eta = U_{(d,-1)}$	$*$
e	$1_{(e,-1)} = V_{(e,-1)}$	$1_{(e,1)} = V_{(e,1)}$
f	$1_{(f,-1)} = V_{(f,-1)} = U_{(f,-1)}$	$*$
g	$1_{(g,-1)} = U_{(g,-1)}$	$1_{(g,1)} = U_{(g,1)}$
h	$1_{(h,-1)} = U_{(h,-1)}$	$1_{(h,1)}[-5] = \theta^{-1}\delta^{-1} = V_{(h,1)}$

In the cases marked by $*$ the string $1_{(p,u)}[i]$ is defined for all $i \in \mathbb{Z}$. There we find:

i	$1_{(d,1)}[i]$	$1_{(f,1)}[i]$
-5	$\gamma^{-1}\beta\alpha^{-1}\lambda^m\kappa^{-1}\iota^{-1}\alpha^{-1}\lambda^m$	$\epsilon^{-1}\delta\theta\iota^{-1}\alpha^{-1}\lambda^m\kappa^{-1}\iota^{-1}$
-4	$\gamma^{-1}\beta\alpha^{-1}\lambda^m\kappa^{-1}\iota^{-1}$	$\epsilon^{-1}\delta\theta\iota^{-1}\alpha^{-1}\lambda^m\kappa^{-1}$
-3	$\gamma^{-1}\beta\alpha^{-1}\lambda^m\kappa^{-1}$	$\epsilon^{-1}\delta\theta\iota^{-1}\alpha^{-1}\lambda^m$
-2	$\gamma^{-1}\beta\alpha^{-1}\lambda^m$	$\epsilon^{-1}\delta\theta\iota^{-1}$
-1	$\gamma^{-1}\beta$	$\epsilon^{-1}\delta\theta$
0	$1_{(d,1)}$	$1_{(f,1)}$
1	$\theta(\iota^{-1}\alpha^{-1}\kappa^{-1})^n\iota^{-1}\alpha^{-1}$	$\zeta\eta^{-1}\gamma^{-1}$
2	$\theta(\iota^{-1}\alpha^{-1}\kappa^{-1})^n\iota^{-1}\alpha^{-1}$ $\cdot\lambda(\iota^{-1}\alpha^{-1}\kappa^{-1})^n$	$\zeta\eta^{-1}\gamma^{-1}\beta(\alpha^{-1}\kappa^{-1}\iota^{-1})\alpha^{-1}$
3	$\theta(\iota^{-1}\alpha^{-1}\kappa^{-1})^n\iota^{-1}\alpha^{-1}$ $\cdot\lambda(\iota^{-1}\alpha^{-1}\kappa^{-1})^n\lambda(\iota^{-1}\alpha^{-1}\kappa^{-1})^n$	$\zeta\eta^{-1}\gamma^{-1}\beta(\alpha^{-1}\kappa^{-1}\iota^{-1})\alpha^{-1}$ $\cdot\lambda(\kappa^{-1}\iota^{-1}\alpha^{-1})^n$

But, $C \cdot 1_{(f,1)}[i]$ defined for $i \in \{-1,1\}$ implies $C = 1_{(f,1)}$, and $C \cdot 1_{(d,1)}[i]$ defined for $i \in \{-1,1\}$ implies $C \in \{\eta, \eta\zeta^{-1}, \delta^{-1}, \delta^{-1}\epsilon\}$, so remain:

i	$1_{(a,1)}[i]$	$1_{(a,-1)}[i]$
-4		$\kappa^{-1}\iota^{-1}\alpha^{-1}\lambda^m\kappa^{-1}$
-3		$\kappa^{-1}\iota^{-1}\alpha^{-1}\lambda^m$
-2	$(\lambda^{-1}(\alpha\iota\kappa)^n)^2$	$\kappa^{-1}\iota^{-1}$
-1	$\lambda^{-1}(\alpha\iota\kappa)^n$	κ^{-1}
0	$1_{(a,1)}$	$1_{(a,-1)}$
1	$\alpha\beta^{-1}$	$\lambda(\kappa^{-1}\iota^{-1}\alpha^{-1})^n$
2	$\alpha\beta^{-1}\gamma\delta^{-1}$	$(\lambda(\kappa^{-1}\iota^{-1}\alpha^{-1})^n)^2$
3	$\alpha\beta^{-1}\gamma\delta^{-1}\epsilon$	
4	$\alpha\beta^{-1}\gamma$	
5	$\alpha\beta^{-1}\gamma\eta\zeta^{-1}$	
6	$\alpha\beta^{-1}\gamma\eta$	
7	α	
8	$\alpha\iota\theta^{-1}\delta^{-1}$	
9	$\alpha\iota\theta^{-1}\delta^{-1}\epsilon$	
10	$\alpha\iota\theta^{-1}$	
11	$\alpha\iota\theta^{-1}\eta\zeta^{-1}$	
12	$\alpha\iota\theta^{-1}\eta$	
13	$\alpha\iota$	
14	$\alpha\iota\kappa\lambda^{-m}$	
15	$\alpha\iota\kappa\lambda^{-m}\alpha\beta^{-1}$	

Now it is not hard to see, that up to equivalence for a string C the expressions $C(i,j)$ can be defined for all $i,j \in \mathbb{Z}$ only if $s(C) = a = e(C)$ and $\sigma(C) = -1, \varepsilon(C) = 1$, thus $C(i,j) = [i]1_{(a,1)} \cdot C \cdot 1_{(a,1)}[j] = (1_{(a,-1)}[i])^{-1} \cdot C \cdot 1_{(a,1)}[j]$. By Proposition 2 we see, that such $C(i,j)$ is defined for all $i,j \in \mathbb{Z}$ iff it is defined for $(i = -3, j = 14)$ and for $(i = 1, j = -1)$. By the corollary of Proposition 2 these strings parameterize the $\mathbb{Z}\mathbb{A}_\infty^\infty$-components of $\Lambda_{m,n}$. Consequently $\Lambda_{m,n}$ has $\mathbb{Z}\mathbb{A}_\infty^\infty$ components if $(m,n) \in \mathbb{N} \times \mathbb{N} \setminus \{(1,1)\}$ and $M(1_{(a,1)})$ lies in a $\mathbb{Z}\mathbb{A}_\infty^\infty$ component if $m,n \geq 2$. Indeed, if say $m \geq 2$, the expressions $(\lambda^{-1}\alpha\iota\kappa\lambda^{-1})(1,-1)$ and $(\lambda^{-1}\alpha\iota\kappa\lambda^{-1})(-3,14)$ are defined, while $1_{(a,1)}(1,-1)$ is defined if $m \geq 2$ and $1_{(a,1)}(-3,14)$ is defined if $n \geq 2$.

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