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Classification of discrete derived categories

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Abstract: The main aim of the paper is to classify the discrete derived categories of bounded complexes of modules over finite dimensional algebras.

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Introduction and main results

Throughout the paper K denotes a fixed algebraically closed field. By an algebra we mean a connected finite dimensional K-algebra (associative, with an identity) and by a module a finite dimensional right module.

For an algebra A, we denote by $\operatorname{mod} A$ the category of A-modules and by $D^b(\operatorname{mod} A)$ the derived category of bounded complexes of A-modules. By an equivalence of two derived categories we mean an equivalence of triangulated categories [10]. Recall from [6, 12] that an A-module T is called a tilting (respectively, cotilting) module provided $\operatorname{Ext}_A^2(T,-) = 0$ (respectively, $\operatorname{Ext}_A^2(-,T) = 0$), $\operatorname{Ext}_A^1(T,T) = 0$ and the number of pairwise nonisomorphic indecomposable direct summands of T equals the rank of the Grothendieck group $K_0(A)$ of A. Two algebras A and B are called tilting-cotilting equivalent if there exists a sequence of algebras $A = A_0, A_1, \ldots, A_m, A_{m+1} = B$ and a sequence of modules $T_{A_i}^{(i)}$ $(0 \leq i \leq m)$ such that $A_{i+1} = \operatorname{End} T_{A_i}^{(i)}$ and $T_{A_i}^{(i)}$ is either a tilting or a

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cotilting A_i-module [3]. It is well-known that if two algebras A and B are tilting-cotilting equivalent then the derived categories $D^b(\operatorname{mod} A)$ and $D^b(\operatorname{mod} B)$ are equivalent [10].

Following [21] a derived category $D^b(\operatorname{mod} A)$ is said to be discrete if for every vector $\mathbf{n} = (n_i)_{i\in\mathbb{Z}}$ of natural numbers there are only finitely many isomorphism classes of indecomposable objects in $D^b(\operatorname{mod} A)$ of homology dimension vector $\mathbf{n}$. An important class of discrete derived categories is formed by the derived categories $D^b(\operatorname{mod} K\Delta)$ of the path algebras $K\Delta$ of Dynkin quivers Δ (of types $\mathbb{A}_m$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$), called derived categories of Dynkin type. It is known that a derived category $D^b(\operatorname{mod} A)$ is equivalent to $D^b(\operatorname{mod} K\Delta)$, for some Dynkin quiver Δ, if and only if A is tilting-cotilting equivalent to $K\Delta$. In particular, for two Dynkin quivers Δ and Δ', the derived categories $D^b(\operatorname{mod} K\Delta)$ and $D^b(\operatorname{mod} K\Delta')$ are equivalent if and only if Δ and Δ' have the same underlying graph. Recently D. Vossieck proved in [21] that the derived category $D^b(\operatorname{mod} A)$ of an algebra A is discrete but not of Dynkin type if and only if A is Morita equivalent to the bound quiver algebra of a gentle bound quiver (in the sense of [2]) having exactly one cycle with different numbers of clockwise and counterclockwise oriented relations. However, the classification of such derived categories has been an open problem.

Denote by Ω the set of all triples (r, n, m) of integers such that $n \geq r \geq 1$ and $m \geq 0$. For each $(r, n, m) \in \Omega$ consider the quiver $Q(r, n, m)$ of the form

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-r-2}} & n-r-1 & & \\
 & & & & {}^{\alpha_0}\nearrow & & & & \searrow^{\alpha_{n-r-1}} & \\
(-m) \xrightarrow{\alpha_{-m}} \cdots \xrightarrow{\alpha_{-2}} (-1) & \xrightarrow{\alpha_{-1}} & 0 & & & & & & & n-r \\
 & & & & {}_{\alpha_{n-1}}\nwarrow & & & & \swarrow_{\alpha_{n-r}} & \\
 & & & & & n-1 & \xleftarrow[\alpha_{n-2}]{} \cdots \xleftarrow[\alpha_{n-r+1}]{} & n-r+1 & &
\end{array}
$$

the ideal $I(r, n, m)$ in the path algebra $KQ(r, n, m)$ of $Q(r, n, m)$ generated by the paths $\alpha_{n-1}\alpha_0$, $\alpha_{n-2}\alpha_{n-1}$, $\ldots$, $\alpha_{n-r}\alpha_{n-r+1}$, and put $\Lambda(r, n, m) = KQ(r, n, m)/I(r, n, m)$. Our first main result is the following.

Theorem A. Let A be a connected algebra and assume that $D^b(\operatorname{mod} A)$ is not of Dynkin type. The following conditions are equivalent:

(i) $D^b(\operatorname{mod} A)$ is discrete.

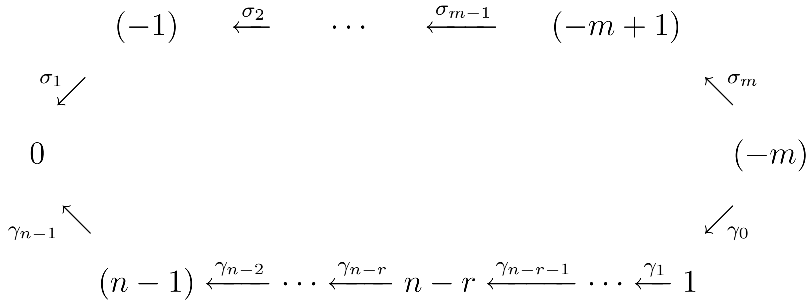
(ii) $D^b(\operatorname{mod} A) \simeq D^b(\operatorname{mod} \Lambda(r, n, m))$, for some $(r, n, m) \in \Omega$.

(iii) A is tilting-cotilting equivalent to $\Lambda(r, n, m)$, for some $(r, n, m) \in \Omega$.

Moreover, for $(r, n, m), (r', n', m') \in \Omega$, $D^b(\operatorname{mod} \Lambda(r, n, m)) \simeq D^b(\operatorname{mod} \Lambda(r', n', m'))$ if and only if $(r, n, m) = (r', n', m')$.

Let $\Omega_f = \{(r, n, m) \in \Omega; n > r\}$. We note that $(r, n, m) \in \Omega_f$ if and only if

$\Lambda(r,n,m)$ is of finite global dimension. We prove in Section 2 that, for each $(r,n,m) \in \Omega_f$, the algebra $\Lambda(r,n,m)$ is tilting-cotilting equivalent to the bound quiver algebra $A(r,n,m) = K\Delta(r,n,m)/J(r,n,m)$, where the quiver $\Delta(r,n,m)$ is of the form



and $J(r,n,m)$ is the ideal in $K\Delta(r,n,m)$ generated by the paths $\gamma_{n-2}\gamma_{n-1}$, $\gamma_{n-3}\gamma_{n-2}$, $\dots$, $\gamma_{n-r-1}\gamma_{n-r}$.

The second aim of the paper is to describe the structure of discrete derived categories which are not of Dynkin type. For $(r,n,m) \in \Omega$, we denote by $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$ the (Gabriel) quiver of the category of indecomposable objects in $D^b(\operatorname{mod}\Lambda(r,n,m))$, that is, the quiver whose vertices are the isomorphism classes of indecomposable objects in $D^b(\operatorname{mod}\Lambda(r,n,m))$ and arrows are given by the irreducible morphisms. We have the additional structure of a translation quiver in $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$ induced by Auslander–Reiten triangles [10, 11], hence $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$ is just the Auslander–Reiten (translation) quiver of $D^b(\operatorname{mod}\Lambda(r,n,m))$. The quiver $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$ is stable if and only if $(r,n,m) \in \Omega_f$. The following theorem describes the structure of the quivers $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$.

Theorem B. (i) For $(r,n,m) \in \Omega_f$, the quiver $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$ has exactly $3r$ components, namely $2r$ components $\mathcal{X}^{(0)}, \dots, \mathcal{X}^{(r-1)}, \mathcal{Y}^{(0)}, \dots, \mathcal{Y}^{(r-1)}$ of type $\mathbb{Z}\mathbb{A}_\infty$, and r components $\mathcal{Z}^{(0)}, \dots, \mathcal{Z}^{(r-1)}$ of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. For each $X \in \mathcal{X}^{(i)}$ we have $\tau^{m+r}X = X[-r]$ and for each $Y \in \mathcal{Y}^{(i)}$ we have $\tau^{n-r}Y = Y[r]$.

(ii) For $(r,n,m) \in \Omega \setminus \Omega_f$, the quiver $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$ consists of precisely $2r$ components, namely r components $\mathcal{X}^{(0)}, \dots, \mathcal{X}^{(r-1)}$ of type $\mathbb{Z}\mathbb{A}_\infty$ and r components $\mathcal{L}^{(0)}, \dots, \mathcal{L}^{(r-1)}$ which are equioriented lines of type $\mathbb{A}_\infty^\infty$. For each $X \in \mathcal{X}^{(i)}$ we have $\tau^{m+r}X = X[-r]$, while the vertices of $\mathcal{L}^{(i)}$ are projective-injective in $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$.

Recall that $n = r$ for $(r,n,m) \in \Omega \setminus \Omega_f$. Theorem B implies in particular that $\Lambda(r,n,m)$ and $\Lambda(r',n',m')$ are derived equivalent if and only if $(r,n,m) = (r',n',m')$. In contrast, the structure of the translation quiver $\Gamma(D^b(\operatorname{mod}\Lambda(r,n,m)))$ reveals only the invariant r.

For $(r,n,m) \in \Omega_f$, we have the Euler integral quadratic form $\chi_{\Lambda(r,n,m)}$ and the (non-symmetric) bilinear homological form $\langle -,- \rangle_{\Lambda(r,n,m)}$ defined on $K_0(D^b(\operatorname{mod}\Lambda(r,n,m))) \simeq K_0(\Lambda(r,n,m)) \simeq \mathbb{Z}^{n+m}$. We have the following.

Theorem C. (i) Let (r,n,m), $(r',n',m') \in \Omega_f$. The bilinear forms $\langle -,-\rangle_{\Lambda(r,n,m)}$ and $\langle -,-\rangle_{\Lambda(r',n',m')}$ are $\mathbb{Z}$-equivalent if and only if $r \equiv r' \pmod 2$ and $\{m+r, n-r\} = \{m'+r', n'-r'\}$. Moreover, if r is even then $\langle -,-\rangle_{\Lambda(r,n,m)}$ is $\mathbb{Z}$-equivalent to the bilinear form of a hereditary algebra of Euclidean type $\tilde{\mathbb{A}}_{m+r,n-r}$.

(ii) Let $(r,n,m) \in \Omega_f$. If r is odd then the Euler form $\chi_{\Lambda(r,n,m)}$ is positive definite of Dynkin type $\mathbb{D}_{n+m}$. If r is even then $\chi_{\Lambda(r,n,m)}$ is positive semi-definite of Dynkin type $\mathbb{A}_{n+m-1}$ and corank 1.

1 Preliminaries

1.1. Let R be a locally bounded category over K [7]. We denote by $\operatorname{mod} R$ the category of all finite dimensional contravariant functors from R to the category of K-vector spaces. If R is bounded (the number of objects in R is finite), then $\operatorname{mod} R$ is equivalent to the category $\operatorname{mod} A$ of finite dimensional right modules over the algebra $A = \bigoplus R$ formed by the quadratic matrices $a = (a_{yx})_{x,y\in R}$ such that $a_{yx} \in R(x,y)$. Conversely, to each basic algebra A we can attach the bounded category R with $A \simeq \bigoplus R$ whose objects are formed by a complete set E of orthogonal primitive idempotents e of A, $R(e,f) = fAe$ and the composition is induced by the multiplication in A. We shall identify a bounded category R with its associated basic algebra $\bigoplus R$. Recall also that every locally bounded category R is the bound quiver category KQ/I, where $Q = Q_R$ is the (locally finite) quiver of R and I is an admissible ideal in the path category KQ of Q. In particular, every finite dimensional K-algebra Λ is Morita equivalent to a bound quiver algebra KQ_Λ/I. For a locally bounded category $R = KQ/I$ and a vertex i of Q, we shall denote by e_i the corresponding primitive idempotent of R, by $S_R(i)$ the corresponding simple R-module, and by $P_R(i)$ (respectively, $I_R(i)$) the projective cover (respectively, injective envelope) of $S_R(i)$ in $\operatorname{mod} R$. Following [19] a locally bounded category R is said to be special biserial if $R \simeq KQ/I$, where the bound quiver (Q,I) satisfies the following conditions:

(1) The number of arrows in Q with a prescribed source or target is at most 2.

(2) For any arrow α of Q there are at most one arrow β and at most one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ are not in I.

1.2. For a locally bounded category R we shall denote by $\Gamma(\operatorname{mod} R)$ the Auslander–Reiten quiver of $\operatorname{mod} R$ and by τ_R and τ_R^- the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We shall identify the vertices of $\Gamma(\operatorname{mod} R)$ with the corresponding indecomposable R-modules. By a component of $\Gamma(\operatorname{mod} R)$ we mean a connected component of $\Gamma(\operatorname{mod} R)$.

1.3. For an algebra Λ we denote by $D^b(\operatorname{mod}\Lambda)$ the bounded derived category of the abelian category of finite dimensional Λ-modules. It has the structure a triangulated category in the sense of Verdier [20]. The corresponding translation functor $D^b(\operatorname{mod}\Lambda) \to D^b(\operatorname{mod}\Lambda)$ assigns to each complex X in $D^b(\operatorname{mod}\Lambda)$ its shift $X[1]$. Accordingly, the distinguished triangles in $D^b(\operatorname{mod}\Lambda)$ are of the form $X \to Y \to Z \to X[1]$. We shall often identify a module from $\operatorname{mod}\Lambda$ with the corresponding complex in $D^b(\operatorname{mod}\Lambda)$ concentrated in degree zero. The homology dimension vector of a complex X from $D^b(\operatorname{mod}\Lambda)$ is the vector $\mathbf{h\text{-}dim}\, X = (\dim_K H^i(X))_{i\in\mathbb{Z}}$, where $H^i(X)$ is the i-th homology space of X. Following [21] the derived category $D^b(\operatorname{mod}\Lambda)$ is said to be discrete provided for every vector $\mathbf{n} = (n_i)_{i\in\mathbb{Z}}$ of natural numbers there are only finitely many isomorphism classes of indecomposable complexes in $D^b(\operatorname{mod}\Lambda)$ of homology dimension vector $\mathbf{n}$. Recall also that by a result due to J. Rickard [16] two derived categories $D^b(\operatorname{mod}A)$ and $D^b(\operatorname{mod}B)$ are equivalent (as triangulated categories) if and only if $A = \operatorname{End}_{D^b(\operatorname{mod}B)}(T)$ for a tilting complex T in $D^b(\operatorname{mod}B)$, that is, a perfect (consisting of finite dimensional projective modules) complex T with $\operatorname{Hom}_{D^b(\operatorname{mod}B)}(T, T[i]) = 0$ for all $i \neq 0$ such that the additive category $\operatorname{add}T$ of T generates $D^b(\operatorname{mod}B)$ as a triangulated category.

1.4. The repetitive category [13] of a bounded category (algebra) Λ is the selfinjective locally bounded category $\hat{\Lambda}$ whose objects are formed by the pairs $(n, x) = x_n$, $x \in \Lambda$, $n \in \mathbb{Z}$, and $\hat{\Lambda}(x_n, y_n) = \{n\} \times \Lambda(x, y)$, $\hat{\Lambda}(x_{n+1}, y_n) = \{n\} \times D\Lambda(y, x)$, and $\hat{\Lambda}(x_p, y_q) = 0$ if $p \neq q, q+1$, where DV denotes the dual space $\operatorname{Hom}_K(V, K)$. The repetitive category $\hat{\Lambda}$ was introduced as a Galois covering of the trivial extension $T(\Lambda) = \Lambda \ltimes D\Lambda$ of Λ by its injective cogenerator $D\Lambda$. Then the category $\operatorname{mod}\hat{\Lambda}$ of finite dimensional right $\hat{\Lambda}$-modules can be regarded as the category of finite dimensional $\mathbb{Z}$-graded modules over $T(\Lambda)$. We view every module M in $\operatorname{mod}\hat{\Lambda}$ as a family $M = (M_n)_{n\in\mathbb{Z}}$ of modules from $\operatorname{mod}\Lambda$ such that $M(x_n) = M_n(x)$ for each $x \in \Lambda$ and $n \in \mathbb{Z}$. The stable module category $\underline{\operatorname{mod}}\,\hat{\Lambda}$ is a triangulated category where the suspension functor Ω^- serves as the translation functor $\underline{\operatorname{mod}}\,\hat{\Lambda} \to \underline{\operatorname{mod}}\,\hat{\Lambda}$, and hence the distinguished triangles in $\underline{\operatorname{mod}}\,\hat{\Lambda}$ are of the form $X \to Y \to Z \to \Omega^- X$. We will usually denote $\Omega^- X$ by $X[1]$. The Auslander–Reiten translation in $\underline{\operatorname{mod}}\,\hat{\Lambda}$ is of the form $\tau = \nu\Omega^2$, where ν is the Nakayama translation induced by the canonical shift $x_n \mapsto x_{n+1}$, $x \in R$, $n \in \mathbb{Z}$, in $\hat{\Lambda}$ (see [10] for details). We have the canonical inclusion $\operatorname{mod}\Lambda \to \underline{\operatorname{mod}}\,\hat{\Lambda}$ which sends a Λ-module X into a $\hat{\Lambda}$-module $M = (M_n)$ concentrated at degree 0 (that is, $M_0 = X$ and $M_n = 0$, $n \neq 0$).

An essential role in our investigations will be played by the Happel functor

$$F\colon D^b(\operatorname{mod}\Lambda) \to \underline{\operatorname{mod}}\,\hat{\Lambda}$$

which is full, faithful, exact, and sends a complex $X = (X^i)_{i\in\mathbb{Z}}$ concentrated in degree 0 to the $\hat{\Lambda}$-module $Y = (Y_i)_{i\in\mathbb{Z}}$ concentrated in degree 0 with $Y_0 = X^0$, see [10, 14] for details. Moreover, F is an equivalence of triangulated categories if and only if $\operatorname{gl.dim}\Lambda < \infty$ [10, 11]. In general, by the image of F we will mean the triangulated subcategory of $\underline{\operatorname{mod}}\,\hat{\Lambda}$ generated by objects of the from $F(X)$, $X \in D^b(\operatorname{mod}\Lambda)$. Note that if $Y \in \underline{\operatorname{mod}}\,\hat{\Lambda}$ is nonzero and Y belongs to the image of F then $Y[n] \not\simeq Y$ for $n \neq 0$.

1.5. Recall that two finite dimensional algebras A and B are called tilting-cotilting equivalent if there is a sequence of algebras $A = A_0, A_1, \ldots, A_m, A_{m+1} = B$ and a sequence of modules $T^{(i)}_{A_i}$, $(0 \leq i \leq m)$ such that $A_{i+1} = \operatorname{End} T^{(i)}_{A_i}$ and $T^{(i)}_{A_i}$ is either a tilting or a cotilting A_i-module. Observe that two Morita equivalent algebras are tilting-cotilting equivalent, because every projective generator is a tilting module. Further, every algebra A is tilting-cotilting equivalent to its opposite algebra A^{op} because the injective cogenerator DA of $\operatorname{mod} A$ is a cotilting A-module and $A^{\mathrm{op}} = \operatorname{End}_A DA$. We need in our considerations APR-tilting modules and APR-cotilting modules introduced in [4]. Namely, for an algebra $A = KQ/I$ and a simple projective noninjective A-module $S_A(i)$, the module $T^i = \tau_A^- S_A(i) \oplus (\bigoplus_{j\in Q_0\setminus\{i\}} P_A(j))$ is a tilting A-module, called the APR-cotilting module associated to $S_A(i)$. Dually, for each simple injective nonprojective A-module $S_A(i)$ the module ${}^iT = \tau_A S_A(i) \oplus (\bigoplus_{j\in Q_0\setminus\{i\}} I_A(j))$ is a cotilting A-module called the APR-cotilting module associated to $S_A(i)$. Finally, recall that if A and B are tilting-cotilting equivalent algebras then the derived categories $D^b(\operatorname{mod} A)$ and $D^b(\operatorname{mod} B)$ are equivalent but in general the converse is not true.

1.6. The one-point extension (respectively, coextension) of an algebra A by an A-module M will be denoted by $A[M]$ (respectively, by $[M]A$). Let $A = KQ/I$ and i be a sink of Q. Following [13] the reflection $S_i^+ A$ of A is defined to be the quotient of the one-point extension $A[I_A(i)]$ by the two-sided ideal generated by the idempotent e_i. Then the sink i of Q is replaced in the quiver of $S_i^+ A$ by a source i'. Dually, for a source j of Q, the reflection $S_j^- A$ of A at j is the quotient of the one-point coextension $[P_A(j)]A$ by the two-sided ideal generated by the idempotent e_j. Moreover, the source j of Q is replaced in the quiver of $S_j^- A$ by a sink j'. It has been proved in [22] that $S_i^+ A$ (respectively, $S_j^- A$) is tilting-cotilting equivalent to A.

1.7. Assume $\Lambda = KQ/I$ is a bound quiver algebra of finite global dimension. Then the Cartan matrix

$$C_\Lambda = (\dim_K \operatorname{Hom}_A(P_\Lambda(i), P_\Lambda(j)))_{i,j\in Q_0}$$

is invertible over $\mathbb{Z}$, and we have a nonsymmetric bilinear form

$$\langle -,- \rangle_\Lambda \colon K_0(\Lambda) \times K_0(\Lambda) \to \mathbb{Z}$$

given by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} C_\Lambda^{-\,\mathrm{t}} \mathbf{y}^{\mathrm{t}}$ for $\mathbf{x}, \mathbf{y} \in K_0(\Lambda) = \mathbb{Z}^{Q_0}$. It has been proved by C. M. Ringel [17] that for modules X and Y from $\operatorname{mod} \Lambda$ we have

$$\langle \operatorname{\mathbf{dim}} X, \operatorname{\mathbf{dim}} Y \rangle_\Lambda = \sum_{i\geq 0} (-1)^i \dim_K \operatorname{Ext}_\Lambda^i (X, Y),$$

where $\operatorname{\mathbf{dim}} Z$ denotes the dimension vector of a module Z in $\operatorname{mod} \Lambda$. The associated integral quadratic form $\chi_\Lambda \colon K_0(\Lambda) \to \mathbb{Z}$, given by $\chi_\Lambda(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle_\Lambda$, for $\mathbf{x} \in K_0(\Lambda)$, is called the Euler form of Λ. Using the isomorphism $K_0(\Lambda) \simeq K_0(D^b(\operatorname{mod} \Lambda))$ induced by the natural inclusion $K_0(\Lambda) \subset K_0(D^b(\operatorname{mod} \Lambda))$ we can consider χ_Λ as the form defined on $K_0(D^b(\operatorname{mod} \Lambda))$. It is known that if an algebra A is tilting-cotilting equivalent to Λ (respectively, $D^b(\operatorname{mod} A) \simeq D^b(\operatorname{mod} \Lambda)$) then the Euler forms χ_A and χ_Λ are $\mathbb{Z}$-equivalent. Moreover, there exists a $\mathbb{Z}$-invertible map $\sigma \colon K_0(A) \to K_0(\Lambda)$ such that $\langle \sigma \mathbf{x}, \sigma \mathbf{y} \rangle_\Lambda =$

$\langle \mathbf{x}, \mathbf{y} \rangle_\Lambda$. Finally, we note that if χ_Λ is positive semi-definite then $\operatorname{rad} \chi_\Lambda = \{\mathbf{x} \in K_0(\Lambda) \mid \chi(\mathbf{x}) = 0\}$ is a subgroup of $K_0(\Lambda)$ such that $K_0(\Lambda)/\operatorname{rad} \chi_\Lambda$ is torsionfree and the form induced on $K_0(\Lambda)/\operatorname{rad} \chi_\Lambda$ by χ_Λ is $\mathbb{Z}$-equivalent to the Euler form χ_H, where H is the path algebra $K\Delta$ of a Dynkin quiver Δ uniquely determined by χ_Λ, called the Dynkin type of χ_Λ. The rank of $\operatorname{rad} \chi_\Lambda$ is called the corank of χ_Λ. The $\mathbb{Z}$-equivalence class of χ_Λ is uniquely determined by its corank and Dynkin type (see [5]).

2 Gentle one-cycle algebras

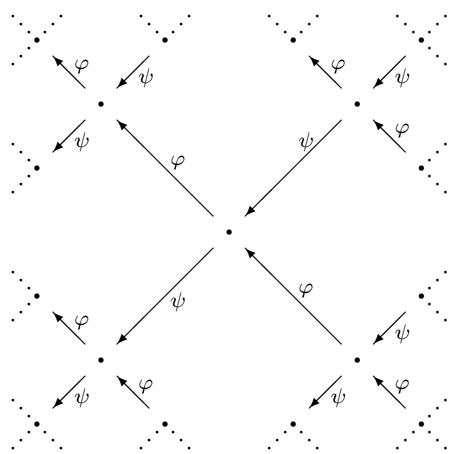
The purpose of this section is to prove the equivalence of the conditions (i), (ii) and (iii) in Theorem A.

Following [2] a bound quiver algebra KQ/I is said to be *gentle* if the bound quiver (Q, I) satisfies the following conditions:

1) Q is connected and the number of arrows in Q with a prescribed source or sink is at most two,
2) I is generated by a set of paths in Q of length two,
3) For any arrow $\alpha \in Q_1$ there are at most one $\beta \in Q_1$ and one $\gamma \in Q_1$ such that $\alpha\beta$ and $\gamma\alpha$ do not belong to I,
4) For any arrow $\alpha \in Q_1$ there are at most one $\xi \in Q_1$ and $\eta \in Q_1$ such that $\alpha\xi$ and $\eta\alpha$ belong to I.

Examples of gentle algebras are the algebras tilting-cotilting equivalent to the hereditary algebras of type $\mathbb{A}_n$ and $\tilde{\mathbb{A}}_n$, classified respectively in [1] and [2].

By a *gentle one-cycle* algebra we mean a gentle algebra $A = KQ/I$ whose quiver contains exactly one cycle, or equivalently $|Q_0| = |Q_1|$. Observe that the bound quiver (Q, I) of a gentle one-cycle algebra $A = KQ/I$ consists of a single cycle together with some branches, each of which is the bound quiver of an algebra tilting-cotilting equivalent to a hereditary algebra of type $\mathbb{A}_t$, that is, a full connected finite bound subquiver of the infinite tree



bound by all possible relations $\varphi\psi = 0 = \psi\varphi$; also, each branch is joined to the cycle at a single point, which we shall call the root of the branch. It has been proved by J. Nehring [15] that the trivial extension $\Lambda \ltimes D\Lambda$ of a non-simply connected algebra Λ is of polynomial growth if and only if Λ is Morita equivalent to a gentle one-cycle algebra. Finally, we say that a gentle one-cycle algebra $A = KQ/I$ satisfies the *clock condition* provided in the unique cycle of (Q, I) the number of clockwise oriented relations equals the number of counterclockwise oriented relations. The following two theorems give characterizations of gentle one-cycle algebras in terms of the derived categories.

Theorem 2.1 ([2]). For an algebra Λ the following conditions are equivalent:

(i) $D^b(\operatorname{mod}\Lambda) \simeq D^b(\operatorname{mod} K\Delta)$ for a quiver Δ of Euclidean type $\tilde{\mathbb{A}}_n$.

(ii) Λ is tilting-cotilting equivalent to a hereditary algebra of type $\tilde{\mathbb{A}}_n$.

(iii) Λ is Morita equivalent to a gentle one-cycle algebra satisfying the clock condition.

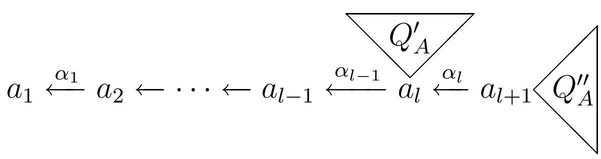
Theorem 2.2 ([21]). The derived category $D^b(\operatorname{mod}\Lambda)$ of an algebra Λ is discrete but not of Dynkin type if and only if Λ is Morita equivalent to a gentle one-cycle algebra not satisfying the clock condition.

Observe that the algebras $\Lambda(r, n, m)$, $(r, n, m) \in \Omega$, defined in the introduction are gentle one-cycle algebras not satisfying the clock conditions. Recall also that two tilting-cotilting equivalent algebras have equivalent derived categories, and two Morita equivalent algebras are trivially tilting-cotilting equivalent. Hence, in order to show the equivalence of the conditions (i), (ii) and (ii) in Theorem A, it remains to prove the following fact.

Proposition 2.3. Let A be a gentle one-cycle algebra which does not satisfy the clock condition. Then there is a triple $(r, n, m) \in \Omega$ such that A is tilting-cotilting equivalent to $\Lambda(r, n, m)$.

Proof 2.4. Let $A = KQ/I$, where the bound quiver (Q, I) contains exactly one cycle and satisfies the conditions (1)–(4) of gentle algebra. A path of length two in Q belonging to I is called a zero-relation. We shall prove that there exists a sequence of algebras $A = A_0$, $A_1, \dots, A_s$, $A_{s+1} = \Lambda(r, n, m)$, for some $(r, n, m) \in \Omega$, such that the algebras A_i and A_{i+1}, $0 \le i \le s$, are tilting-cotilting equivalent. This will be done in several steps.

(a) In the first step we prove that A is tilting-cotilting equivalent to a gentle one-cycle algebra $A_1 = KQ^{(1)}/I^{(1)}$ such that all external branches of the unique cycle are not bound, and consequently are linear quivers without zero-relations. Assume that one of the external branches of (Q, I) is bound be a zero-relation. By passing, if necessary, to the opposite algebra, we may assume that (Q, I) is of the following form



where $\alpha_l\alpha_{l-1} \in I$, $\alpha_{l-1}\alpha_{l-2} \notin I$, $\dots$, $\alpha_2\alpha_1 \notin I$, one of Q'_A and Q''_A is a branch, while the other contains the cycle. We define a module $T_A = \bigoplus_{b\in Q_0} T(b)$, where $T(a_i) = P(a_l)/P(a_i)$, for $i \in \{1,\dots,l-1\}$, and $T(b) = P(b)$ for $b \in Q_0 \setminus \{a_1,\dots,a_{l-1}\}$. Then T_A is a tilting A-module and $B = \operatorname{End} T_A = KQ_B/J$, where the bound quiver (Q_B, J) has the form

$$\boxed{Q'_B} \; a_l \overset{\beta_l}{\longleftarrow} a_1 \longleftarrow \cdots \longleftarrow a_{l-1} \overset{\beta_1}{\longleftarrow} a_{l+1} \; \boxed{Q''_B}$$

$Q'_B = Q'_A$ is bound by the same relations as Q'_A, while $Q''_B = Q''_A$ is bound by the same relations as Q''_A. Moreover, the linear quiver $a_l \leftarrow a_{l-1} \leftarrow \cdots \leftarrow a_1 \leftarrow a_{l+1}$ is not bound, and $\nu\beta_1 \in J$, for some $\nu \in Q''_B = Q''_A$, if and only if $\nu\alpha_l \in I$. We refer for details to the proof of [3, Lemma 2.4]. Observe that we have replaced the branch of (Q, I) containing the sink a_1 by a branch having the same number of vertices, but exactly one zero-relation less. Thus by an obvious induction on the number of zero-relations occurring in the branches of (Q, I) we reduce $A = KQ/I$ to a gentle one cycle algebra $A_1 = KQ^{(1)}/I^{(1)}$ whose branches are not bound by zero-relations.

(b) The second step in our procedure consists in replacing the algebra $A_1 = KQ^{(1)}/I^{(1)}$ by a gentle one-cycle algebra $A_2 = KQ^{(2)}/I^{(2)}$, tilting-cotilting equivalent to A_1, and whose all branches are equioriented linear quivers without zero-relations. This is done by a suitable iterated application of APR-tilting (respectively, APR-cotilting) modules at the simple projective (respectively, simple injective) modules corresponding to sinks (respectively, sources) of the linear branches $(Q^{(1)}, I^{(1)})$.

(c) In the third step we replace the algebra $A_2 = KQ^{(2)}/I^{(2)}$ by a gentle one-cycle algebra $A_3 = KQ^{(3)}/I^{(3)}$ which is tilting-cotilting equivalent to A_2, all zero-relations are on the unique cycle of $(Q^{(3)}, I^{(3)})$, and the branches of $(Q^{(3)}, I^{(3)})$ are equioriented linear quivers. We have some cases to consider. Assume first that $(Q^{(2)}, I^{(2)})$ admits a bound subquiver of the form

$$\begin{array}{ccc} & & c \\ & & \nearrow \\ a_1 \to a_2 \to \cdots \to a_t & \xrightarrow{\alpha} b & \\ & & {\scriptstyle\beta}\searrow \\ & & d \end{array}$$

where b, c, d lie on the cycle, $\alpha\beta \in I^{(2)}$, and a_1 is a source of $Q^{(2)}$. Suppose the cycle of $(Q^{(2)}, I^{(2)})$ contains a bound subquiver

$$b \to c \to \cdots\cdots \to u \xrightarrow{\gamma} v \xrightarrow{\sigma} w$$

with $\gamma\sigma \in I^{(2)}$, and the quiver $b \to c \to \cdots\cdots \to u \xrightarrow{\gamma} v$ is not bound. Then the iterated reflection $S^-_{a_t} \cdots S^-_{a_2} S^-_{a_1} A_2$ is a gentle one-cycle algebra given by the bound quiver obtained from $(Q^{(2)}, I^{(2)})$ by replacing the branch $a_1 \to a_2 \to \cdots \to a_t$ by a subpath of an equioriented branch $v \to \cdots \to a_1 \to a_2 \to \cdots \to a_t$ rooted to the cycle in the middle point of the path $u \xrightarrow{\gamma} v \xrightarrow{\sigma} w$ belonging to $I^{(2)}$. Moreover, $S^-_{a_t} \cdots S^-_{a_2} S^-_{a_1} A_2$ is tilting-cotilting equivalent to A_2 (see 1.6). Assume now that the cycle of $(Q^{(2)}, I^{(2)})$ contains a subquiver of the form

$$b \to c \to c_1 \to \cdots \to c_q \leftarrow c_{q+1}$$

which is not bound, and possibly is of the reduced form $b = c_{-1} \leftarrow c_0 = c$. Then $S^-_{a_t} \cdots S^-_{a_2} S^-_{a_1} A_2$ is a gentle one-cycle algebra given by the bound quiver obtained from $(Q^{(2)}, I^{(2)})$ by replacing the branch $a_1 \to a_2 \to \cdots \to a_t$ by a subpath of an equioriented line

$$a_t \leftarrow \cdots \leftarrow a_2 \leftarrow a_1 \leftarrow \cdots \leftarrow u \xleftarrow{\gamma} c_q \xleftarrow{\sigma} c_{q+1}$$

bound by $\sigma\gamma = 0$, with c_q and c_{q+1} lying on the cycle, and the remaining ones not on the cycle. Further, assume that $(Q^{(2)}, I^{(2)})$ contains a bound quiver of the form (Σ, R)

$$\begin{array}{ccc} & & d \\ & & \diagup \\ b_1 \leftarrow b_2 \leftarrow \cdots \leftarrow b_r \xleftarrow{\eta} a & & \\ & & {}_{\xi}\nwarrow \\ & & c \end{array}$$

with $\xi\eta \in I^{(2)}$, and a, c, d lying on the cycle. Then the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A_2)$ admits a full translation subquiver

$$\begin{array}{ccccccccc}
P(b_1) & & \tau^- P(b_1) & & \cdots & & \tau^{-r+1} P(b_1) & & \tau^{-r} P(b_1) \\
& \searrow \; \nearrow & & \searrow & & \nearrow & & \searrow \; \nearrow & \\
& P(b_2) & & \ddots & & \iddots & & \tau^{-r+1} P(b_2) & \\
& & \searrow & & \searrow \; \nearrow & & \nearrow & & \\
& & \ddots & & \tau^- P(b_{r-1}) & & \iddots & & \\
& & & \searrow \; \nearrow & & \searrow \; \nearrow & & & \\
& & & P(b_r) & & \tau^- P(b_r) & & & \\
& & & & \searrow \; \nearrow & & & & \\
& & & & P(a) & & & &
\end{array}$$

where $\tau^{-r}P(b_1)$ is the direct summand of the radical of $P(c)$. Let $T_{A_2} = \bigoplus_{x\in(Q_{A_2})_0} T(x)$, where $T(b_i) = \tau^{-r+i-1}P(b_i)$ for $i \in \{1,\dots,r\}$, and $T(x) = P(x)$ for $x \in (Q_{A_2})_0 \setminus \{b_1,\dots,b_r\}$. Then T_{A_2} is a tilting A_2-module and $\operatorname{End} T_{A_2}$ is given by the bound quiver obtained from $(Q^{(2)}, I^{(2)})$ by replacing the bound quiver (Σ, R) by the following linear quiver

$$d \text{ --- } a \leftarrow b_r \leftarrow \cdots \leftarrow b_2 \leftarrow b_1 \leftarrow c$$

without relations. Observe that $\operatorname{End} T_{A_2}$ is a gentle one-cycle algebra, and is clearly tilting-cotilting equivalent to A_2. Finally assume that $(Q^{(2)}, I^{(2)})$ contains a bound subquiver (Δ, J) of the form

$$\begin{array}{c}
e \\
\downarrow\alpha \\
a_1 \to a_2 \to \cdots \to a_t \xrightarrow{\gamma} d \xrightarrow{\sigma} b_r \to \cdots \to b_2 \to b_1 \\
\downarrow\beta \\
c
\end{array}$$

with $r, t \ge 1$, c, d, e lying on the cycle, and $\alpha\beta$, $\gamma\sigma \in I^{(2)}$. Then $\Gamma(\operatorname{mod} A_2)$ admits a full translation subquiver

$$\begin{array}{ccccccccc}
P(b_1) & & \tau^- P(b_1) & & \cdots & & \tau^{-r+1}P(b_1) & & \tau^{-r}P(b_1) \\
& \searrow & \nearrow & \searrow & & \nearrow & & \searrow \ \nearrow & \\
& P(b_2) & & \ddots & & \iddots & & \tau^{-r+1}P(b_2) & \\
& & \searrow & & \searrow \ \nearrow & & & \nearrow & \\
& & \ddots & & \tau^- P(b_{r-1}) & & \iddots & & \\
& & & \searrow \ \nearrow & & \searrow \ \nearrow & & & \\
& & & P(b_r) & & \tau^- P(b_r) & & & \\
& & & & \searrow \ \nearrow & & & & \\
& & & & P(d) & & & & \\
& & & & \nearrow \ \searrow & & & & \\
& & & P(c) & & \tau^- P(c) & & &
\end{array}$$

where $\tau^{-r}P(b_1) = \operatorname{rad} P(a_t)$, $\dots$, $P(a_i) = \operatorname{rad} P(a_{i-1})$, $2 \le i \le t-1$, and $P(d)/P(c) = \operatorname{rad} P(e)$. Let $T'_{A_2} = \bigoplus_{x\in(Q_{A_2})_0} T'(x)$, where $T'(b_i) = \tau^{-r+i-1}P(b_i)$ for $i \in \{1,\dots,r\}$, and $T'(x) = P(x)$ for $x \in (Q_{A_2})_0 \setminus \{b_1,\dots,b_r\}$. Then T'_{A_2} is a tilting A_2-module and $\operatorname{End} T'_{A_2}$ is given by the bound quiver obtained from $(Q^{(2)}, I^{(2)})$ by replacing the bound quiver (Δ, J) by the bound quiver of the form

$$\begin{array}{ccccccccccccccc} &&&&&&&&&&&&&& e \\ &&&&&&&&&&&&&& \downarrow\alpha \\ a_1 & \to & a_2 & \to & \cdots & \to & a_t & \to & b_1 & \to & b_2 & \to \cdots \to & b_r & \to & d \\ &&&&&&&&&&&&&& \downarrow\beta \\ &&&&&&&&&&&&&& c \end{array}$$

bound only by $\alpha\beta = 0$. Therefore, applying the above procedure to all branches of $(Q^{(2)}, I^{(2)})$ which are not rooted to the cycle in the middle point of a zero-relation (lying entirely on the cycle), we obtain the required gentle one-cycle algebra $A_3 = KQ^{(3)}/I^{(3)}$, tilting-cotilting equivalent to A_2, and whose all zero-relations lie on the cycle.

(d) The fourth step in our procedure consists in replacing A_3 by a gentle one-cycle algebra $A_4 = KQ^{(4)}/I^{(4)}$ such that all (equioriented) branches of $(Q^{(4)}, I^{(4)})$ are oriented toward the cycle, that is, have a source not lying on the cycle. Assume $(Q^{(3)}, I^{(3)})$ contains a bound subquiver of the form

$$\begin{array}{ccccccccc} e \\ \alpha\downarrow \\ d & \to & b_r & \to & \cdots & \to & b_2 & \to & b_1 \\ \beta\downarrow \\ c \end{array}$$

Taking as above the tilting A_3-module $T'_{A_3} = \bigoplus_{x\in(Q_{A_3})_0} T'(x)$, where we put $T'(b_i) = \tau^{-r+i-1}P(b_i)$ for $i \in \{1, \dots, r\}$, and $T'(x) = P(x)$ for $x \in (Q_{A_3})_0 \setminus \{b_1, \dots, b_r\}$, we obtain a gentle one-cycle algebra $\operatorname{End} T'_{A_3}$ given by the bound quiver obtained from $(Q^{(3)}, I^{(3)})$ by replacing the above bound subquiver by the following one

$$\begin{array}{ccccccccc} &&&&&&&& e \\ &&&&&&&& \alpha\downarrow \\ b_1 & \to & b_2 & \to & \cdots & \to & b_r & \to & d \\ &&&&&&&& \beta\downarrow \\ &&&&&&&& c \end{array}$$

and bound only by $\alpha\beta = 0$, and which is tilting-cotilting equivalent to A_3. Applying the iterated reflections (as above) to all branches of $(Q^{(3)}, I^{(3)})$ which are not oriented toward the cycle, we obtain the required gentle one-cycle algebra $A_4 = KQ^{(4)}/I^{(4)}$.

(e) The fifth step in our procedure consists of removing in $(Q^{(4)}, I^{(4)})$ all consecutive zero-relations oriented in opposite directions on the cycle, together with (eventual) branches rooted in the midpoints of those relations. Assume $(Q^{(4)}, I^{(4)})$ admits a full subquiver of the form

$$
\begin{array}{ccccccccc}
 & & & & & c & & & \\
 & & & & & \downarrow\alpha & & & \\
 & a_1 & — & \cdots & — & a_{t-1} \overset{\beta}{\longleftarrow} a_t & \leftarrow & \cdots \leftarrow & a_l \\
 & \swarrow & & & & & & & \\
a_0 = b_0 & & & & & & & & \\
 & \nwarrow & & & & & & & \\
 & b_1 & — & \cdots & — & b_{s-1} \overset{\sigma}{\longleftarrow} b_s & \leftarrow & \cdots \leftarrow & b_k \\
 & & & & & \uparrow\gamma & & & \\
 & & & & & d & & &
\end{array}
$$

bound only by $\alpha\beta = 0$ and $\gamma\sigma = 0$, the vertices c, a_t, …, a_1, $a_0 = b_0$, b_1, …, b_s, d lie on the cycle, and possibly $l = t$ or $k = s$. Let H be the path algebra of the full linear subquiver of the above quiver formed by all vertices except c and d. Then H is a hereditary algebra of Dynkin type $\mathbb{A}_{l+k-1}$ and the Auslander–Reiten quiver $\Gamma(\operatorname{mod} H)$ contains a complete section Σ containing the simple modules $S(a_t)$ and $S(b_s)$, belonging to the opposite border orbits in $\Gamma(\operatorname{mod} H)$. Let T'_{A_4} be the direct sum of modules lying on Σ, considered as A_4-modules. Consider the A_4-module

$$T_{A_4} = T'_{A_4} \oplus \bigoplus_{x \in Q_0^{(4)} \setminus (Q_H)_0} P(x).$$

Then T_{A_4} is a tilting A_4-module and $\operatorname{End} T_{A_4}$ is a gentle one-cycle algebra given by the bound quiver obtained from $(Q^{(4)}, I^{(4)})$ by replacing the above bound subquiver by a quiver of the form

$$c \to u_1 \to u_2 \to \cdots \to u_i \to w \leftarrow v_j \leftarrow \cdots \leftarrow v_2 \leftarrow v_1 \leftarrow d$$

with $i + j = l + k$, and not bound. Therefore, we incorporated the linear quivers $a_l \to \cdots \to a_{t+1}$ and $b_k \to \cdots \to b_{s+1}$ inside the cycle and erased simultaneously the two zero-relations with midpoints a_t and b_s (thus a clockwise and a counterclockwise zero-relations on the cycle). Applying systematically the above procedure we erase completely all the consecutive zero-relations of opposite directions on the cycle. Thus we obtain a gentle one-cycle algebra $A_5 = KQ^{(5)}/I^{(5)}$, where all zero-relations in $(Q^{(5)}, I^{(5)})$ are either clockwise oriented or counterclockwise oriented zero-relations on the cycle, all branch of $(Q^{(5)}, I^{(5)})$ are lines oriented toward to the cycle and rooted in the midpoints of zero-relations, and A_5 is tilting-cotilting equivalent to A_4.

(f) Our next objective is to replace A_5 by a gentle one-cycle algebra $A_6 = KQ^{(6)}/I^{(6)}$, tilting-cotilting equivalent to A_5, and such that all zero-relations in $(Q^{(6)}, I^{(6)})$ are clockwise oriented zero-relations on the cycle. Suppose all zero-relations $(Q^{(5)}, I^{(5)})$ are counterclockwise oriented zero-relations on the cycle. Observe that the opposite algebra A_5^{op} is tilting-cotilting equivalent to A_5 (see 1.5). Moreover, A_5^{op} is a gentle one-cycle algebra where all zero-relations are clockwise oriented zero-relations on the cycle but all (equioriented) branches are oriented outside the cycle. Applying now the procedure described in (d), we obtain the required gentle one-cycle algebra $A_6 = KQ^{(6)}/I^{(6)}$, obtained from A_5^{op} by reversing orientations of all arrows in the branches.

(g) We now replace A_6 by a gentle one-cycle algebra $A_7 = KQ^{(7)}/I^{(7)}$, tilting-cotilting equivalent to A_6, such that all zero-relations in $(Q^{(7)}, I^{(7)})$ are consecutive clockwise oriented zero-relations on the cycle, and all branches are oriented toward the cycle. Assume that the cycle of $(Q^{(6)}, I^{(6)})$ admits a full bound subquiver Σ of the form

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c = u_0 - u_1 - \cdots - u_{l-1} - u_l = d \xrightarrow{\gamma} e \xrightarrow{\sigma} f$$

with $l \geq 0$, and bound only by $\alpha\beta = 0 = \gamma\sigma$. We have two cases to consider.

Suppose first that the above walk contains a subquiver of the form $u_{i-1} \to u_i \leftarrow u_{i+1}$, for some $i \in \{0, \ldots, l-1\}$ (where $u_{-1} = b$). Consider the path algebra H of the quiver given by the vertices $c = u_0,\ u_1,\ \ldots,\ u_{l-1},\ u_l = d$. Then $\Gamma(\operatorname{mod} H)$ admits a complete section of the form

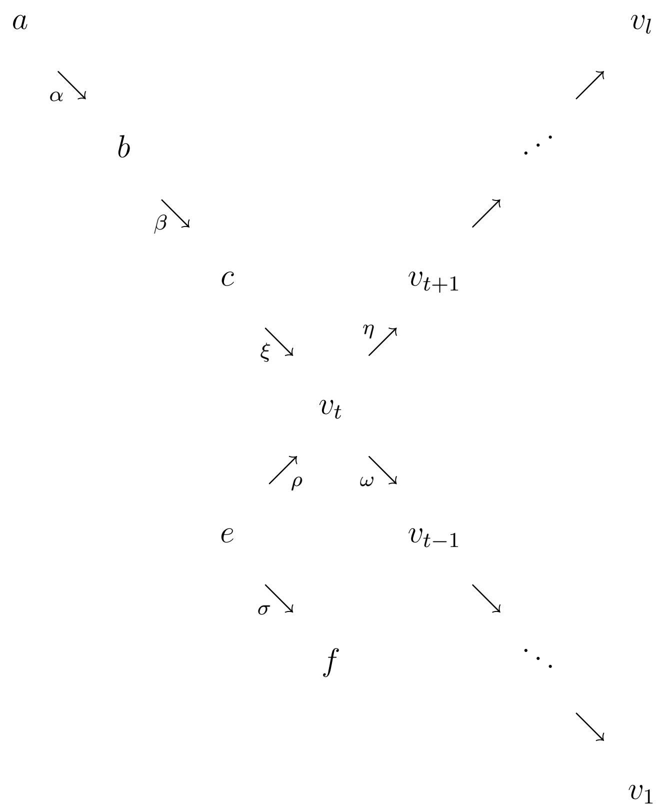
$$\begin{array}{ccccc}
P(c) & & & & \\
 & \searrow & & & \\
 & & \ddots & & \\
 & & & \searrow & \\
 & & & & V \\
 & & & \nearrow & \\
 & & \cdot^{\cdot^{\cdot}} & & \\
 & \nearrow & & & \\
P(d) & & & &
\end{array}$$

Denote by T'_{A_6} the direct sum of modules, considered as A_6-modules, lying on this section, by P the direct sum of all projective A_6-modules $P(x)$, for $x \in Q_0^{(6)} \setminus (Q_H)_0$, and put $T_{A_6} = T'_{A_6} \oplus P$. Then $B = \operatorname{End} T_{A_6}$ is a gentle one-cycle algebra $K\Delta/J$, where (Δ, J) is obtained from $(Q^{(6)}, I^{(6)})$ by replacing Σ by a quiver Σ' of the form

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c \leftarrow v_1 \leftarrow \cdots \leftarrow v_t \to v_{t+1} \to \cdots \to v_{l-1} \to v_l = d \xrightarrow{\gamma} e \xrightarrow{\sigma} f$$

and bound only by $\alpha\beta = 0 = \gamma\sigma$. Let $C = S^-_{v_l} \cdots S^-_{v_{t+1}} S^-_{v_1} \cdots S^-_{v_t} B$ be the iterated reflection. Then C is a gentle one-cycle algebra $K\Delta'/J'$, where (Δ', J') is obtained from

(Δ, J) by replacing the above quiver Σ' by a quiver Σ'' of the form



bound by $\alpha\beta = 0$, $\beta\xi = 0$, $\xi\eta = 0$, $\rho\omega = 0$, and $\nu\rho = 0$ for the arrow ν in $Q^{(6)}$ (if exists) with sink e and different from γ. Observe that the vertices a, b, c, v_t, e and f lie on the cycle of (Δ', J'), while the quivers $v_{t-1} \to \cdots \to v_1$ and $v_{t+1} \to \cdots \to v_l$ are branches. Applying now the procedure from (c) we may replace the algebra C by a gentle one-cycle algebra $D = K\Delta''/J''$, where (Δ'', J'') is obtained from (Δ', J') by replacing the above quiver Σ'' by the quiver Σ'''

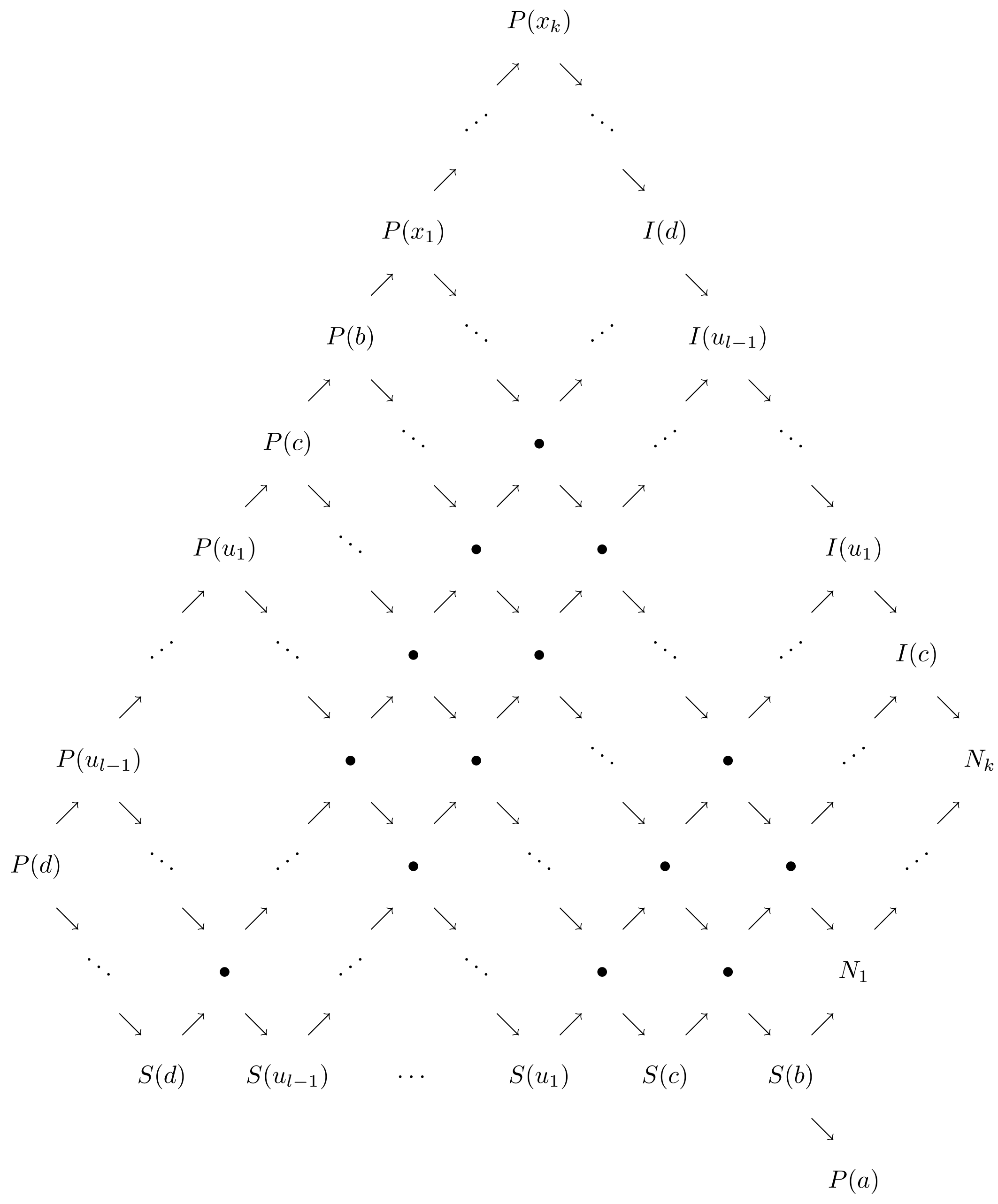
$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} v_l \to \cdots \to v_{t+1} \to v_t \leftarrow v_{t-1} \leftarrow \cdots \leftarrow v_1 \xleftarrow{\varphi} e \xrightarrow{\sigma} f$$

bound by $\alpha\beta = 0 = \beta\gamma$. Moreover, if we have in (Δ', J') a path $w_p \to \cdots \to w_2 \to w_1 \xrightarrow{\psi} e$ then also $\psi\varphi \in J''$. Finally, applying again the procedure from (c) we may replace D by a gentle one-cycle algebra $E = K\Delta'''/J'''$, tilting-cotilting equivalent to D (hence also to A_6) given by a bound quiver obtained from the bound quiver (Δ'', J'') by insertion the path $w_p \to \cdots \to w_1 \to e$ into the cycle. Observe that in our process we replaced the zero-relation $\gamma\sigma = 0$ in $(Q^{(7)}, I^{(7)})$ by a zero-relation $\beta\gamma = 0$ which is consecutive to $\alpha\beta = 0$, all zero-relations in (Δ''', J''') are clockwise oriented zero-relations on the cycle, and all branches are lines oriented toward the cycle and rooted to the cycle in the midpoints of zero-relations.

Assume now that Σ is the equioriented quiver

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c = u_0 \to u_1 \to \cdots \to u_{l-1} \to u_l = d \xrightarrow{\gamma} e \xrightarrow{\sigma} f,$$

with $l \geq 0$ and bound only by $\alpha\beta = 0 = \gamma\sigma$. Observe that we may have branches in $(Q^{(6)}, I^{(6)})$ rooted to the cycle in the vertices b and e. Denote by $\overline{\Sigma}$ the subquiver of $(Q^{(6)}, I^{(6)})$ consisting of Σ and the branch $x_k \to x_{k-1} \to \cdots \to x_1 \to x_0 = b$ rooted to the cycle in the vertex b, where possible $k = 0$ ($\overline{\Sigma} = \Sigma$) if such a branch do not exist. Then the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A_6)$ admits a full translation subquiver of the form



Let M be the direct sum of the indecomposable A_6-modules $I(u_{l-1})$, …, $I(u_1)$, $I(c)$, N_k, …, N_1, $S(b)$ (respectively, $I(u_{l-1})$, …, $I(u_1)$, $I(c)$, $S(b)$, if $\overline{\Sigma} = \Sigma$). Further, denote by P the direct sum of the indecomposable projective A_6-modules $P(z)$, for all $z \in Q_0^{(6)} \setminus \{d, u_{l-1}, \ldots, u_1, c, b, x_1, \ldots, x_{k-1}\}$, and put $T = M \oplus P$. Observe that T is a direct sum of $|Q_0^{(6)}|$ pairwise nonisomorphic indecomposable A_6-modules. Moreover, it follows from our choice of M that we have $\mathrm{Ext}^1_{A_6}(T, T) = \mathrm{Ext}^1_{A_6}(M, T) = D\,\overline{\mathrm{Hom}}_{A_6}(T, \tau_{A_6} M) = 0$, and $\mathrm{Hom}_{A_6}(D(A_6), \tau_{A_6}T) = \mathrm{Hom}_{A_6}(D(A_6), \tau_{A_6}M) = 0$, and so $\mathrm{pd}_{A_6} T \leq 1$. Thus T is a tilting A_6-module. A simple checking shows that $F = \mathrm{End}_{A_6}(T)$ is a gentle one-cycle algebra $K\Delta/J$, where (Δ, J) is obtained from $(Q^{(6)}, I^{(6)})$ by replacing the subquiver $\overline{\Sigma}$ by the subquiver

$$a \xrightarrow{\alpha} b \leftarrow x_1 \leftarrow \cdots \leftarrow x_k \rightarrow u_0 \rightarrow \cdots \rightarrow u_{l-1} \xrightarrow{\beta} d \xrightarrow{\gamma} e \xrightarrow{\sigma} f$$

if $\overline{\Sigma} \neq \Sigma$, and by the subquiver

$$a \xrightarrow{\alpha} b \rightarrow u_0 \rightarrow \cdots \rightarrow u_{l-1} \xrightarrow{\beta} d \xrightarrow{\gamma} e \xrightarrow{\sigma} f$$

is $\overline{\Sigma} = \Sigma$, and bound only by zero-relations $\beta\gamma = 0 = \gamma\sigma$ (in both cases). Observe that in this process we replaced the zero-relation $\alpha\beta = 0$ by the zero-relation $\beta\gamma = 0$ which is consecutive to $\gamma\sigma = 0$, and inserted the branch $x_k \rightarrow \cdots \rightarrow x_1 \rightarrow x_0 = b$ into the cycle, if such a subquiver of $(Q^{(6)}, I^{(6)})$ exists.

Iterating the above two types of procedures, we obtain a gentle one-cycle algebra $A_7 = KQ^{(7)}/I^{(7)}$, tilting-cotilting equivalent to A_6, and such that all zero-relations of $(Q^{(7)}, I^{(7)})$ are consecutive clockwise oriented zero-relations on the cycle, and all branches of $(Q^{(7)}, I^{(7)})$ are lines oriented toward the cycle and rooted to the cycle in midpoints of zero-relations.

(h) Assume now that the cycle of $(Q^{(7)}, I^{(7)})$ is not an oriented cycle with $I^{(7)}$ generated by all paths of length 2 on it. We shall prove that then A_7 is tilting-cotilting equivalent to an algebra $A_8 = A(r, n, m) = K\Delta(r, n, m)/J(r, n, m)$, where $\Delta(r, n, m)$ is the quiver

$$\begin{array}{ccccccccc}
 & (-1) & \xleftarrow{\sigma_2} & \cdots & \xleftarrow{\sigma_{m-1}} & (-m+1) & & & \\
\sigma_1 \swarrow & & & & & & \nwarrow \sigma_m & & \\
0 & & & & & & & (-m) & \\
\gamma_{n-1} \nwarrow & & & & & & \swarrow \gamma_0 & & \\
 & (n-1) & \xleftarrow{\gamma_{n-2}} \cdots \xleftarrow{\gamma_{n-r}} & n-r & \xleftarrow{\gamma_{n-r-1}} & \cdots \xleftarrow{\gamma_1} & 1 & &
\end{array}$$

for some $n > r \geq 1$ and $m \geq 0$, equivalently $(r, n, m) \in \Omega_f$, and $J(r, n, m)$ is generated by the paths $\gamma_{n-r-1}\gamma_{n-r}, \ldots, \gamma_{n-2}\gamma_{n-1}$. It follows from our assumption that the cycle of $(Q^{(7)}, I^{(7)})$ admits a subquiver

$$a_{r+1} \xrightarrow{\beta_r} a_r \xrightarrow{\beta_{r-1}} a_{r-1} \rightarrow \cdots \rightarrow a_3 \xrightarrow{\beta_2} a_2 \xrightarrow{\beta_1} a_1 \xrightarrow{\beta_0} a_0 \overset{\alpha}{\text{---}} b$$

with $r \geq 1$ and such that $\beta_r\beta_{r-1}, \ldots, \beta_2\beta_1, \beta_1\beta_0 \in I^{(7)}$ are all zero-relations in $(Q^{(7)}, I^{(7)})$. Moreover, beside the cycle, we may have in the quiver $(Q^{(7)}, I^{(7)})$ lines oriented toward the cycle and rooted to the cycle in the vertices $a_r, \ldots, a_2, a_1$. We first show that A_7 is tilting-cotilting equivalent to a gentle one-cycle $\Lambda = K\Delta/J$ where (Δ, J) has the same bound cycle as $(Q^{(7)}, I^{(7)})$ but additionally at most one external line, and such a line is oriented toward the cycle and rooted in the vertex a_1. Thus we shall insert all lines rooted in the vertices $a_r, \ldots, a_2$ into a line rooted in a_1. Suppose t is the maximal element from $\{1, \ldots, r\}$ such that there is a nontrivial line rooted in the vertex a_r, and assume $t \geq 2$. Let $w_p \to \cdots \to w_2 \to w_1$ be the branch rooted to the cycle in a_t, that is, there exists an arrow $w_1 \to a_t$ different from β_t. Taking the iterated reflection $S^-_{w_1} S^-_{w_2} \cdots S^-_{w_p} A_7$ we obtain a gentle one-cycle algebra given by the bound quiver obtained from $(Q^{(7)}, I^{(7)})$ by replacing the line $w_p \to \cdots \to w_2 \to w_1 \to a_t$ by the line $a_{t-1} \xrightarrow{\xi} w_p \to \cdots \to w_2 \to w_1$, and moreover we create a zero-relation $\eta\xi = 0$ if there exists in $(Q^{(7)}, I^{(7)})$ an arrow $c \xrightarrow{\eta} a_{t-1}$ different from β_{t-1}. Applying now the corresponding procedures from (c) and (d) we may replace the algebra $S^-_{w_1} S^-_{w_2} \cdots S^-_{w_p} A_7$ by a gentle one-cycle algebra having the same bound cycle as $(Q^{(7)}, I^{(7)})$ but the lines rooted only in the vertices $a_{t-1}, \ldots, a_1$. Hence, by an obvious induction we obtain the required gentle one-cycle algebra $\Lambda = K\Delta/J$. Suppose (Δ, J) admits a subquiver $x_k \to x_{m-1} \to \cdots \to x_1 \xrightarrow{\gamma} x_0 = a_1$ with $\gamma \neq \beta_1$. Applying now the constructions from (g), we may replace Λ by a gentle one-cycle algebra $\Lambda' = K\Delta'/J'$, tilting-cotilting equivalent to Λ (and hence to A_7), such that (Δ', J') consists of a gentle cycle bound by r consecutive clockwise oriented zero-relations and having m consecutive counterclockwise oriented arrows. Applying now APR-tiling and APR-cotilting modules at the simple projective and simple injective Λ'-modules respectively, we obtain an algebra A_8 isomorphic to an algebra $A(r,n,m) = K\Delta(r,n,m)/J(r,n,m)$, for some $(r,n,m) \in \Omega_f$, which is tilting-cotilting equivalent to A_7. We finally note that $A_8 = A(r,n,m) = \operatorname{End} T_{A_9}$, where $A_9 = \Lambda(r,n,m)$ is the algebra $KQ(r,n,m)/I(r,n,m)$ described in the introduction and T_{A_9} is the tilting A_9-module constructed in the second part of (g). In particular, A_8 is tilting-cotilting equivalent to $A_9 = \Lambda(r,n,m)$.

(i) Finally, assume that the cycle of $(Q^{(7)}, I^{(7)})$ has cyclic orientation and $I^{(7)}$ is generated by all paths of lengths 2 on the cycle. Applying arguments as above (changing of equioriented lines), we conclude that A_7 is tilting-cotilting equivalent the gentle one-cycle algebra $A_8 = KQ^{(8)}/I^{(8)}$, where $(Q^{(8)}, I^{(8)})$ has the same bound cycle as $(Q^{(7)}, I^{(7)})$ but at most one external line, and this line is not bound and oriented toward the cycle. Observe that A_8 is isomorphic to an algebra $\Lambda(r,n,m) = KQ(r,n,m)/I(r,n,m)$.

Therefore, we have proved that A is tilting-cotilting equivalent to $\Lambda(r,n,m)$, for some $(r,n,m) \in \Omega$. This finishes the proof of the proposition.

3 Structure of $\Gamma(D^b(\operatorname{mod} \Lambda(r,n,m)))$

Fix $(r,n,m) \in \Omega$ and let $\Lambda = \Lambda(r,n,m)$. We also denote by Q the quiver $Q(r,n,m)$. Our aim in this section is to describe the quiver $\Gamma(D^b(\operatorname{mod} \Lambda))$. In particular, we are interested in the action of the suspension functor on $\Gamma(D^b(\operatorname{mod} \Lambda))$.

Recall that we have the Happel functor $F\colon D^b(\operatorname{mod} \Lambda) \to \underline{\operatorname{mod}}\,\hat{\Lambda}$ which is full and faithful. Moreover, F is an equivalence of triangulated categories if the global dimension of Λ is finite, that is, if $r < n$. We know that $\hat{\Lambda}$ is special biserial (see [2]) and the Auslander–Reiten quiver of $\underline{\operatorname{mod}}\,\hat{\Lambda}$ consists of $2r$ components $\mathcal{X}^{(0)}, \ldots, \mathcal{X}^{(r-1)}, \mathcal{Y}^{(0)}, \ldots, \mathcal{Y}^{(r-1)}$ of type $\mathbb{Z}\mathbb{A}_\infty$ and r components $\mathcal{Z}^{(0)}, \ldots, \mathcal{Z}^{(r-1)}$ of type $\mathbb{Z}\mathbb{A}_\infty^\infty$ (see [9, Propostion (3.1)]). However, in order to determine which parts of them belong to the image of F we need a more precise knowledge about their structure. This information will be also useful in the next section.

First we give a precise description of $\hat{\Lambda}$. Let $\hat{Q}$ be the quiver whose vertices are (i,k), $i \in \mathbb{Z}$, $k = -m, \dots, n-1$. For each $i \in \mathbb{Z}$ and $k = -m, \dots, n-1$, we have in $\hat{Q}$ an arrow $\alpha_{i,k} : (i,k) \to (i,k+1)$. Next, for $i \in \mathbb{Z}$ and $k = n-r+1, \dots, n-1$, we have an arrow $\alpha^*_{i,k} : (i,k+1) \to (i+1,k)$ in $\hat{Q}$. Finally, we have an arrow $\alpha^*_{i,n-r} : (i,n-r+1) \to (i+1,-m)$ in $\hat{Q}$ for any $i \in \mathbb{Z}$. In all above formulas (i,n) denotes the vertex $(i,0)$. It is known that $\hat{Q}$ is the quiver of $\hat{\Lambda}$.

Let $\omega_{i,-m}$ be the path $\alpha_{i,-m} \cdots \alpha_{i,n-r}$ and we put $\omega_{i,k} = \alpha_{i,k} \cdots \alpha_{i,n-r}\ \alpha^*_{i,n-r}\ \alpha_{i+1,-m} \cdots \alpha_{i+1,k-2}$, $k = -m+1, \dots, n-r+1$. Let $\hat{I}$ be the ideal in $K\hat{Q}$ generated by all the relations of the forms

$$\begin{aligned}
&\alpha_{i,k}\alpha_{i,k+1},\ k = n-r, \dots, n-1,\\
&\alpha^*_{i,k}\alpha^*_{i+1,k-1},\ k = n-r+1, \dots, n-1,\\
&\alpha^*_{i,n-r}\alpha^*_{i+1,n-1} \text{ if } m = 0,\\
&\alpha_{i,-1}\alpha^*_{i+1,n-1} \text{ if } m > 0,\\
&\alpha_{i,n-r+1}\alpha^*_{i,n-r+1} - \alpha^*_{i,n-r}\omega_{i+1,-m} \text{ if } r > 1,\\
&\alpha_{i,k}\alpha^*_{i,k} - \alpha^*_{i,k-1}\alpha_{i+1,k-1},\ k = n-r+2, \dots, n-1,\\
&\alpha^*_{i,n-1}\alpha_{i+1,n-1} - \alpha_{i,0}\omega_{i,1} \text{ if } r > 1,\\
&\alpha^*_{i,n-1}\omega_{i+1,-m} - \alpha_{i,0}\omega_{i,1} \text{ if} r = 1,\\
&\alpha_{i,k}\omega_{i,k+1}\alpha_{i+1,k}, k = -m, \dots, -1, 1, \dots, n-r,
\end{aligned}$$

Then $\hat{\Lambda} \simeq K\hat{Q}/\hat{I}$ (see for example [18]). We may also identify Λ with the full subcategory of $\hat{\Lambda}$ formed by $(0,k)$, $k = -m, \dots, n-1$.

For each string ω in $\hat{\Lambda}$ we denote by M_ω the corresponding string $\hat{\Lambda}$-module. If $\omega = e_{(i,k)}$ is the trivial path at the vertex (i,k) then we write $M_{i,k}$ instead of $M_{e_{(i,k)}}$.

Fix $k \in \{0, \dots, r-1\}$. We denote the vertices of $\mathcal{X}^{(k)}$ by $X^{(k)}_{i,j}$, $i \le j$, $i,j \in \mathbb{Z}$, in such a way that $\tau X^{(k)}_{i,j} = X^{(k)}_{i-1,j-1}$ and we have arrows $X^{(k)}_{i,j} \to X^{(k)}_{i,j+1}$ and $X^{(k)}_{i,j} \to X^{(k)}_{i+1,j}$ (provided $i+1 \le j$). Similarly, we denote the vertices of $\mathcal{Y}^{(k)}_{i,j}$, $i \ge j$, $i,j \in \mathbb{Z}$, in such a way that $\tau Y^{(k)}_{i,j} = Y^{(k)}_{i-1,j-1}$ and we have arrows $Y^{(k)}_{i,j} \to Y^{(k)}_{i+1,j}$ and $Y^{(k)}_{i,j} \to Y^{(k)}_{i,j+1}$ (provided $i \ge j+1$). Finally, we denote the vertices of $\mathcal{Z}^{(k)}$ by $Z^{(k)}_{i,j}$, $i,j \in \mathbb{Z}$, in such a way that $\tau Z^{(k)}_{i,j} = Z^{(k)}_{i-1,j-1}$ and we have arrows $Z^{(k)}_{i,j} \to Z^{(k)}_{i+1,j}$ and $Z^{(k)}_{i,j} \to Z^{(k)}_{i,j+1}$. Using the general Auslander–Reiten theory for special biserial algebras the above numbering can be arranged in such a way we have the following chains of morphism coming from the natural ordering of strings

$$\begin{aligned}
X^{(k)}_{i,i} \longrightarrow X^{(k)}_{i,i+1} \longrightarrow \cdots \longrightarrow Z^{(k)}_{i,i-1} \longrightarrow Z^{(k)}_{i,i} \longrightarrow Z^{(k)}_{i,i+1} \longrightarrow&\\
\cdots \longrightarrow X^{(k)}_{i,i-1}[1] \longrightarrow X^{(k)}_{i-1,i-1}[1],&\\
Y^{(k)}_{i,i} \longrightarrow Y^{(k)}_{i+1,i} \longrightarrow \cdots \longrightarrow Z^{(k)}_{i-1,i} \longrightarrow Z^{(k)}_{i,i} \longrightarrow Z^{(k)}_{i+1,i} \longrightarrow&\\
\cdots \longrightarrow Y^{(k)}_{i-1,i}[1] \longrightarrow Y^{(k)}_{i-1,i-1}[1].&
\end{aligned}$$

Moreover, we have distinguished triangles

$$X^{(k)}_{i,i+d} \longrightarrow Z^{(k)}_{i,j} \longrightarrow Z^{(k)}_{i+d+1,j} \longrightarrow X^{(k)}_{i,i+d}[1], \tag{1}$$

$$Y^{(k)}_{i+d,i} \longrightarrow Z^{(k)}_{i,j} \longrightarrow Z^{(k)}_{i,j+d+1} \longrightarrow X^{(k)}_{i+d,i}[1], \tag{2}$$

which will play an important role. We may also assume that $X^{(k)}_{i,j}[1] = X^{(k+1)}_{i,j}$, $Y^{(k)}_{i,j}[1] = Y^{(k+1)}_{i,j}$ and $Z^{(k)}_{i,j}[1] = Z^{(k+1)}_{i,j}$ for $k = 0, \dots, r-2$. (We will see in Lemmas 3.1 and 3.2 that with this convention we have $X^{(r-1)}_{i,j}[1] = X^{(0)}_{i+r+m,j+r+m}$ and $Y^{(r-1)}_{i,j}[1] = Y^{(0)}_{i+r-n,j+r-n}$.) The above numbering is uniquely determined by the above conditions if we assume that

$$Z^{(0)}_{0,0} = S_\Lambda(0),$$

and thus

$$X^{(0)}_{0,0}[1] = \begin{cases} M_{\omega_{-1,0}} & \text{if } m = 0 \text{ and } r = 1 \\ M_{\alpha_{-1,n-r+1}} & \text{if } m = 0 \text{ and } r > 1 \\ S_\Lambda(-1) & \text{if } m > 0, \end{cases} \tag{3}$$

$$Y^{(0)}_{0,0}[1] = \begin{cases} M_{\alpha^*_{0,1}} & \text{if } r = 1 = n, \\ S_\Lambda(n-1) & \text{if } r = 1 \text{ and } n > 1, \\ M_{\omega_{0,n-1}} & \text{if } r = 2, \\ M_{\alpha^*_{0,n-2}} & \text{if } r > 2. \end{cases} \tag{4}$$

It is known (see [9]) that the modules $X^{(k)}_{i,i}$ and $Y^{(k)}_{i,i}$ are of the form $M_{i,k}$, $k = -m, \dots, -1, 1, \dots, n-r$, $M_{\alpha_{i,k}}$, $M_{\alpha^*_{i,k}}$, $k = n-r+1, \dots, n-1$, and $M_{\omega_{i,k}}$, $k = -m, \dots, n-r+1$. Thus in order to describe the action of the suspension functor on $\Gamma(\underline{\mathrm{mod}}\,\hat{\Lambda})$ we need to calculate the action of $\tau = \tau_{\hat{\Lambda}}$ and the suspension functor on the above modules.

Using the above description of $\hat{\Lambda}$ and our convention $M[-1] = \Omega M$ we can easily calculate the following:

$$\begin{aligned}
M_{i,k}[-1] &= M_{\omega_{i,k+1}},\ k = -m, \dots, -1,\ m \geq 1, \\
M_{i,k}[-1] &= M_{\omega_{i,k+1}},\ k = 1, \dots, n-r,\ n \geq r+1, \\
M_{\alpha_{i,n-r+1}}[-1] &= M_{\omega_{i+1,-m}},\ r \geq 2, \\
M_{\alpha_{i,k}}[-1] &= M_{\alpha_{i+1,k-1}},\ k = n-r+2, \dots, n-1,\ r \geq 3, \\
M_{\alpha^*_{i,k}}[-1] &= M_{\alpha^*_{i,k+1}},\ k = n-r+1, \dots, n-2,\ r \geq 3, \\
M_{\alpha^*_{i,n-1}}[-1] &= M_{\omega_{i,1}},\ r \geq 2, \\
M_{\omega_{i,k}}[-1] &= M_{i+1,k},\ k = -m, \dots, -1,\ m \geq 1, \\
M_{\omega_{i,0}}[-1] &= \begin{cases} M_{\alpha_{i+1,n-1}} & r \geq 2 \\ M_{\omega_{i+1,-m}} & r = 1 \end{cases}, \\
M_{\omega_{i,k}}[-1] &= M_{i+1,k},\ k = 1, \dots, n-r,\ n \geq r+1, \\
M_{\omega_{i,n-r+1}}[-1] &= \begin{cases} M_{\omega_{i,1}} & r = 1 \\ M_{\alpha^*_{i,n-r+1}} & r \geq 2 \end{cases}.
\end{aligned}$$

Since $\tau = \nu\Omega^2$, $\nu M_{i,k} = \nu M_{i-1,k}$, $\nu M_{\alpha_{i,k}} = M_{\alpha_{i-1,k}}$, $\nu M_{\alpha^*_{i,k}} = M_{\alpha^*_{i-1,k}}$ and $\nu M_{\omega_{i,k}} = M_{\omega_{i-1,k}}$, we can calculate the rules for τ which are a little bit more tricky and we will not present them in all details. Note that we have $M_{0,k} = S_\Lambda(k)$, $k = -m, \dots, -1, 1, \dots, n-r$, $M_{\omega_{0,-m}} = P_\Lambda(-m)$ and $M_{\alpha_{0,k}} = P_\Lambda(k)$, $k = n-r+1, \dots, n-1$. As the result we get

$$\begin{aligned}
\tau S_\Lambda(k)[j] &= S_\Lambda(k+1)[j], k = -m, \dots, -2,\ m \geq 2,\\
\tau S_\Lambda(1)[j] &= \begin{cases} P_\Lambda(n-1)[j] & r \geq 2 \\ P_\Lambda(m)[j] & r = 1 \end{cases},\ m \geq 2,\\
\tau S_\Lambda(k)[j] &= S_\Lambda(k+1)[j], k = 1, \dots, n-r-1,\ n \geq r+2,\\
\tau S_\Lambda(n-r)[j] &= S_\Lambda(1)[j+r],\ n \geq r+1,\\
\tau P_\Lambda(-m)[j] &= \begin{cases} P_\Lambda(-m)[j-1] & m = 0, r = 1 \\ P_\Lambda(n-1)[j-1] & m = 0, r \geq 2 \\ S_\Lambda(-m)[j-1] & m \geq 1 \end{cases},\ m \geq 0,\\
\tau P_\Lambda(k)[j] &= P_\Lambda(k-1)[j-1],\ k = n-r+2, \dots, n-1,\ r \geq 3,\\
\tau P_\Lambda(n-r+1)[j] &= P_\Lambda(-m)[j-1],\ r \geq 2.
\end{aligned}$$

Each module of one of the forms $M_{i,k}$, with $k = -m, \dots, -1$, $M_{\alpha_{i,k}}$, with $k = n-r+1, \dots, n-1$, $M_{\omega_{i,k}}$, with $k = -m, \dots, -1$, is the shift of one of the modules $S_\Lambda(k)$, $k = -m, \dots, 0$, $P_\Lambda(-m)$, $P_\Lambda(k)$, $k = n-r+1, \dots, n-1$. It follows from the formulas

$$\begin{aligned}
M_{\omega_{i,-m}}[-2k+1] &= M_{i+k,-m+k-1},\ k = 1, \dots, m,\\
M_{\omega_{i,-m}}[-2k] &= M_{\omega_{i+k,-m+k}},\ k = 1, \dots, m,\\
M_{\omega_{i,-m}}[-2m-k] &= M_{\alpha_{i+m+1,n-k}},\ k = 1, \dots, r-1,\\
M_{\omega_{i,-m}}[-2m-r] &= M_{\omega_{i+m+1,-m}}.
\end{aligned}$$

Taking into account the above calculations and the assumption (3) we get the following statement about the components $\mathcal{X}^{(k)}$.

Lemma 3.1. We have the following formulas

$$\begin{aligned}
X^{(k)}_{q(r+m)+m,q(r+m)+m} &= P_\Lambda(-m)[qr+k],\\
X^{(k)}_{q(r+m)+p,q(r+m)+p} &= S_\Lambda(-1-p)[qr+k-1],\ p = 0, \dots, m-1, m > 0,\\
X^{(k)}_{q(r+m)-p,q(r+m)-p} &= P_\Lambda(n-p)[qr+k-p],\ p = 1, \dots, r-1,
\end{aligned}$$

$k = 0, \dots, r-1$, $q \in \mathbb{Z}$. In particular, $X^{(k)}_{i,j}[r] = \tau^{-m-r} X^{(k)}_{i,j}$ for any $k = 0, \dots, r-1$, $i, j \in \mathbb{Z}$, $i \leq j$.

Similarly as above, one can show that for $r < n$ each module of one of the forms $M_{i,k}$, $k = 1, \dots, n-r$, $M^*_{\alpha_{i,k}}$, $k = n-r+1, \dots, n-1$, $M_{\omega_{i,k}}$, $k = 1, \dots, n-r+1$, is the shift of one of the modules $S_\Lambda(k)$, $k = 1, \dots, n-r$. On the other hand, if $r = n$ we have

$$M_{\alpha^*_{i,k}}[r] = M_{\alpha^*_{i,k}},\ k = n-r+1, \dots, n-1,\ r \geq 2,$$

$$M_{\omega_{i,1}}[r] = M_{\omega_{i,1}},$$

and

$$\begin{aligned}
\tau M_{\alpha^*_{i,k}} &= M_{\alpha^*_{i-1,k+2}},\ k = n-r+1,\dots,n-3,\ r \geq 4,\\
\tau M_{\alpha^*_{i,n-2}} &= M_{\omega_{i-1,1}},\ r \geq 3,\\
\tau M_{\alpha^*_{i,n-1}} &= M_{\alpha^*_{i-1,n-r+1}},\ r \geq 2,\\
\tau M_{\omega^*_{i,1}} &= \begin{cases} M_{\omega_{i-1,1}} & r = 1,2 \\ M_{\alpha^*_{i-1,n-r+2}} & r \geq 3 \end{cases}.
\end{aligned}$$

Hence, we get the following information about the components $\mathcal{Y}^{(k)}$ using the assumption (4).

Lemma 3.2. If $r < n$ then we have the following formulas

$$Y^{(k)}_{q(n-r)+p,q(n-r)+p} = S_\Lambda(n-r-p)[r-2-qr+k],\ p = 0,\dots,n-r-1,$$

If $r = n$ then the modules $Y^{(k)}_{i,i}$, $k = 0,\dots,r-1$, $i \in \mathbb{Z}$, coincide with the modules $M_{\alpha^*_{i,k}}$, $k = n-r+1,\dots,n-1$, $M_{\omega_{i,1}}$, $i \in \mathbb{Z}$. In both cases we get $Y^{(k)}_{i,j}[r] = \tau^{n-r} Y^{(k)}_{i,j}$ for any $k = 0,\dots,r-1$, $i,j \in \mathbb{Z}$, $i \geq j$.

Part (i) of Theorem B follows immediately from the above lemmas, since the Happel functor F is an equivalence if $r < n$. For part (ii) note first that the components $\mathcal{X}^{(0)}$, …, $\mathcal{X}^{(r-1)}$ are contained in the image of F. It follows, because each module $X^{(k)}_{i,i}$ is the shift of a Λ-module and we the $X^{(k)}_{i,j}$, $i \neq j$, are iterated extension of some $X^{(k)}_{l,l}$.

On the other hand, we have $Y[r] \simeq Y$ for $Y \in \mathcal{Y}^{(0)} \vee \cdots \vee \mathcal{Y}^{(r-1)}$, hence the components $\mathcal{Y}^{(0)}$, …, $\mathcal{Y}^{(r-1)}$ are not contained in the image of F. Using triangles

$$\begin{gathered}
X^{(k)}_{i,-1} \longrightarrow Z^{(k)}_{i,0} \longrightarrow Z^{(k)}_{0,0} = S_\Lambda(0)[k] \longrightarrow X^{(k)}_{i,0}[1],\ i < 0,\\
X^{(k)}_{0,i} \longrightarrow Z^{(k)}_{0,0} = S_\Lambda(0)[k] \longrightarrow Z^{(k)}_{i+1,0} \longrightarrow X^{(k)}_{0,i}[1],\ i \geq 0,
\end{gathered}$$

we get that the modules $Z^{(k)}_{i,0}$, $k = 0,\dots,r-1$, $i \in \mathbb{Z}$, belong to the image of F. Finally, using triangles

$$\begin{gathered}
Y^{(k)}_{-1,j} \to Z^{(k)}_{i,j} \to Z^{(k)}_{i,0} \to Y^{(k)}_{-1,j}[1],\ j > 0, j < 0, i \in \mathbb{Z},\\
Y^{(k)}_{j-1,0} \to Z^{(k)}_{i,0} \to Z^{(k)}_{i,j} \to Y^{(k)}_{j-1,0}[1],\ j > 0, i \in \mathbb{Z},
\end{gathered}$$

we obtain that the modules $Z^{(k)}_{i,j}$, $k = 0,\dots,r-1$, $i,j \in \mathbb{Z}$, $j \neq 0$, do not belong to the image of F.

4 Properties of the Euler form

Fix $(r,n,m) \in \Omega_f$ and put $\Lambda = \Lambda(r,n,m)$. We will also use notation introduced in the previous section. Our aim in this section is to describe the properties of the Euler form $\chi = \chi_\Lambda$ and dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$. We put $\langle -,-\rangle = \langle -,-\rangle_\Lambda$. One can easily calculate that

$$\langle \mathbf{x},\mathbf{y}\rangle_\Lambda = \sum_{i=-m}^{n-1} x_i y_i - \sum_{i=-m}^{n-1} x_i y_{i+1} + \sum_{k=2}^{r+1}\big[(-1)^k \sum_{i=n-r}^{n+1-k} x_i y_{i+k}\big],$$

where $y_n = y_0$ and $y_{n+1} = y_1$. Consequently

$$\chi_\Lambda(\mathbf{x}) = \langle \mathbf{x},\mathbf{x}\rangle_\Lambda = \sum_{i=-m}^{n-1} x_i^2 - \sum_{i=-m}^{n-1} x_i x_{i+1} + \sum_{k=2}^{r+1}\big[(-1)^k \sum_{i=n-r}^{n+1-k} x_i x_{i+k}\big],$$

where $x_n = x_0$ and $x_{n+1} = x_1$.

We introduce the following notation:

$$\begin{aligned}
\mathbf{s}_i &= -\operatorname{\mathbf{dim}} X_{i,i}^{(0)},\ i = 0,\dots,m+r-1,\\
\mathbf{t}_i &= -\operatorname{\mathbf{dim}} Y_{i,i}^{(0)},\ i = 0,\dots,n-r-1,\\
\mathbf{h}_1 &= \mathbf{s}_0 + \cdots + \mathbf{s}_{m+r-1},\\
\mathbf{h}_2 &= \mathbf{t}_0 + \cdots + \mathbf{t}_{n-r-1}.
\end{aligned}$$

Since the objects $X_{i,i}^{(0)}$ and $Y_{i,i}^{(0)}$ have been described in Lemmas 3.1 and 3.2 we can give more direct formulas for $\mathbf{s}_i$ and $\mathbf{t}_i$. In particular, we have $\mathbf{h}_2 = \mathbf{h}_1$ if r is even. We will write just $\mathbf{h}$ for this common value in this case. If r is odd then $\mathbf{h}_2 = -\mathbf{h}_1 - 2\mathbf{e}_0$, where $\mathbf{e}_i = \operatorname{\mathbf{dim}} S_\Lambda(i)$, $i = -m,\dots,n-1$. Moreover, we get the following basis in $K_0(\Lambda)$

$$\begin{aligned}
\mathbf{d}_1 &= \mathbf{e}_0,\\
\mathbf{d}_i &= \mathbf{s}_{i-2},\ i = 2,\dots,m+r,\\
\mathbf{d}_i &= \mathbf{t}_{i-m-r-1},\ i = m+r+1,\dots,m+n-1,\\
\mathbf{d}_{m+n} &= \mathbf{h}_1.
\end{aligned}$$

In order to describe the dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ we introduce the following construction. The shift functor $T\colon D^b(\operatorname{mod}\Lambda) \to D^b(\operatorname{mod}\Lambda)$ acts on $\Gamma(D^b(\operatorname{mod}\Lambda))$ in a natural way. Let $\Sigma = \Sigma_\Lambda$ be the quiver obtained from $\Gamma(D^b(\operatorname{mod}\Lambda))$ by dividing by T^2. Since $\operatorname{\mathbf{dim}} X = \operatorname{\mathbf{dim}} X[2]$ with each vertex x of Σ we can associate the dimension vector of the corresponding object of $D^b(\operatorname{mod}\Lambda)$, which we will call the dimension vector of x.

Assume first r is even. Recall, we assumed that $X_{i,j}^{(k)}[1] = X_{i,j}^{(k+1)}$, $Y_{i,j}^{(k)}[1] = Y_{i,j}^{(k+1)}$ and $Z_{i,j}^{(k)}[1] = Z_{i,j}^{(k+1)}$, $k = 0,\dots,r-2$. Finally, from Lemmas 3.1 and 3.2 we get $X_{i,j}^{(r-1)}[1] = X_{i+r+m,j+r+m}^{(0)}$ and $Y_{i,j}^{(r-1)}[1] = Y_{i-r+n,i-r+n}^{(0)}$. As the consequence, using triangle (1) and (2) we obtain that $Z_{i,j}^{(r-1)}[1] = Z_{i+r+m,j-r+n}^{(0)}$. Hence, we get that in this case Σ is the disjoint union of four stable tubes, two of them of rank $m+r$ and two of them of rank $n-r$, and two components of type $\mathbb{Z}\tilde{\mathbb{A}}_{n-r,m+r}$. The dimension vectors of vertices lying on the mouth of tubes of rank $m+r$ are by definition $\mathbf{s}_0, \dots, \mathbf{s}_{m+r-1}$ and $-\mathbf{s}_0, \dots, -\mathbf{s}_{m+r-1}$, respectively, while the dimension vectors of vertices lying on the mouth of tubes of rank

$n-r$ are $\mathbf{t}_0, \dots, \mathbf{t}_{n-r-1}$ and $-\mathbf{t}_0, \dots, -\mathbf{t}_{n-r-1}$. Finally, using the triangles (1) and (2) for $i = 0 = j$ we get in the components of type $\mathbb{Z}\tilde{\mathbb{A}}_{n-r,m+r}$ sections of the forms

$$\begin{array}{ccc}
\mathbf{e}_0 \to \mathbf{e}_0 + \mathbf{s}_0 \to \cdots \to \mathbf{e}_0 + \mathbf{s}_0 + \cdots + \mathbf{s}_{m+r-2} & & \\
\searrow \qquad\qquad\qquad\qquad\qquad\qquad \searrow & & \\
\qquad \mathbf{e}_0 + \mathbf{t}_0 \to \cdots \to \mathbf{e}_0 + \mathbf{t}_0 + \cdots + \mathbf{t}_{n-r-2} \to \mathbf{e}_0 + \mathbf{h} & &
\end{array}$$

and

$$\begin{array}{ccc}
-\mathbf{e}_0 \to -\mathbf{e}_0 - \mathbf{s}_0 \to \cdots \to -\mathbf{e}_0 - \mathbf{s}_0 - \cdots - \mathbf{s}_{m+r-2} & & \\
\searrow \qquad\qquad\qquad\qquad\qquad\qquad \searrow & & , \\
\qquad -\mathbf{e}_0 - \mathbf{t}_0 \to \cdots \to \ \mathbf{e}_0 - \mathbf{t}_0 - \cdots - \mathbf{t}_{n-r-2} \ \to -\mathbf{e}_0 - \mathbf{h} & &
\end{array}$$

respectively, where we replaced the vertices by their dimension vectors.

We get the following description of dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ in this case.

Lemma 4.1. If r is even and X is an indecomposable object in the derived category $D^b(\operatorname{mod}\Lambda)$ then $\mathbf{dim}\, X$ is of one of the forms

$$p\mathbf{h},\ p \in \mathbb{Z},$$
$$p\mathbf{h} + \sum_{i=k}^{k+l-1} \mathbf{s}_i,\ 0 \le k \le m+r-1, 0 < l \le m+r-1, p \in \mathbb{Z},$$
$$p\mathbf{h} + \sum_{i=k}^{k+l-1} \mathbf{t}_i,\ 0 \le k \le n-r-1, 0 < l \le n-r-1, p \in \mathbb{Z},$$
$$\pm(\mathbf{e}_0 + p\mathbf{h} + \sum_{i=0}^{k-1} \mathbf{s}_i + \sum_{i=0}^{l-1} \mathbf{t}_i),\ 0 \le k \le m+r-1, 0 \le l \le n-r-1, p \in \mathbb{Z},$$

where $\mathbf{s}_{m+r+i} = \mathbf{s}_i$ and $\mathbf{t}_{n-r+i} = \mathbf{t}_i$. On the other hand, if $\mathbf{x}$ is one of the above dimension vectors then:

(a) there exist up to shift $n+m$ isomorphism classes of indecomposable objects X in $D^b(\operatorname{mod}\Lambda)$ such that $\mathbf{dim}\, X = \mathbf{x}$ if $\mathbf{x} = p\mathbf{h}$, $p \in \mathbb{Z}$,

(b) there exists a uniquely determined up to shift indecomposable object X in $D^b(\operatorname{mod}\Lambda)$ such that $\mathbf{dim}\, X = \mathbf{x}$, otherwise.

Proof 4.2. It follows from the well-known properties of stable tubes and quivers of the form $\mathbb{Z}\tilde{\mathbb{A}}_{p,q}$.

Suppose now r is odd. Similarly as above we get now that the quiver Σ consists of two tubes of ranks $2(m+r)$ and $2(n-r)$, respectively, and one component of type $\mathbb{Z}\tilde{\mathbb{A}}_{2(n-r),2(m+r)}$. The vertices lying on the mouth of the tube of rank $2(m+r)$ have dimension vectors $\mathbf{s}_0, \dots, \mathbf{s}_{m+r-1}, -\mathbf{s}_0, \dots, -\mathbf{s}_{m+r-1}$, the vertices lying on the mouth of the tube of rank $2(n-r)$ have dimension vectors $\mathbf{t}_0, \ldots, \mathbf{t}_{n-r-1}, -\mathbf{t}_0, \ldots, \mathbf{t}_{n-r-1}$, and in the component of type $\mathbb{Z}\mathbb{A}_{2(n+m)}$ we have a section

$$\begin{array}{ccc}
\mathbf{e}_0 & \to \mathbf{e}_0+\mathbf{s}_0 \to \cdots \to \mathbf{e}_0+\mathbf{s}_0+\cdots+\mathbf{s}_{m+r-2} \to & \mathbf{e}_0+\mathbf{h}_1 \\
\downarrow & & \downarrow \\
\mathbf{e}_0+\mathbf{t}_0 & & \mathbf{e}_0+\mathbf{s}_1+\cdots+\mathbf{s}_{m+r-1} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathbf{e}_0+\mathbf{t}_0+\cdots+\mathbf{t}_{n-r-2} & & \mathbf{e}_0+\mathbf{s}_{m+r-1} \\
\downarrow & & \downarrow \\
\mathbf{e}_0+\mathbf{h}_2 & \to \mathbf{e}_0+\mathbf{t}_1+\cdots+\mathbf{t}_{n-r-1} \to \cdots \to \mathbf{e}_0+\mathbf{t}_{n-r-1} \to & \mathbf{e}_0
\end{array},$$

where again we replaced vertices by their dimension vectors.

By the same arguments as above we get the following.

Lemma 4.3. If r is odd and X is an indecomposable object in $D^b(\operatorname{mod}\Lambda)$ then $\mathbf{dim}\,X$ is of one of the forms

$$\pm\sum_{i=k}^{k+l-1}\mathbf{s}_i,\ 0\le k\le m+r-1, 0<l\le m+r-1,$$
$$\pm\sum_{i=k}^{k+l-1}\mathbf{t}_i,\ 0\le k\le n-r-1, 0<l\le (n-r)-1,$$
$$\mathbf{e}_0+\sum_{i=0}^{k-1}\mathbf{s}_i+\sum_{i=0}^{l-1}\mathbf{t}_i,\ 0\le k\le 2(m+r)-1, 0\le l\le 2(n-r)-1,$$

and 0, where $\mathbf{s}_{m+r+i}=-\mathbf{s}_i$, $\mathbf{s}_{2(m+r)+i}=\mathbf{s}_i$, $i=0,\ldots,m+r-1$, $\mathbf{t}_{n-r+i}=-\mathbf{s}_i$, $\mathbf{s}_{2(n-r)+i}=\mathbf{s}_i$, $i=0,\ldots,n-r-1$.

On the other hand, if $\mathbf{x}$ is one of the above dimension vectors then there exists up to shift infinitely many indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ with dimension vector $\mathbf{x}$.

Let $\Gamma=\Gamma_{m+r,n-r}$ be the path algebra of the quiver

$$\begin{array}{ccccc}
 & 2 & \leftarrow\cdots\leftarrow & m+r & \\
\swarrow & & & & \nwarrow \\
1 & & & & n+m\cdot \\
\nwarrow & & & & \swarrow \\
 & m+r+1 & \leftarrow\cdots\leftarrow & n+m-1 &
\end{array}$$

We have the following.

Lemma 4.4. Assume r is even. Let $\sigma\colon K_0(\Lambda) \to K_0(\Gamma)$ be the map given by

$$\sigma(\mathbf{d}_i) = \mathbf{dim}\, S_\Gamma(i),\ i = 1, \dots, m+n-1,$$
$$\sigma(\mathbf{d}_{m+n}) = \sum_{j=1}^{m+n} \mathbf{dim}\, S_\Gamma(j).$$

Then σ is the isomorphism of $K_0(\Lambda)$ and $K_0(\Gamma)$ such that $\langle \sigma\mathbf{x}, \sigma\mathbf{y}\rangle_\Gamma = \langle \mathbf{x}, \mathbf{y}\rangle$ for $\mathbf{x}, \mathbf{y} \in K_0(\Lambda)$.

Proof 4.5. It is easily to check by direct calculations that the vectors $\sigma\mathbf{d}_i$, $i = 1, \dots, m+n$, form a basis of $K_0(\Gamma)$ and $\langle \sigma\mathbf{d}_i, \sigma\mathbf{d}_j\rangle_\Gamma = \langle \mathbf{d}_i, \mathbf{d}_j\rangle_\Lambda$, $i, j = 1, \dots, m+n$.

The following description of χ is the immediate consequence of the above lemma.

Corollary 4.6. If r is even then χ is $\mathbb{Z}$-equivalent to the form of the Euclidean diagram of type $\tilde{\mathbb{A}}_{n+m-1}$. In particular, χ is positive semidefinite with corank 1 and of Dynkin type $\mathbb{A}_{n+m-1}$.

We also get the following description of dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ in terms of the Euler form.

Proposition 4.7. Let r be even. If $\mathbf{x}$ is the dimension vector of an indecomposable object in $D^b(\operatorname{mod}\Lambda)$ then $\chi_\Lambda(\mathbf{x}) \in \{0, 1\}$. On the other hand, given $\mathbf{x} \in K_0(\Lambda)$ we have:

(a) if $\chi(\mathbf{x}) = 0$ then there exist up to shift $n+m$ isomorphism classes of indecomposable objects X in $D^b(\operatorname{mod}\Lambda)$ such that $\mathbf{dim}\, X = \mathbf{x}$,

(b) if $\chi(\mathbf{x}) = 1$ then there exists a uniquely determined up to shift indecomposable object X in $D^b(\operatorname{mod}\Lambda)$ such that $\mathbf{dim}\, X = \mathbf{x}$.

Proof 4.8. The proposition follows from the description of dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ presented in Lemma 4.1, the formula for the isomorphism $\sigma\colon K_0(\Lambda) \to K_0(\Gamma)$ given in Lemma 4.4, and well-know description of 0-roots and 1-roots of the form χ_Γ.

Now we turn our attention to the case r odd.

Proposition 4.9. If r is odd then χ is $\mathbb{Z}$-equivalent to the form of the Dynkin diagram of type $\mathbb{D}_{n+m}$, hence is positive definite.

Proof 4.10. Since r is odd we can rewrite χ in the form

$$\chi(\mathbf{x}) = \frac{1}{2}\Big[x_{-m}^2 + \sum_{i=-m}^{-1} (x_i - x_{i+1})^2 + \sum_{i=1}^{n-r-1} (x_i - x_{i+1})^2 + \sum_{i=n-r+1}^{n-1} x_i^2 + (x_{n-r} - x_{n-r+1} + \cdots + x_1)^2\Big].$$

Hence $\chi(\mathbf{x}) \geq 0$ and $\chi(\mathbf{x}) = 0$ if and only if the following equations are satisfied

$$\begin{aligned} x_{-m} &= 0, \\ x_i - x_{i+1} &= 0,\ i = -m, \dots, -1, \\ x_i - x_{i+1} &= 0,\ i = 1, \dots, n-r-1, \\ x_i &= 0,\ i = n-r+1, \dots, n-1, \\ x_{n-r} - x_{n-r+1} + \cdots + x_1 &= 0. \end{aligned}$$

As a consequence we get $x_{-m} = \cdots = x_0 = 0$ and there exists $a \in \mathbb{Z}$ such that $x_i = a$, $i = 1, \dots, n-r$. Finally, taking into account the last equation, we get $2a = 0$, and so $a = 0$. Hence χ is positive definite.

Using the same arguments as above we can show that, for each $\mathbf{a} \in \mathbb{Z}^{n+m}$ with $\sum_{i=1}^{n+m} a_i$ is even, there exists a unique solution $\mathbf{x} \in \mathbb{Z}^{n+m}$ of the system

$$\begin{aligned} x_{-m} &= a_1, \\ x_i - x_{i+1} &= a_{i+m+2},\ i = -m, \dots, -1, \\ x_i - x_{i+1} &= a_{i+m+1},\ i = 1, \dots, n-r-1, \\ x_i &= a_{i+m},\ i = n-r+1, \dots, n-1, \\ x_{n-r} - x_{n-r+1} + \cdots + x_1 &= a_{n+m}. \end{aligned} \tag{5}$$

In particular, χ has exactly $2(n+m-1)(n+m)$ roots. Indeed, $\chi(\mathbf{x}) = 1$ if and only if $\mathbf{x}$ is a solution of the system (5), where $|a_k| = |a_l| = 1$ for some $k < l$ and $a_i = 0$, $i \neq k, l$. As the consequence we get that χ is of type $\mathbb{D}_{n+m}$ since the Dynkin type of a positive definite form is uniquely determined by the number of roots.

The connection of the Euler form χ with dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ is described by the following proposition.

Proposition 4.11. Let r be odd. If $\mathbf{x}$ is a dimension vector of an indecomposable object in $D^b(\operatorname{mod}\Lambda)$, then $\chi(\mathbf{x}) \in \{0, 1, 2\}$. Moreover, for each 1-root $\mathbf{x}$ of χ, there exists an indecomposable object X in $D^b(\operatorname{mod}\Lambda)$ such that $\mathbf{dim}\, X = \mathbf{x}$.

Proof 4.12. Since we have a description of the dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ given in Lemma 4.3, by direct calculations we obtain

$$\begin{aligned} &\chi(0) = 0, \\ &\chi(\pm \sum_{i=k}^{k+l-1} \mathbf{s}_i) = 1,\ 0 \leq k \leq m+r-1, 0 < l < m+r-1, \\ &\chi(\pm \sum_{i=k}^{k+m+r-1} \mathbf{s}_i) = 2,\ 0 \leq k \leq m+r-1, \end{aligned}$$

$$\chi(\pm\sum_{i=k}^{k+l-1}\mathbf{t}_i)=1,\ 0\le k\le n-r-1, 0<l<n-r-1,$$
$$\chi(\pm\sum_{i=k}^{k+n-r-1}\mathbf{s}_i)=2,\ 0\le k\le n-r-1,$$
$$\chi(\mathbf{e}_0+\sum_{i=0}^{k-1}\mathbf{s}_i+\sum_{i=0}^{l-1}\mathbf{t}_i)=1,\ 0\le k\le 2(m+r)-1, 0\le l\le 2(n-r)-1,$$

where $\mathbf{s}_{m+r+i}=-\mathbf{s}_i$, $\mathbf{s}_{2(m+r)+i}=\mathbf{s}_i$, $i=0,\dots,m+r-1$, $\mathbf{t}_{n-r+i}=-\mathbf{s}_i$, $\mathbf{s}_{2(n-r)+i}=\mathbf{s}_i$, $i=0,\dots,n-r-1$, and hence the first part follows. The second part also follows, since we have exactly $2(m+r)(m+r-1)$ different dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ which are 1-roots.

The statement of the above proposition is not true for 2-roots of χ, that is in general not each 2-root is a dimension vector of an indecomposable object in $D^b(\operatorname{mod}\Lambda)$. Indeed, we have $2^4\binom{m+n}{4}+2(m+n)$ 2-roots of χ, while there are only $2(m+n)$ dimension vectors of indecomposable objects in $D^b(\operatorname{mod}\Lambda)$ which are 2-roots (these numbers are equal if and only if $m+n<4$).

We finish our consideration by pointing out how much information can be derived from the knowledge of the Auslander–Reiten quiver and the bilinear Ringel from. Let $\Phi=\Phi_\Lambda$ be the Coxeter transformation of Λ. Moreover, for nonzero integers a and b, denote by $\gcd(a,b)$ the greatest common divisor of a and b, and by $\operatorname{lcm}(a,b)$ the least common multiplicity of a and b.

Lemma 4.13. Let r be odd. Then there are $m+n-2+2\gcd(m+r,n-r)$ Φ-orbits of 1-roots of χ. There are $m+r-1$ Φ-orbits with $2(m+r)$ elements, $n-r-1$ Φ-orbits with $2(n-r)$ elements and $2\gcd(m+r,n-r)$ Φ-orbits with $2\operatorname{lcm}(m+r,n-r)$ elements.

Proof 4.14. Using the formula $\Phi(\mathbf{dim}\,X)=\mathbf{dim}\,\tau_{D^b(\operatorname{mod}\Lambda)}X$, which holds for any object $X\in D^b(\operatorname{mod}\Lambda)$, and the knowledge of the Auslander–Reiten quiver $D^b(\operatorname{mod}\Lambda)$ we easily get the following

$$\Phi\mathbf{s}_0=-\mathbf{s}_{m+r-1},$$
$$\Phi\mathbf{s}_i=\mathbf{s}_{i-1},\ i=1,\dots,m+r-1,$$
$$\Phi\mathbf{t}_0=-\mathbf{t}_{n-r-1},$$
$$\Phi\mathbf{t}_i=\mathbf{t}_{i-1},\ i=1,\dots,n-r-1.$$

It follows immediately from the above formulas that, for each $l=1,\dots,m+r-1$, $l\ne m+r$, the vectors $\sum_{i=k}^{k+l-1}\mathbf{s}_i$, $k=0,\dots,2(m+r)-1$, form a Φ-orbit, and, for each $l=1,\dots,n-r-1$, the vectors $\sum_{i=k}^{k+l-1}\mathbf{t}_i$, $k=0,\dots,2(n-r)-1$, form a Φ-orbit. Recall that we use the convention $\mathbf{s}_{m+r+i}=-\mathbf{s}_i$, $\mathbf{s}_{2(m+r)+i}=\mathbf{s}_i$, $i=0,\dots,m+r-1$, $\mathbf{t}_{n-r+i}=-\mathbf{s}_i$, $\mathbf{s}_{2(n-r)+i}=\mathbf{s}_i$, $i=0,\dots,n-r-1$.

In order to analyze the action of Φ on the dimension vectors of the form $\mathbf{e}_0+\sum_{i=0}^{k-1}\mathbf{s}_i+$

$\sum_{i=0}^{l-1} \mathbf{t}_i$ note that

$$\Phi \mathbf{e}_0 = \mathbf{e}_0 + \mathbf{s}_{m+r-1} + \mathbf{t}_{n-r-1} = \mathbf{e}_0 + \sum_{i=0}^{2(m+r)-2} \mathbf{s}_i + \sum_{i=0}^{2(n-r)-2} \mathbf{t}_i,$$

since $\Phi^{-1}\mathbf{e}_0 = \mathbf{e}_0 + \mathbf{s}_0 + \mathbf{t}_0$. Consequently, we get

$$\Phi(\mathbf{e}_0 + \sum_{i=0}^{k-1} \mathbf{s}_i) = \mathbf{e}_0 + \sum_{i=0}^{k-2} \mathbf{s}_i + \sum_{i=0}^{n-r-1} \mathbf{t}_i, \ k \geq 1,$$
$$\Phi(\mathbf{e}_0 + \sum_{i=0}^{l-1} \mathbf{t}_i) = \mathbf{e}_0 + \sum_{i=0}^{m+r-2} \mathbf{s}_i + \sum_{i=0}^{l-2} \mathbf{t}_i, \ l \geq 1,$$
$$\Phi(\mathbf{e}_0 + \sum_{i=0}^{k-1} \mathbf{s}_i + \sum_{i=0}^{l-1} \mathbf{t}_i) = \mathbf{e}_0 + \sum_{i=0}^{k-2} \mathbf{s}_i + \sum_{i=0}^{l-2} \mathbf{t}_i, \ l, k \geq 1.$$

Note that the above dimension vectors are in a natural correspondence with the elements of the set $\mathcal{R} = \mathcal{R}_{2(m+r),2(n-r)} = \{0, \dots, 2(m+r)-1\} \times \{0, \dots, 2(n-r)-1\}$. According to the above formulas the action of Φ induces the action on $\mathcal{R}$ given by the formula $(i, j) \mapsto (i-1, j-1)$, where the result on the first coordinate is taken modulo $m+r$ and the result on the second coordinate is taken module $n-r$. It is an easy combinatorics to notice that this action has exactly $\gcd(2(m+r), 2(n-r)) = 2\gcd(m+r, n-r)$ orbits, each of them with $\frac{4(m+r)(n-r)}{2\gcd(m+r,n-r)} = 2\,\mathrm{lcm}(m+r, n-r)$ elements.

Note that it follows from the above lemma that in general the bilinear form $\langle -, - \rangle$ is not $\mathbb{Z}$-equivalent to the form $\langle -, - \rangle_D$, where D is a hereditary algebra of type $\mathbb{D}_{n+m}$. Indeed, there are $m+n$ orbits of the action of Φ_D on 1-roots of χ_D and each orbit has exactly $2(m+n-1)$ elements.

Proposition 4.15. If $(r', n', m') \in \Omega_f$ then the bilinear forms $\langle -, - \rangle$ and $\langle -, - \rangle_{\Lambda(r',n',m')}$ are $\mathbb{Z}$-equivalent if and only if $r \equiv r' \pmod 2$ and either $m+r = m'+r'$ and $n-r = n'-r'$ or $m+r = n'-r'$ and $n-r = m'+r'$.

Proof 4.16. It follows from Corollary 4.6 and Proposition 4.9 that the bilinear forms $\langle -, - \rangle$ and $\langle -, - \rangle_{\Lambda(r',n',m')}$ can be $\mathbb{Z}$-equivalent only if $r \equiv r' \pmod 2$. If r and r' are even then the claim follows from Lemma 4.4, since the bilinear forms of the algebras $\Gamma_{p,q}$ and $\Gamma_{p',q'}$ are $\mathbb{Z}$-equivalent if and only if either $p = p'$ and $q = q'$ or $p = q'$ and $q = p'$.

Assume now that both r and r' are odd. If neither one of the conditions formulated in the proposition is satisfied then using the previous lemma we get that the actions of the corresponding Coxeter transformations on 1-roots differ, hence the forms $\langle -, - \rangle$ and $\langle -, - \rangle_{\Lambda(r',n',m')}$ cannot be $\mathbb{Z}$-equivalent.

Finally, assume that either $m+r = m'+r'$ and $n-r = n'-r'$ or $m+r = n'-r'$ and $n-r = m'+r'$. Then $n'+m' = n+m$. If $m+r = m'+r'$ and $n-r = n'-r'$ then the map $G\colon K_0(\Lambda) \to K_0(\Lambda(r', n', m'))$ given by $G(\mathbf{d}_i) = \mathbf{d}'_i$ is an isomorphism of abelian groups such that $\langle G\mathbf{x}, G\mathbf{y} \rangle_{\Lambda(r',n',m')} = \langle \mathbf{x}, \mathbf{y} \rangle$, where $\mathbf{d}'_1, \dots, \mathbf{d}'_{n+m}$ is the basis of $K_0(\Lambda(r', n', m'))$

defined in the analogous way as the basis $\mathbf{d}_1, \ldots, \mathbf{d}_{n+m}$ of $K_0(\Lambda)$. Similarly, if $m + r = n' - r'$ and $n - r = m' + r'$ then we define the map $H\colon K_0(\Lambda) \to K_0(\Lambda(r', n', m'))$ by the formulas

$$\begin{aligned} H(\mathbf{d}_1) &= \mathbf{d}'_1, \\ H(\mathbf{d}_i) &= \mathbf{d}'_{m'+r'+i-1},\ i = 2, \ldots, m + r, \\ H(\mathbf{d}_i) &= \mathbf{d}'_{i-m-r+1},\ i = m + r + 1, \ldots, m + n - 1, \\ H(\mathbf{d}_{n+m}) &= -\mathbf{d}'_{n+m} - 2\mathbf{d}'_1. \end{aligned}$$

A direct checking shows that H is the required isomorphism.

An important information which follows from the above proposition is the following. Given $(r', n', m') \in \Omega_f$ such that the Auslander–Reiten quivers of $D^b(\operatorname{mod}\Lambda)$ and $D^b(\Lambda(r, n, m))$ are isomorphic as the translation quivers, and the bilinear forms $\langle -, - \rangle$ and $\langle -, - \rangle_{\Lambda(r', n', m')}$ are $\mathbb{Z}$-equivalent, then either $(r', n', m') = (r, n, m)$ or $(r', n', m') = (r, m + 2r, n - 2r)$. Obviously the second possibility may appear only if $n \geq 2r$.

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