Geometric methods in representation theory of finite dimensional algebras

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1. Introduction

It is natural to study some problems in representation theory from a geometric point of view. Indeed, for a given finite dimensional k-algebra Λ and $z \in \mathbb{N}$, the possible Λ-module structures on k^z may be viewed as solutions of certain algebraic equations determined by the structure constants of Λ. In fact this idea is almost as old as "modern" representation theory, see [**Ar**], [**Vo**]. Further investigations in this direction include [**Ga**], [**Ma**], [**P1**]–[**P4**], [**Sc1**], [**MeS**], as well as some work focussed on the representation theory of quivers like [**Kc1**]–[**Kc3**], [**Sc2**], [**CB3**]. Representation theory also motivated the analysis more geometric questions, for example [**Bo3**]–[**Bo6**], see these works also for further references. The "roots" of the the present paper are [**Ga**], [**Dr**] and [**P1**]–[**P4**]. We want to give here an overview of some newer results in this direction. For some of these results it is convenient to look at the corresponding schemes rather than to the more popular varieties. Thus the object of our interest will be the schemes alg_d of d-dimensional associative algebra structures and mod_Λ^z of z-dimensional Λ-module structures, both over some algebraically closed field k. Indeed, since these schemes are defined more naturally than the varieties, we have a closer relation between their local structure and homological data, the price is, that we have to deal with non-reduced structures which we still can not interpret satisfactory.

Since on the schemes alg_d and mod_Λ^z operate some general linear groups by transport of structure, the functorial point of view for schemes as it was introduced in [**DeGa**] is the adequate one. For the convenience of the reader we try to give a minimal introduction to this language in section 2, which is based on [**Ja**]. We recommend this last book also as reference for more precise information.

In section 3 we study the relation between the homological invariants $\mathrm{Ext}_\Lambda^i(M, M)$ and the local structure of mod_Λ^z at the corresponding point. These results are based on elementary semicontinuity considerations and deformation theory. Also we present similar results for the relation between Hochschild cohomology groups and alg_d, see [**GeP**]. It seems, that these results are at least partially well-known folklore, but we know no adequate reference.

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In section 4 our starting point is the classical result of Gabriel [**Ga**] that finite representation type is open; this implies that deformations (see 3.5.3) of representation finite algebras are also of finite representation type. We will see, that in this last statement "finite representation type" may be replaced by "tame". Let us point out already here, that our result is based heavily on the tame-wild theorem of Drozd in a similar way as the result of Gabriel is based on Brauer-Thrall II.

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2. Schemes and algebraic groups

Let k be an algebraically closed field; we should point out however, that for most of the theory presented here, k could be also a commutative ring.

2.1. **k-functors.** A k-*functor* is a functor from the category k-alg of commutative k-algebras to the category of sets. Let us admit, that this definition is somehow sloppy since we want to consider also the *category* with these functors as objects, and therefore the morphisms between them must be *sets*, thus we should replace k-alg by something smaller, see [**DeGa**, Conventions Générales]. A *subfunctor* of a k-functor X is a k-functor Y with $\mathrm{Y}(A) \subset \mathrm{X}(A)$ and $\mathrm{Y}(\varphi) = \mathrm{X}(\varphi)|_{\mathrm{Y}(A)}$ for all k-algebras A, A' and all $\varphi \in$ k-alg(A, A'). The morphisms $\mathrm{Mor}(\mathrm{X}, \mathrm{Y})$ between two k-functors are the corresponding natural transformations. For a diagram $\mathrm{X}_1 \xrightarrow{f_1} \mathrm{S} \xleftarrow{f_2} \mathrm{X}_2$ of k-functors we define the fibred product $\mathrm{X}_1 \times_{\mathrm{S}} \mathrm{X}_2$ by $(\mathrm{X}_1 \times_{\mathrm{S}} \mathrm{X}_2)(A) = \{(x_1, x_2) \in \mathrm{X}_1(A) \times \mathrm{X}_2(A) \mid f_1(A)(x_1) = f_2(A)(x_2)\}$. Special cases include the direct product and fibres of morphisms.

2.2. **Affine schemes.** The representable functors $\mathrm{Sp}_{\mathrm{k}}R :=$ k-alg$(R, -)$ are called *affine schemes*; if R is a finitely generated k-algebra we call $\mathrm{Sp}_{\mathrm{k}}R$ an *affine algebraic scheme*, if R contains no nilpotent elements, $\mathrm{Sp}_{\mathrm{k}}R$ is called *reduced*. We write $\mathbb{A}^n := \mathrm{Sp}_{\mathrm{k}}\mathrm{k}[T_1, \dots, T_n]$ and observe $\mathbb{A}^n(R) \cong R^n$ for all $R \in$ k-alg. For any k-functor X we have the Yoneda-isomorphism $\mathrm{Mor}(\mathrm{Sp}_{\mathrm{k}}R, \mathrm{X}) \xrightarrow{\sim} \mathrm{X}(R)$. We set $\mathrm{k}[\mathrm{X}] := \mathrm{Mor}(\mathrm{X}, \mathbb{A}^1)$ and observe that $\mathrm{k}[\mathrm{Sp}_{\mathrm{k}}R] \xrightarrow{\sim} R$. For an affine scheme X the *closed* (resp. *open*) *subfunctors* of X are by definition of the form $V(I)$ (resp. $D(I)$) for some ideal I of k[X], where

$$V(I)(A) := \{x \in \mathrm{X}(A) \mid f(A)(x) = 0 \ \forall f \in I\},$$
$$D(I)(A) := \{x \in \mathrm{X}(A) \mid \sum_{f \in I} A f(A)(x) = A\}.$$

Remark . (1) $V(I) \cong \mathrm{Sp}_{\mathrm{k}}\mathrm{k}[\mathrm{X}]/I$, while $D(I) \cong D(\sqrt{I})$. If A is a field, we have $D(I)(A) = X(A) \setminus V(I)(A)$, otherwise this equation may be false.

(2) Since k is algebraically closed, we have for the affine scheme X by the Hilbert Nullstellensatz: $\mathrm{X}(\mathrm{k}) \cong \{I \subset \mathrm{k}[X] \mid I \text{ is a maximal ideal}\}$, furthermore an open subfunctor Y of X is uniquely determined by Y(k).

2.3. **Schemes.** Let X be a k-functor. A subfunctor Y of X is called *open* (resp. *closed*), if for any affine scheme X′ over k and any morphism $f : \mathrm{X}' \to \mathrm{X}$ the subfunctor $f^{-1}(Y) \subset \mathrm{X}'$ is open (resp. closed). This is compatible with the corresponding definition for affine schemes. A family $(\mathrm{Y}_j)_{j \in J}$ of open subfunctors of X is an *open covering* if $\mathrm{X}(A) = \cup_{j \in J} Y_j(A)$ for any $A \in$ k-alg which is a *field*.

Remark . If Y, Y' are open subfunctors of X then we have $\mathrm{Y} = \mathrm{Y}'$ iff $\mathrm{Y}(A) = \mathrm{Y}'(A)$ for any $A \in$ k-alg which is a field.

By definition a k-functor X is *local* if for every k-functor Y and for every open covering $(\mathrm{Y}_j)_{j\in J}$ of Y the following sequence of sets is exact:

$$(*) \quad \mathrm{Mor}(\mathrm{Y},\mathrm{X}) \xrightarrow{\alpha} \prod_{j\in J} \mathrm{Mor}(Y_j, X) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{j,j'\in J} \mathrm{Mor}(Y_j \cap Y_{j'}, X), \qquad \text{where}$$

$$\alpha(f) = (f\mid_{Y_j})_{j\in J}$$
$$\beta((f_j)_{j\in J}) = (f_j\mid_{Y_j\cap Y_{j'}})_{j,j'\in J}$$
$$\gamma((f_j)_{j\in J}) = (f_{j'}\mid_{Y_j\cap Y_{j'}})_{j,j'\in J}$$

Note that this means essentially that the functor $\mathrm{Mor}(?, X)$ is a sheave in some sense. One can prove that X is already a local functor if $(*)$ is exact for all *affine schemes* $\mathrm{Y} = \mathrm{Sp}_{\mathrm{k}} R$ and all special open coverings of $\mathrm{Sp}_{\mathrm{k}} R$ of the form $\big(D((f_j))\big)_{j\in J}$, thus affine schemes are local functors.

Definition . A k-functor is called a *scheme* (over k) if it is local and if it admits an open covering by affine schemes; a scheme is called *algebraic* if it admits an open covering by affine algebraic schemes.

Remark . Affine schemes are schemes, as well as open and closed subfunctors and fibred products of schemes.

2.4. Tangent spaces. Let X be a scheme over k, and $x \in \mathrm{X}(\mathrm{k})$. By definition the *tangent space* of X at x is given by

$$T_{\mathrm{X},x} := \{t \in \mathrm{X}(\mathrm{k}[\varepsilon]) \mid X(p)(t) = x\}$$

where $\mathrm{k}[\varepsilon]$ is the algebra of dual numbers and $p\colon \mathrm{k}[\varepsilon] \to \mathrm{k}$ the canonical projection. Thus, if X is algebraic, $T_{\mathrm{X},x}$ will be a finitely generated k-module. A morphism $f\colon \mathrm{X} \to \mathrm{Y}$ of schemes induces for each $x \in \mathrm{X}(\mathrm{k})$ a linear map

$$\mathrm{d}\, f_x \colon T_{\mathrm{X},x} \to T_{\mathrm{Y},f(x)}$$

the *differential* of f at x.

Remark . If an algebraic scheme X is not reduced, $\dim_{\mathrm{k}} T_{\mathrm{X},x}$ will be generically bigger than the dimension of X.

2.5. Group schemes and operations. A k-*group functor* is a functor from k-alg to the category of groups; thus if we compose such a functor with the forgetful functor we obtain a k-functor. A k-*group scheme* is a k-group functor, that gives a scheme if we compose it with the forgetful functor. In the following we will consider only *affine* k-group schemes without stating this always.

Example . The k-group functor G_a defined by $\mathrm{G}_a(A) = (A, +)$ is the additive group over k. The *general linear group* of rank n over k is the k-group scheme defined by

$$\mathrm{Gl}_n(A) = \{\text{group of invertible } n \times n \text{ matrices over } A\}$$

A special case is $\mathrm{G}_m := \mathrm{Gl}_1$, the multiplicative group over k.

Let G be a k-group functor. A *right operation* of G on a k-functor X is a morphism $\alpha\colon \mathrm{X}\times\mathrm{G}\to\mathrm{X}$, such that for each $A\in$ k-alg the map $\alpha(A)\colon \mathrm{X}(A)\times\mathrm{G}(A)\to\mathrm{X}(A)$ is a right operation of $\mathrm{G}(A)$ on $\mathrm{X}(A)$.

If Y is a subfunctor of X, then the stabilizer of Y in G is the subgroup functor defined by

$$\mathrm{Stab}_G(Y)(A) := \bigl\{g\in\mathrm{G}(A)\mid \alpha(A')\bigl(\mathrm{Y}(A'),\mathrm{G}(\rho_{A'})(g)\bigr)\subset Y(A') \qquad \text{for all } A\text{-algebra structures } A\xrightarrow{\rho_{A'}}A'\bigr\}.$$

When no confusion arises, we will write $x^g := \alpha(A')(\mathrm{X}(\rho)(x),g)$ if $x\in\mathrm{X}(A)$, $g\in\mathrm{G}(A')$ and $A\xrightarrow{\rho}A'$ defines a A-algebra structure. For $x\in\mathrm{X}(\mathrm{k})$, we write $\mathrm{Stab}_{\mathrm{G}}(x):=\mathrm{Stab}_{\mathrm{G}}(\mathrm{Y}_x)$, where Y_x is the smallest subfunctor of X containing x.

If Y is a closed subfunctor of the k-*scheme* X, then $\mathrm{Stab}_{\mathrm{G}}(\mathrm{Y})$ is a closed subfunctor of G. This is not trivial and uses the hypothesis that k is a field.

For $x\in\mathrm{X}(\mathrm{k})$ we define the orbit map $\pi_x\colon\mathrm{G}\to\mathrm{X}$ by

$$\pi_x(A)\colon\mathrm{G}(A)\to\mathrm{X}(A),\ g\mapsto\alpha(A)(X(\rho_A)(x),g)=x^g$$

for all k-algebras $\mathrm{k}\xrightarrow{\rho_A}A$. Unfortunately, although G and X are schemes, the image functor $\mathrm{O'}_x$ of π_x (i.e. $\mathrm{O'}_x(A):=\pi_x(\mathrm{G}(A))$) will in general not be a scheme, but only a *faisceau*, see [**Ja**, I, 5.2]. The orbit O_x of x is by definition $F_\lambda\,\mathrm{O'}_x$, where F_λ is the left adjoint of the inclusion functor from the category of k-faisceaux into the category of k-functors. It turns out, that in our situation O_x is a *subscheme* of X, containing $\mathrm{O'}_x$. Furthermore O_x is reduced if G and $\mathrm{Stab}_{\mathrm{G}}(x)$ are reduced. One can show (see [**DeGa**, III,§1,1.15]), that if k is algebraically closed, $\mathrm{O}_x(\mathrm{k})=\mathrm{O'}_x(\mathrm{k})$. We must refer to [**Ja**, I,5] for a more explicit treatment of this matter.

Remark . Look with the above notation at the differential of the orbit map

$$(\mathrm{d}\,\pi_x)_{\mathrm{id}}\colon T_{\mathrm{G},\mathrm{id}}\to T_{\mathrm{O}_x,x}\xrightarrow{\text{incl.}}T_{\mathrm{X},x}$$

It is not hard to see, that $\ker(\mathrm{d}\,\pi_x)_{\mathrm{id}}=T_{\mathrm{Stab}_G(x),\mathrm{id}}$. Thus, if $\mathrm{Stab}_G(x)$ is reduced at id, the restriction $(\mathrm{d}\,\pi_x)_{\mathrm{id}}\colon T_{\mathrm{G},\mathrm{id}}\to T_{\mathrm{O}_x,x}$ is surjective.

3. The scheme of module structures

3.1. Cohomology of modules. For later reference we write down some standard calculations related with the bar-resolution. Let Λ be a (finite dimensional) associative k-algebra and M a finite dimensional Λ-module; for M we have then the bar-resolution

$$\cdots\xrightarrow{\partial^1_M}\Lambda\otimes_{\mathrm{k}}\Lambda\otimes_{\mathrm{k}}M\xrightarrow{\partial^0_M}\Lambda\otimes_{\mathrm{k}}M\xrightarrow{\partial_M}M\to 0$$

with

$$\partial^k_M(v_0\otimes\cdots\otimes v_{k+1}\otimes m)=\sum_{i=0}^{k}(-1)^i v_0\otimes\cdots\otimes v_iv_{i+1}\otimes\cdots\otimes v_k\otimes m + (-1)^{k+1}v_0\otimes\cdots\otimes v_k\otimes v_{k+1}m$$

Now, suppose that Λ is given by $\lambda: V\otimes V\to V$ where V is the underlying vectorspace of Λ, similarly let M be given by $\mu\colon V\otimes W\to W$ and a further module

N by $\nu : V \otimes X \to X$, then we obtain from the above resolution the following complex K^*:

$$0 \to \mathrm{Hom}_{\mathrm{k}}(W, X) \xrightarrow{d^0_{\lambda,\mu,\nu}} \mathrm{Hom}_{\mathrm{k}}(V \otimes W, X) \xrightarrow{d^1_{\lambda,\mu,\nu}} \mathrm{Hom}_{\mathrm{k}}(V \otimes V \otimes W, X) \to \cdots$$

with

$$\begin{aligned}(d^l_{\lambda,\mu,\nu}\varphi_l)(v_1 \otimes \cdots \otimes v_{l+1} \otimes w) = \nu(v_1 \otimes \varphi_l(v_2 \otimes \cdots \otimes v_{l+1} \otimes w)) \\ + \sum_{i=1}^{l} (-1)^i \varphi_l(v_1 \otimes \cdots \otimes \lambda(v_i \otimes v_{i+1}) \otimes \cdots \otimes w) \\ + (-1)^{l+1} \varphi_l(v_1 \otimes \cdots v_l \otimes \mu(v_{l+1} \otimes w))\end{aligned}$$

and $\mathrm{H}^i(K^*) \cong \mathrm{Ext}^i_\Lambda(M, N)$.

3.2. Definition. Let $W = \mathrm{k}^z$. In the affine space $\mathrm{Hom}_{\mathrm{k}}(V \otimes W, W)$ we may look at the (Zariski-) closed subset of Λ-module structures on W. Similarly the scheme mod^z_Λ is defined by

$$\mathrm{mod}^z_\Lambda(R) := \{R \otimes \Lambda\text{-module structures on } R \otimes W\}$$

for every $R \in$ k-alg; this is an affine scheme, indeed $\mathrm{mod}^z_\Lambda = \mathrm{Sp}_{\mathrm{k}} M_z$ where

$$\begin{aligned} M_z &= \mathrm{k}[X^{(j)}_{(h,i)} \mid_{\substack{h=1,\ldots,d \\ i,j=1,\ldots,z}}]/I \\ &\text{with } I := \left(\sum_{s=1}^{d} \lambda^s_{g,h} X^{(j)}_{(s,i)} - \sum_{t=1}^{z} X^{(j)}_{(g,t)} X^{(t)}_{(h,i)},\ X^{(j)}_{(1,i)} - \delta_{i,j} \text{ for } \substack{g,h=1,\ldots,d \\ i,j=1,\ldots,z} \right)\end{aligned}$$

(choose a base $1_\Lambda = v_1, \ldots, v_d$ of Λ s.t. $v_h v_i = \sum_j \lambda^j_{h,i} v_j$; in fact, by [**Bo4**] we may use any presentation of Λ in order to determine mod^z_Λ.) Observe, that $\mathrm{mod}^z_\Lambda(\mathrm{k})$ is just the set of Λ-module structures on W mentioned above.

The general linear group Gl_z operates on mod^z_Λ by transport of structure, i.e. $\mu^g(v \otimes w) = g(\mu(v \otimes g^{-1}w))$. Thus the orbits of $\mathrm{Gl}_z(\mathrm{k})$ on $\mathrm{mod}^z_\Lambda(\mathrm{k})$ represent the isoclasses of z-dimensional Λ-modules.

As a general convention we will use small greek letters $(\mu, \nu, \ldots)$ for the elements of $\mathrm{mod}^z_\Lambda(R)$ and the corresponding roman capitals $(M, N, \ldots)$ for the respective $R \otimes \Lambda$-modules.

3.3. The tangent space. By definition, the tangent space $T_{\mathrm{mod}^z_\Lambda,\mu}$ of mod^z_Λ at some k-rational point μ is given by the $\mathrm{k}[\varepsilon] \otimes \Lambda$-module structures $\tilde{\mu}$ on $\mathrm{k}[\varepsilon] \otimes W$, that reduce to μ modulo (ε), thus we may identify canonically $T_{\mathrm{mod}^z_\Lambda,\mu} = \ker(d^1_{\lambda,\mu,\mu})$.

Now, consider the orbit map $\pi_\mu \colon \mathrm{Gl}_z \to \mathrm{mod}^z_\Lambda$ corresponding to μ. We calculate with the above identification: $(\mathrm{d}\,\pi_\mu) = d^0_{\lambda,\mu,\mu}$. Since the stabilizer of a module structure, being an open subscheme of an affine space, is always reduced, we also may identify $\mathrm{Im}\, d^0_{\lambda,\mu,\mu} = T_{\mathrm{O}_\mu,\mu}$. In other words, we have proved, ([**Ga**]):

$$T_{\mathrm{mod}^z_\Lambda,\mu}/T_{\mathrm{O}_\mu,\mu} \xrightarrow[\text{can.}]{\sim} \mathrm{Ext}^1_\Lambda(M, M)$$

Remark . In general mod^z_Λ is not reduced, look for example at the 0-dimensional scheme $\mathrm{mod}^1_{\mathrm{k}[x]/(x^2)}$. Thus, if we consider the variety $\mathrm{mod}^{z,\mathrm{red}}_\Lambda$, we obtain only an inclusion instead of the above isomorphism. We would like to have a characterization of the selfextensions of M which correspond to tangent vectors of $\mathrm{mod}_\Lambda^{z,\mathrm{red}}$ at μ. Related with this is the question: When is mod_Λ^z already reduced?

3.4. Upper semicontinuity. The following lemma is an easy exercise, using 3.1, see [**Sc1**]; compare also [**GeP**] for the corresponding statements about Hochschild cohomology of algebras.

LEMMA . *For a given finite dimensional* k*-algebra* Λ *and* $z \in \mathbb{N}$ *we have:*

a) *The function*

$$\delta^i \colon \mathrm{mod}_\Lambda^z(\mathrm{k}) \to \mathbb{N}_0, \mu \mapsto \dim_\mathrm{k} \mathrm{Ext}_\Lambda^i(M,M)$$

is upper semicontinuous (with respect to Zariski-topology) for all $i \in \mathbb{N}_0$.

b) *If* $\mathrm{Ext}_\Lambda^n(M,M) = 0$ *for some* z*-dimensional* Λ*-module* M*, there is an open neighborhood* U *of* μ *in* $\mathrm{mod}_\Lambda^z(\mathrm{k})$ *and there exist integers* c_μ, d_μ *such that we have for all* $\nu \in U$*:*

(i) $\dim_\mathrm{k} \ker d_{\lambda,\nu,\nu}^{n-1} = c_\mu$

(ii) $\sum_{i=0}^{n-1} (-1)^i \dim_\mathrm{k} \mathrm{Ext}_\Lambda^i(N,N) = d_\mu$

(iii) $\mathrm{Ext}_\Lambda^n(N,N) = 0$

3.5. Deformations. We need some elementary deformation theory of modules in the language of schemes, compare [**GhS2**, 9].

3.5.1. A *formal deformation* of $\mu \in \mathrm{mod}_\Lambda^z(\mathrm{k})$ is an element $\tilde{\mu} \in \mathrm{mod}_\Lambda^z(\mathrm{k}[[T]])$ s.t. $\mathrm{mod}_\Lambda^z(\tilde{\pi})(\tilde{\mu}) = \mu$, where $\tilde{\pi}$ is the canonical projection from $\mathrm{k}[[T]]$, the algebra of formal power series to k, i.e. we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sp}_\mathrm{k}\mathrm{k} \arrow[r, equal] \arrow[d, "\mathrm{Sp}_\mathrm{k}\tilde{\pi}"'] & \mathrm{Sp}_\mathrm{k}\mathrm{k} \arrow[d, "\mu^\#"] \\ \mathrm{Sp}_\mathrm{k}\mathrm{k}[[T]] \arrow[r, "\tilde{\mu}^\#"'] & \mathrm{mod}_\Lambda^z \end{tikzcd}$$

Two formal deformations $\tilde{\mu}_1, \tilde{\mu}_2$ of μ are *equivalent* if there exists $\tilde{g} \in \mathrm{Gl}_z(\mathrm{k}[[T]])$ s.t. $\mathrm{Gl}_z(\tilde{\pi})(\tilde{g}) = \mathrm{id}_z$ and $\tilde{\mu}_2 = \tilde{\mu}_1^{\tilde{g}}$; a formal deformation of μ is *trivial*, if it is equivalent to μ itself.

REMARK . Sometimes the formal deformations are also called *one-parameter deformations*, we want to explain this. First, if we have a formal deformation $\tilde{\mu}$ of μ, we can factorize $\tilde{\mu}^\#$ as $\mathrm{Sp}_\mathrm{k}\mathrm{k}[[T]] \xrightarrow{\mathrm{Sp}_\mathrm{k}\iota} \mathrm{Sp}_\mathrm{k}\hat{R} \xrightarrow{\hat{\mu}^\#} \mathrm{mod}_\Lambda^z$ where $\hat{R}$ is the image of the induced map $\tilde{\mu}^* \colon \mathrm{k}[\mathrm{mod}_\Lambda^z] \to \mathrm{k}[[T]]$. Thus $\hat{R}$ is a finitely generated k-algebra, an integral domain and, as we will see, at most of (Krull) dimension 1. For this end it is sufficient to study the local ring $\tilde{R} := \hat{R}_{\hat{R}\cap T\mathrm{k}[[T]]}$ which by construction is dominated by $\mathrm{k}[[T]]$. Take $0 \neq x \in \mathrm{rad}\,\tilde{R}$ and consider the morphism of k-algebras

$$\bar{\iota} \colon \tilde{R}/x\tilde{R} \to \mathrm{k}[[T]]/x\mathrm{k}[[T]]$$

induced by the inclusion, which is still injective (we exclude the trivial case $\hat{R} = \mathrm{k}$). Now, $\mathrm{k}[[T]]/x\mathrm{k}[[T]] \cong \mathrm{k}[T]/(T^n)$, thus also $\tilde{R}/x\tilde{R}$ is a finite-dimensional k-algebra and consequently by Krull's principal ideal theorem $\tilde{R}$ has dimension 1.

On the other hand, if we have an (irreducible) curve $C \subset \mathrm{mod}_\Lambda^z$ passing through μ, this gives rise to a formal deformation of μ, in fact we get

$$\mathrm{k}[\mathrm{mod}_\Lambda^z] \xrightarrow{\text{proj.}} \mathrm{k}[C] \xrightarrow{\text{incl.}} \widetilde{\mathrm{k}[C]} \xrightarrow{f} \mathrm{k}[[T]],$$

where $\widetilde{\mathrm{k}[C]}$ is the normalization of $\mathrm{k}[C]$ and f is induced by the I-adic completation of $\widetilde{\mathrm{k}[C]}$ at some maximal ideal I lying over the ideal of μ.

3.5.2. Similarly, the *infinitesimal deformations* of μ may be identified with $T_{\mathrm{mod}^z_\Lambda,\mu}$, compare 3.3, while the trivial infinitesimal deformations can be identified with $\operatorname{Im} d^0_{\lambda,\mu,\mu}$, thus $\operatorname{Ext}^1_\Lambda(M,M)$ classifies the infinitesimal deformations of μ.

3.5.3. Let $\tilde{\mu}$ be a formal deformation of μ, then μ is called a *jump deformation*, if there exists $\mu_1 \in \mathrm{mod}^z_\Lambda(\mathrm{k})$ and $g \in \mathrm{Gl}_z(\mathrm{k}((T)))$ with $\tilde{\mu} = \mu_1^g$. In this case we say by abuse of language that μ_1 is a *(jump) deformation* of μ.

Remark . μ_1 is a (jump) deformation of μ iff μ lies in the Zariski closure of the orbit $\mu_1^{\mathrm{Gl}_z(\mathrm{k})} \subset \mathrm{mod}^z_\Lambda(\mathrm{k})$, compare [**Bo3**, Introduction]. We leave the proof to the reader; hint: [**Kr2**, III.2.3.1].

3.6. Rigidity. Related with the different notions of deformations there are also several concepts of rigidity. Let $\mu \in \mathrm{mod}^z_\Lambda(\mathrm{k})$, then μ is called

a) *absolutely rigid* if $\operatorname{Ext}^1_\Lambda(M,M)=0$ or equivalently if all infinitesimal deformations are trivial,
b) *analytically rigid* if all formal deformations are trivial,
c) *geometrically rigid* if the orbit $\mu^{\mathrm{Gl}_z}(\mathrm{k})$ is open in $\mathrm{mod}^z_\Lambda(\mathrm{k})$,
d) *semi-rigid* if all jump deformations are isomorphic to μ itself.

In our situation we have the following implications:

$$\text{a)} \Longrightarrow \text{b)} \Longleftrightarrow \text{c)} \Longrightarrow \text{d)}$$

The implication a) $\Longrightarrow$ b) is essentially the same as the well-known case of algebras, see [**Gh1**].

b) $\Longrightarrow$ c) is based on the observation, that if μ is not geometrically rigid, then there exists a curve $C \subset \mathrm{mod}^z_\Lambda$ whith $\overline{\mu^{\mathrm{Gl}_z(\mathrm{k})}} \cap C(\mathrm{k}) = \mu$; with the construction of the remark in 3.5.1 we obtain a non-trivial deformation.

c) $\Longrightarrow$ b) is based on the fact, that the stabilizer groups of module structures are always smooth, see 3.3, compare also [**GeP**].

c) $\Longrightarrow$ d) is trivial.

Example . (1) The module of the example in remark 3.3 is geometrically rigid but not absolutely rigid.

(2) A simple regular module over a tame hereditary algebra are semi-rigid but not geometrically rigid (if it lies in a homogenous tube).

3.7. Proposition. *If* $\operatorname{Ext}^2_\Lambda(M,M)=0$*, then* μ *is a smooth point of the scheme* mod^z_Λ.

This is an easy consequence of 3.4 and the fact, that $\operatorname{Ext}^2_\Lambda(M,M)=0$ implies, that every infinitesimal deformation can be lifted to a formal deformation (and thus $T_{\mathrm{mod}^z_\Lambda,\mu} = T_{\mathrm{mod}^{z,\mathrm{red}}_\Lambda,\mu}$), compare [**GeP**].

Corollary . a) *If* $\operatorname{Ext}^2_\Lambda(M,M)=0$ *the local dimension of the* k*-scheme* mod^z_Λ *at* μ *is* $z^2 - \dim_{\mathrm{k}} \operatorname{End}_\Lambda(M) + \dim_{\mathrm{k}} \operatorname{Ext}^1_\Lambda(M,M)$.

b) *If* $\operatorname{Ext}^2_\Lambda(M,M)=0$ *and* $\operatorname{End}_\Lambda(M) \cong \mathrm{k}$ *there is an open neighborhood* U *of* μ *in* $\mathrm{mod}^z_\Lambda(\mathrm{k})$ *such, that all the orbits passing through* U *have codimension* $\dim_{\mathrm{k}} \operatorname{Ext}^1_\Lambda(M,M)$.

3.8. Algebras. There are parallel results for algebras. The scheme alg_d is defined as follows: If $V = \mathrm{k}^d$, then

$$\mathrm{alg}_d(R) = \{\text{associative unitary } R\text{-algebra structures on } R \otimes V\}$$

It is well known, though less obvious, that alg_d is also an affine scheme. We have the operation of Gl_d on alg_d by transport of structure, and the orbits of the k-rational points correspond to the isoclasses of d-dimensional associative k-algebras. If we calculate the Hochschild cohomology groups as in [**H**] we find:

- $T_{\mathrm{alg}_d,\lambda} \xrightarrow{\sim} \ker d^2_\lambda$ canonically,
- the differential of the orbit map $\pi_\lambda : \mathrm{Gl}_d \to \mathrm{alg}_d$ of λ may be identified with d^1_λ and $T_{\mathrm{Stab_k}(\lambda),\mathrm{id}} \xrightarrow{\sim} \ker d^1_\lambda$.

Thus we find by similar considerations as in 3.3 a canonical isomorphism $T_{\mathrm{alg}_d,\lambda}/T_{\mathrm{O}_\lambda,\lambda} \cong \mathrm{H}^2(\Lambda)$ if $\mathrm{Stab}_{\mathrm{Gl}_d}(\lambda)$ is reduced; this is always true if $\mathrm{char}\,\mathrm{k} = 0$, while for positive characteristic there are counterexamples, see [**GhS1**].

The different notions of deformations, resp. rigidity carry over almost literally from our considerations of modules, but here geometrical rigidity implies formal rigidity only if the stabilizer is smooth.

Finally, the low Hochschild cohomology groups give us some local information about alg_d:

PROPOSITION . *Let Λ be a d-dimensional* k-*algebra and λ the corresponding point of* alg_d

a) *If* $\mathrm{H}^1(\Lambda) = 0$ *then* $\mathrm{Stab}_{\mathrm{Gl}_z}(\lambda)$ *is smooth and λ is semirigid.*
b) *If* $\mathrm{H}^3(\Lambda) = 0$*, then λ is a smooth point of* alg_d.
c) *If* $\mathrm{H}^1(\Lambda) = 0 = \mathrm{H}^3(\Lambda)$*, there exists an open neighborhood U of λ in* $\mathrm{alg}_d(\mathrm{k})$*, such that the codimension of all* $\mathrm{Gl}_d(\mathrm{k})$ *orbits passing through U is* $\dim_{\mathrm{k}} \mathrm{H}^2(\Lambda)$.

For the proofs, which are based on the methods presented here, we refer to [**GeP**]. There we study also an important example: If Λ is strongly simply connected and tame of polynomial growth we have $\mathrm{H}^1(\Lambda) = 0 = \mathrm{H}^3(\Lambda)$, in this situation we construct even explicitly a $\dim_{\mathrm{k}} \mathrm{H}^2(\Lambda)$-parametric family of algebras, containing Λ.

4. Representation type

4.1. Definition. Let Λ be a finite dimensional k-algebra. Then Λ is of

- *finite representation type* iff the number isoclasses of indecomposable modules is finite,
- *tame* if for every $z \in \mathbb{N}_0$ the indecomposable modules of dimension z may be parametrized by a finite number of rational curves,
- *wild* if the representation theory of Λ is as complicated as the representation theory of $\mathrm{k}\langle x, y\rangle$, the (non commutative) free associative algebra in two variables.

We refer to [**CB1**] for more precise definitions of tame and wild. Recall, that by Brauer-Thrall II, if Λ is of infinite representation type there are in some dimension infinitely many isoclasses of indecomposable modules. Indeed, the proofs of this result (see [**Ba**], [**Bo1**],[**Bo2**], [**Fi**]) provide much more information. Similarly, the theorem of Drozd (see [**Dr**], [**CB1**] and [**GNRSV**]) asserts, that if Λ is not tame, then it is already wild. Both theorems are deep and the proofs are quite complicated.

4.2. Representation type and dimension. We define the following closed subfunctor of mod_Λ^z:

$$\mathrm{mod}_\Lambda^{z,t}(R) := \{\mu \in \mathrm{mod}_\Lambda^z(R) \mid \mathrm{rank}_R \operatorname{End}_{R\otimes\Lambda}(M) \geq t\}$$

for all k-algebras R.

Proposition . *Let Λ be a finite dimensional* k*-algebra.*

a) Λ *is representation finite* $\Longleftrightarrow \dim \mathrm{mod}_\Lambda^{z,t} \leq z^2 - t$ *for all* $z \in \mathbb{N}$, $1 \leq t \leq z^2$.

b) Λ *is tame* $\Longleftrightarrow \quad \dim \mathrm{mod}_\Lambda^{z,t} \leq z^2 + z - t$ *for all* $z \in \mathbb{N}$, $1 \leq t \leq z^2$.

The proofs "$\Longrightarrow$" are in both cases direct consequences of the definition of the respective representation type, while the other directions require Brauer-Thrall II and Drozd's theorem respectively.

4.3. Upper semicontinuity. Look at the scheme $algmod_d^z$ which is defined by

$$algmod_d^z(R) := \{(\lambda,\mu) \mid \lambda \text{ is an } R\text{-algebra structure on } R\otimes \mathrm{k}^d, \\ \mu \text{ is a } R\otimes\Lambda\text{-module structure on } R\otimes \mathrm{k}^z\}$$

for all k-algebras R. We have the canonical projection $\pi : algmod_d^z \to \mathrm{alg}_d$ with fibre $\pi^{-1}(\lambda) = \mathrm{mod}_\Lambda^z$ for $\lambda \in \mathrm{alg}_d(\mathrm{k})$; moreover we have the obvious operation of Gl_z on $algmod_d^z$. The following result [**Ga**, 3.2] is important for us:

Lemma . *The morphism* $\pi(\mathrm{k})\colon algmod_d^z(\mathrm{k}) \to \mathrm{alg}_d(\mathrm{k})$ *sends closed and* $\mathrm{Gl}_z(\mathrm{k})$ *invariant sets to closed sets.*

The proof uses certain Grassmanians naturally related with mod_Λ^z, for a proof avoiding the scheme-theoretic language see [**LRS**]. A direct consequence is:

Proposition . *The function* $\delta_d^{z,t} : \mathrm{alg}_d(\mathrm{k}) \to \mathbb{N}_0,\ \lambda \mapsto \dim \mathrm{mod}_\Lambda^{z,t}$ *is upper semicontinuous.*

4.4. Theorem. ([**Ge2**]) *Deformations of tame (resp. representation finite) algebras are tame (resp. representation finite).*

This is an direct consequence of the propositions 4.2 and 4.3.

Example . Look at the algebras $\Lambda_1 := \mathrm{k}\langle a,b\rangle/\langle a^2 - bab, a^3, b^2\rangle$ (observe: $(\bar a\bar b)^2 = 0 = (\bar b\bar a)^2$ in Λ_1) and $\Lambda_0 := \mathrm{k}\langle a,b\rangle/\langle a^2, b^2, (ab)^2, (ba)^2\rangle$, then Λ_1 is a (jump) deformation of Λ_0, furthermore Λ_0 is special biserial and thus well-known to be tame, so we may conclude by our theorem that Λ_1 is tame. In fact, in [**CB2**] by other methods there is given a complete description of the indecomposable modules of Λ_1, and it is quite interesting to compare the modules of a given dimension over Λ_1 and Λ_0. Similarly, Λ_0 deforms into the quaternion algebra, $\Lambda_1' := \mathrm{k}\langle a,b\rangle/\langle a^2 - bab, b^2 - aba, (ab)^2, (ba)^2\rangle$ which gives a tameness proof for this algebra – but in this case there is no description of the indecomposable modules available.

Let us mention also, that the above theorem was an important step towards the classification of the tame distributive algebras, see [**Ge3**], [**DxGe**].

Remark . (1) Recently Crawley-Boevey proved a similar result for families of algebras defined by quivers and relations whose dimension is not necesarily fixed.

(2) We can prove also this type of result for an other class of problems: Deformations of tame bimodule problems are tame, but we do not want to enter here into the technical details.

(3) It is still an open question if tame is an open property.

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