A Scheme for Shallow Water Flow with Area Variation

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Abstract. We consider the shallow water equations for flows through channels with variable area. The system is obtained by depth/width averaging of the Euler equations and forms a hyperbolic set of balance laws. Exact steady-state solutions are available and are controlled by the relative positions of the bottom crest and channel throat. We present a Roe-type upwind scheme for the system. Considerations of conservation, near steady-state accuracy, velocity regularization and positivity near dry states are discussed. Numerical solutions are presented illustrating the merits of the scheme for a variety of flows including sub-, trans- and supercritical flows and drainage problems, with emphasis on effects of the interplay between topography and geometry on the solution.

Keywords: Hyperbolic conservation laws, upwind schemes, source terms.
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INTRODUCTION

We consider the shallow water equations over bottom topography of elevation $B(x)$ and through rectangular channels with variable area of width $\sigma(x)$. The model is an average flow model describing nearly horizontal flows, and may be derived from the three dimensional Euler equations by depth/width averaging. Denoting by $h(x,t)$ the depth of the water layer, and by $u(x,t)$ the average velocity (see Figure1), and assuming that the pressure is given by hydrostatic balance $p = p(y) = \rho g(h+B-y)$, with g the gravitational constant, the shallow water system is given by

$$\begin{pmatrix} \sigma h \\ \sigma hu \end{pmatrix}_t + \begin{pmatrix} \sigma hu \\ \sigma hu^2 + \frac{1}{2} g\sigma h^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -gh\sigma B'(x) + \frac{gh^2}{2}\sigma'(x) \end{pmatrix} \tag{1}$$

Recent years have seen growing interest in development of numerical methods for shallow water systems (see for example [1, 2, 3, 4, 5, 6] and references therein), in particular [1, 2, 4] are concerned with flows through channels with variable geometry. Desirable properties include recognizing and respecting steady states in order to accurately compute near steady state flows, and ensuring positivity of the computed solution in order to handle near dry states, such as arise in reservoir drainage or flooding problems. We have derived an upwind scheme for the shallow water system, and have implemented it to a variety of test problems to illustrate its robustness in converging accurately to steady state, in computing accurately small perturbations thereof, and in reservoir drainage problems.

THE MODEL

Equation (1) is a hyperbolic conservation law with geometric source terms accounting for the effects of bottom topography and variable channel width. The eigenstructure of the system is given by the matrices

$$R = \begin{pmatrix} 1 & 1 \\ u-c & u+c \end{pmatrix} \quad , \quad \Lambda = \begin{pmatrix} u-c & 0 \\ 0 & u+c \end{pmatrix}$$

where we use $c^2 = gh$. The system loses hyperbolicity if $h = 0$, for which both eigenvectors coincide.

System (1) admits smooth steady-state solutions satisfying

$$Q = \sigma hu = constant \ , \qquad E = \frac{u^2}{2} + g(h+B) = constant \ , \tag{2}$$

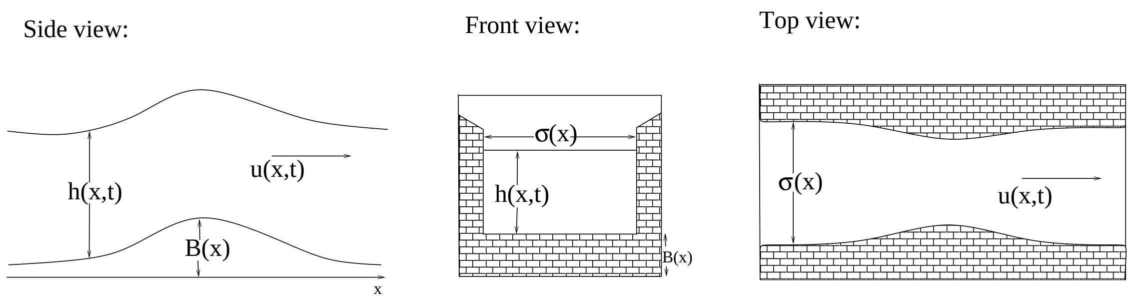

FIGURE 1. Schematic for the shallow water equations through rectangular channels with variable area.

of which steady state of rest is a simple example, with $u = 0$, $h + B = constant$. More general steady state solutions can be obtained by rootfinding using (2), and are classified into subcritical, supercritical, and transcritical depending on the magnitude of the Froude number F

$$F^2 = \frac{u^2}{gh}.$$

THE SCHEME

We write equation (1) in the form

$$W_t + F(W)_x = S,$$

where $F(W)$ is the flux function and S denotes the geometric source terms. We use a Roe-type upwind scheme to approximate the flux gradient terms and approximate the source term in an upwinded manner, by projecting S onto the eigenvectors of the Jacobian matrix $A = F'(W)$ [7]. The Jacobian matrix is linearized about the Roe averages

$$\bar{h} = \frac{h_L + h_R}{2}, \qquad \bar{u} = \frac{\sqrt{\sigma_L h_L} u_L + \sqrt{\sigma_R h_R} u_R}{\sqrt{\sigma_L h_L} + \sqrt{\sigma_R h_R}}$$

with $\bar{c}^2 = g\bar{h}$. The source term is approximated by

$$\hat{\sigma} = \frac{\sigma_L + \sigma_R}{2}, \qquad \hat{h} = \frac{h_L + h_R}{2}, \qquad \hat{B}' = \frac{\Delta B}{\Delta x}, \qquad \hat{\sigma}' = \frac{\Delta \sigma}{\Delta x}$$

Here $\Delta(\cdot) = (\cdot)_{j+1} - (\cdot)_j$. The scheme is given by

$$W_j^{n+1} = W_j^n - \frac{\Delta t}{\Delta x} \left\{ A^+_{j-\frac{1}{2}} \left(W_j^n - W_{j-1}^n \right) + A^-_{j+\frac{1}{2}} \left(W_{j+1}^n - W_j^n \right) \right\} \tag{3}$$

$$A^+ \Delta W = \sum_{\lambda_k > 0} (\alpha_k \lambda_k - \beta_k) r_k \quad , \quad A^- \Delta W = \sum_{\lambda_k < 0} (\alpha_k \lambda_k - \beta_k) r_k,$$

where $\Delta W = \sum_k \alpha_k r_k$, $\Delta x S = \sum_k \beta_k r_k$, r_k and λ_k are the eigenvectors/eigenvalues of the Jacobian matrix, and the wave strengths are given by

$$\alpha_1 = \frac{(\bar{u} + \bar{c})\Delta(\sigma h) - \Delta(\sigma h u)}{2\bar{c}}, \quad \alpha_2 = -\frac{(\bar{u} - \bar{c})\Delta(\sigma h) - \Delta(\sigma h u)}{2\bar{c}},$$

$$\beta_1 = g \frac{\hat{\sigma} \hat{h} \Delta B - \hat{h}^2 \Delta \sigma}{2\bar{c}}, \qquad \beta_2 = -g \frac{\hat{\sigma} \hat{h} \Delta B - \hat{h}^2 \Delta \sigma}{2\bar{c}}.$$

It can be easily verified that in the absence of a geometric source term, the linearization is conservative and satisfies $A \Delta W = \Delta F$. It can also be verified that for data that corresponds to steady state of rest, $\alpha_k \lambda_k - \beta_k$ vanishes for $k = 1, 2$.

This implies that a steady state of rest will be recognized and respected by the scheme, a property often referred to as being a 'well-balanced' scheme.

Roe's scheme is known not to be entropy satisfying. We have implemented the Harten and Hyman entropy fix [8]. When $h << 1$, recovering the velocity using $u = (\sigma hu)/(\sigma h)$ is prone to large errors. Following [5], we use a regularized expression

$$u = \frac{2Q}{\sigma\,(h + \max(h, \varepsilon))}.$$

for small ε (typically $\varepsilon = O(10^{-5})$. For drainage problems, we have also experimented with velocity stabilization based on the expected steady state energy, E_{SS}, which seemed to give good convergence properties: if $h < \varepsilon$,

$$u = sign(Q)\sqrt{2\ max\,(E_{SS} - g(h+B), 0)}.$$

NUMERICS

Test 1: *Convergence to Steady State*: In this example, the bottom topography is given by a parabolic bump of hight 0.2, and a channel with parabolic contraction of maximum contraction 0.8. We impose $Q = 4.42$ at inflow and $h = 2$ at outflow, and integrate the equations for large t until steady state is reached. Figure 2 shows solutions corresponding to three related computations. On the left, the channel is straight $(\sigma \equiv 1)$, the flow accelerates as it goes over the bump but remains subcritical; in the middle, the throat of the geometry is right at the crest of the topography. As the flow now needs to pass through a narrower passage, it needs to go faster. It accelerates and becomes supercritical at the narrowest point, then drops back to subcritical through a shock; and on the right, the throat of the geometry is off center, leaving more room for the flow to pass and making it possible for the flow to accelerate but still remain subcritical. All figures show exact and computed solutions, with excellent agreement. At the final time, relative errors in Q and E are of the order of 10^{-7} and 10^{-5} respectively.

Test 2: *Propagation of Small Perturbations to Steady State*: In this test, a steady state is initialized and a small perturbation of order 10^{-2} is imposed. We apply open boundary conditions at both ends, and follow the propagation of the disturbance, over a parabolic topography bump and an centered parabolic geometrical contraction, until the perturbation leaves the domain and the unperturbed steady state is recovered. Figure 3 shows two related computations corresponding to steady state of rest in the top row and steady state of (nonrest) subcritical flow. In both cases, the small perturbation is resolved accurately, and leaves behind a clean steady state with relative errors in both Q and E within 10^{-6} of the expected values of the unperturbed state.

Test 3: *Drainage of a reservoir*: Here we compute the drainage of a reservoir following a dam break. The topography is a parabolic bump and the geometry is an off-center parabolic contraction similar to that in *Test 1*. The initial water level is 0.8. We impose symmetry boundary conditions on the left and allow the water to drain through the right boundary. The equations are integrated until the water drains, except for water that gets trapped in the trough. While the current version of the scheme is not positive, we note that it exhibits *remarkable* robustness near dry state as illustrated by this test.

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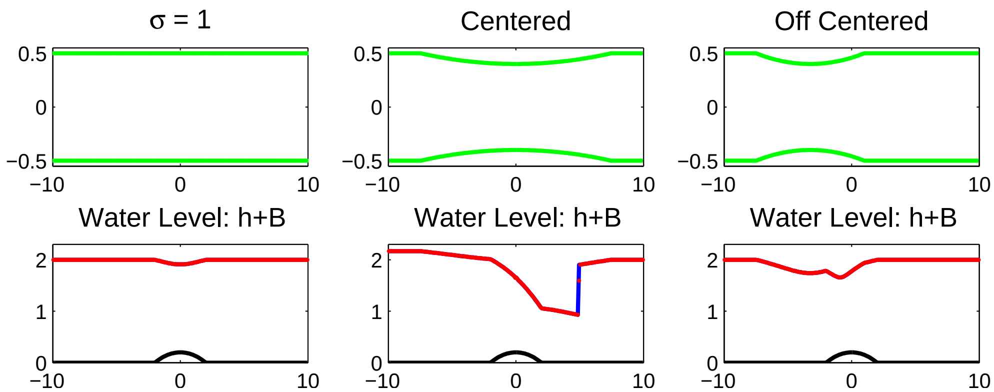

FIGURE 2. Convergence to Steady States. Straight channel (left), center (middle) and off-center (right column) geometry.

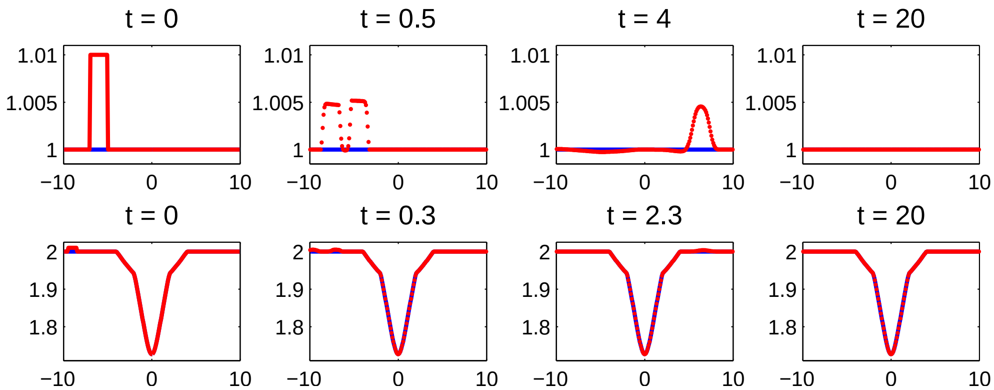

FIGURE 3. Perturbation of steady states of rest (first row) and subcritical flow (second row).

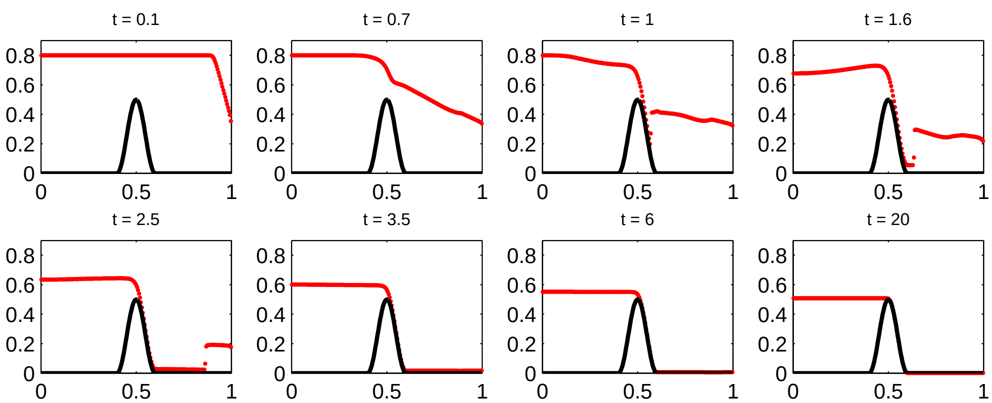

FIGURE 4. Drainage of reservoir following a dam break.

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