HW 1 Sample homework solutions.

Section 1.1 Problem # 11 Verify by substitution that each given function is a solution of the given differential equation:

$$x^2y'' + 5xy' + 4y = 0, \quad y_1 = \frac{1}{x^2}, \quad y_2 = \frac{\ln x}{x^2}$$

Solution:

If $y = y_1 = x^{-2} \Rightarrow y' = -2x^{-3}$ and $y'' = 6x^{-4}$

$$\Rightarrow x^2y'' + 5xy' + 4y = x^2(6x^{-4}) + 5x(-2x^{-3}) + 4(x^{-2}) = 0$$

If $y = y_2 = x^{-2}\ln x \Rightarrow y' = x^{-3} - 2x^{-3}\ln x$

$$y'' = -5x^{-4} + 6x^{-4}\ln x$$

Then
$$x^2y'' + 5xy' + 4y = x^2(-5x^{-4} + 6x^{-4}\ln x) + 5x(x^{-3} - 2x^{-3}\ln x) + 4(x^{-2}\ln x)$$
$$= (-5x^{-2} + 5x^{-2}) + (6x^{-2} - 10x^{-2} + 4x^{-2})\ln x = 0$$

Problem # 30 Write a D. E. of the form $\frac{dy}{dx} = f(x,y)$ having the function g as its solution:

"The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.

Solution: Here $m = y'$ and the slope of the graph $y = x^2 + k$ at x is $m' = \frac{d}{dx}(x^2 + k) = 2x$

The orthogonality relation $mm' = -1$ gives the differential equation $2xy' = -1$

Problem #39: Use your knowledge of derivatives to ~~solve~~:
find one solution of:

$$xy' + y = 3x^2$$

Solution: We reason that if $y = kx^2$, then each term in the differential equation is a multiple of x^2.
The choice $k=1$ balances the equation, and provides the solution

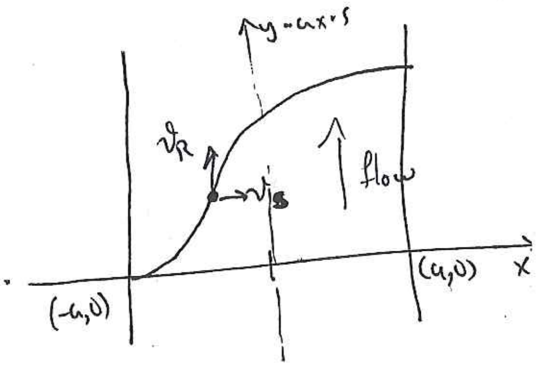
$$y(x) = x^2.$$

Problem 1.2 #40

Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_s = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function:

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.



A:

Let $y = y(x)$ be the vertical component of the swimmer's position at any given point x.

The equation for y becomes:

$$\frac{dy}{dx} = \frac{v_R}{v_s} = \frac{v_0}{v_s}\left(1 - \frac{x^4}{a^4}\right)$$, where a, v_0, v_s are the consta specified above

$\Rightarrow$ Integrating both sides we get:

$$y = \frac{v_0}{v_s}\left(x - \frac{x^5}{5a^4}\right) + C$$, C = constant of integration.

Since the swimmer started at the position $(-a, 0)$ (left bank of the river)

Then we have the initial condition $y(-a) = 0$

$$\Rightarrow 0 = \frac{v_0}{v_s}\left(-a - \frac{(-a)^5}{5a^4}\right) + C = \frac{v_0}{v_s}\left(-a + \frac{a}{5}\right) + C = -\frac{4}{5}\frac{v_0}{v_s}a + C \Rightarrow C = \frac{4}{5}\frac{v_0}{v_s}a$$

$$\Rightarrow y = \frac{v_0}{v_s}\left(x - \frac{x^5}{5a^4}\right) + \frac{4}{5}\frac{v_0}{v_s}a$$

$$y(a) = \frac{v_0}{v_s}\left(a - \frac{a}{5}\right) + \frac{4}{5}\frac{v_0}{v_s}a = \frac{8}{5}\frac{v_0}{v_s}a = \frac{8}{5}\,\frac{9\text{ mi/h}}{3\text{ mi/h}}\,0.5\text{ mi} = \frac{12}{5}\text{ mi}$$

$\Rightarrow$ The swimmer was drifted 2.4 miles downstream.