4.4 #18 Find a basis for the solution space of the following homogeneous linear system:

$$x_1 + 3x_2 + 4x_3 + 5x_4 = 0$$
$$2x_1 + 6x_2 + 9x_3 + 5x_4 = 0$$

First get the coefficient matrix : 2×5:

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 9 & 5 \end{bmatrix}$$

Use elementary row operations to get the reduced echelon form:

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 9 & 5 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$$\xrightarrow{-4R_2 + R_1} \begin{bmatrix} 1 & 3 & 0 & 25 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

Leading variables: x_1, x_3

Free variables: x_2, x_4

Set x_2, x_4 equal to parameters r, s

$x_2 = r$, $x_4 = s$ $\quad x_3 = 5x_4 = 5 \cdot s$

$x_1 + 3x_2 + 25x_4 = 0 \quad x_1 = -3r - 25s$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3r - 25s \\ r \\ 5s \\ s \end{pmatrix} = r \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -25 \\ 0 \\ 5 \\ 1 \end{pmatrix}$$

Therefore, a basis for the solution space is $S = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -25 \\ 0 \\ 5 \\ 1 \end{pmatrix} \right\}$.

4.3 #11: w as a linear combination of v_1, v_2 where

$w = (1,0,0,-1)$, $v_1 = (7,-6,4,5)$, $v_2 = (3,-3,2,3)$

Try to find c_1, c_2 s.t. $w = c_1 v_1 + c_2 v_2$

$$\Rightarrow c_1 \begin{pmatrix} 7 \\ -6 \\ 4 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$\Rightarrow$ The augmented coefficient matrix is:

$$\begin{bmatrix} 7 & 3 & 1 \\ -6 & -3 & 0 \\ 4 & 2 & 0 \\ 5 & 3 & -1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 & 1 \\ -6 & -3 & 0 \\ 4 & 2 & 0 \\ 5 & 3 & -1 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2]{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \\ 5 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{-2R_1+R_3 \\ -5R_1+R_5}]{2R_1+R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{bmatrix} \xrightarrow[\substack{R_2+R_3 \\ 3R_2+R_4}]{(-1)R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No inconsistencies, 2 leading variable

Solve by back substitution:

$c_2 = -2$, $c_1 = 1$ $\Rightarrow w = v_1 - 2v_2$.