0327

Practice Exam 2:

1. Given that two vectors u and v are linearly independent, are $u-v$ and v linearly dependent or linearly independent? Prove your answer

Answer: $u-v$ and v are linearly independent

Proof: Assume that $c_1(u-v)+c_2v=0$ for some c_1, c_2.

We want to show $c_1=c_2=0$.

$$0=c_1(u-v)+c_2v = c_1u-c_1v+c_2v$$
$$=c_1u+(c_2-c_1)v$$

Since u and v are linearly independepts

$\Rightarrow c_1=0, \quad c_2-c_1=0 \quad \Rightarrow c_2=c_1=0$

Therefore, $u-v$ and v are linearly independent.

Problem 2 (a) For what vectors b does $Ax=b$ have a solution, with A given by

$$A=\begin{bmatrix} 6 & 3 & 3 \\ 2 & 5 & -1 \\ -4 & -8 & 1 \end{bmatrix}.$$

Answer:

The linear system has at least one solution when I can find x_1, x_2, x_3 such that:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = Ax = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 5 & -1 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 6x_1 + 3x_2 + 3x_3 \\ 2x_1 + 5x_2 - x_3 \\ -4x_1 - 8x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 \\ 2x_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -8x_2 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The system has a solution when b is a linear combination of the column vectors of A.

That is, the linear system has at least one solution if and only if b belongs to the column space of A, $b \in \mathrm{Col}(A)$.

We will study this space in more detail in part (b)

(b) Find a basis for the vector space spanned by the columns of A.

Answer: First, we need to proceed by row operations to reduce A to its echelon form:

$$\begin{bmatrix} 6 & 3 & 3 \\ 2 & 5 & -1 \\ -4 & -8 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 5 & -1 \\ -4 & -8 & 1 \end{bmatrix} \xrightarrow[2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & -2 \\ 0 & -6 & 3 \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{3}R_3]{\frac{1}{2}R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Columns 1 and 2 are the pivot columns.

$\Rightarrow \left\{ \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -8 \end{pmatrix} \right\}$ forms a basis for Col(A).

(c) Find all possible solutions for $b = \begin{bmatrix} 0 \\ 1 \\ -3/2 \end{bmatrix}$

The augmented coefficient matrix is:

$$\begin{bmatrix} 6 & 3 & 3 & 0 \\ 2 & 5 & -1 & 1 \\ -4 & -8 & 1 & -3/2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 5 & -1 & 1 \\ -4 & -8 & 1 & -3/2 \end{bmatrix}$$

$$\xrightarrow[2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 4 & -2 & 1 \\ 0 & -6 & 3 & -3/2 \end{bmatrix} \xrightarrow[-\frac{1}{3}R_1]{\frac{1}{2}R_2} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & -1 & \frac{1}{2} \\ 0 & 2 & -1 & \frac{1}{2} \end{bmatrix} \rightarrow$$

$$\xrightarrow{-R_1+R_2} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading ~~coeff~~ vars: x_1, x_2
Free variables: x_3

$x_3 = s$, $2x_2 - x_3 = \frac{1}{2}$ $\quad 2x_2 = s + \frac{1}{2}$ $\quad x_2 = \frac{1}{2}s + \frac{1}{4}$

$2x_1 + x_2 + x_3 = 0$ $\quad 2x_1 = -\frac{1}{2}s - \frac{1}{4} - s = -\frac{3}{2}s - \frac{1}{4} \Rightarrow x_1 = -\frac{3}{4}s - \frac{1}{8}$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}s - \frac{1}{8} \\ \frac{1}{2}s + \frac{1}{4} \\ s \end{pmatrix} = \begin{pmatrix} -1/8 \\ 1/4 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3/4 \\ 1/2 \\ 1 \end{pmatrix}$$

Problem 3 Find the determinant of the following matrix using elementary row operations:

$$A = \begin{bmatrix} 1 & 2 & -2 & 5 \\ -1 & 2 & 3 & 4 \\ 1 & 3 & 1 & -2 \\ -1 & -3 & 0 & -4 \end{bmatrix}$$

Answer:

$$\xrightarrow[\substack{-R_1+R_3 \\ R_1+R_4}]{R_1+R_2} \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 1 & 9 \\ 0 & 1 & 3 & -7 \\ 0 & -1 & -2 & 1 \end{bmatrix} = B \Rightarrow \det A = \det B$$

$$\xrightarrow{\text{swap}(R_2, R_3)} \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & 1 & 3 & -7 \\ 0 & 4 & 1 & 9 \\ 0 & -1 & -2 & 1 \end{bmatrix} = C \Rightarrow \det B = -\det C$$

$\parallel$ $\det A$

$$\xrightarrow[R_2+R_4]{-4R_2+R_3} \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & -11 & 37 \\ 0 & 0 & 1 & -6 \end{bmatrix} \xrightarrow{\text{swap}(R_2, R_3)} \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & -11 & 37 \end{bmatrix} = D$$

$\det C = -\det D$

$\det A = \det D$

$$\xrightarrow{11R_3+R_4} \begin{bmatrix} 1 & 2 & -2 & 5 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & -29 \end{bmatrix} = E \qquad \det A = \det D = \det E$$

E is upper triangular $\Rightarrow \det A = \det E = -29$, the product of diagonal elements.

Problem 4: Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = \begin{bmatrix}1\\0\\2\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}$. Find a basis of $\mathbb{R}^4$ containing the vectors v_1, v_2.

Answer: Consider $\{v_1, v_2, e_1, e_2, e_3, e_4\}$

where $e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, $e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$,

$v_1, v_2, e_1, e_2, e_3, e_4$ span $\mathbb{R}^4$, and are linearly dependent.

Need to find the redundant vectors.

Form the matrix that has those vectors as its column vectors, and identify the pivot columns by reducing it to the echelon form:

$$\begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\2&2&0&0&1&0\\1&1&0&0&0&1\end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\1&1&0&0&\frac{1}{2}&0\\1&1&0&0&0&1\end{bmatrix}$$

$$\xrightarrow{-R_3+R_4} \begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\1&1&0&0&\frac{1}{2}&0\\0&0&0&0&-\frac{1}{2}&1\end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\0&-1&-1&0&\frac{1}{2}&0\\0&0&0&0&-\frac{1}{2}&1\end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix}1&2&1&0&0&0\\0&1&0&1&0&0\\0&0&-1&1&\frac{1}{2}&0\\0&0&0&0&-\frac{1}{2}&1\end{bmatrix}$$

The pivot columns are the columns 1, 2, 3 and 5

$\Rightarrow \{v_1, v_2, e_1, e_3\}$ forms a basis for $\mathbb{R}^4$.

Problem 5. Let A and B be $n \times n$ matrices. Show that AB is invertible if and only if both A and B are invertible.

Answer:

Using the relation $\det(AB) = \det A \det B$ we can easily see that the determinant of AB is non-zero if and only if both $\det A$ and $\det B$ are non-zero.

Since any ~~one~~ $n \times n$ matrix is invertible if and only if its determinant is not zero, then the analysis above proves that AB is invertible if and only if A and B are invertible.