0510.1
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Practice exam 3:

Problem #1 Consider the system of equations

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} -3 & 5 \\ -5 & 3 \end{bmatrix} \underline{x}$$

(a) Find the general solution

(b) Write the solution in terms of real functions only.

Answer:

$$A = \begin{bmatrix} -3 & 5 \\ -5 & 3 \end{bmatrix}$$

E-values: $\begin{vmatrix} -3-\lambda & 5 \\ -5 & 3-\lambda \end{vmatrix} = (-3-\lambda)(3-\lambda) + 25 = (\lambda-3)(\lambda+3)+25$

$= \lambda^2 - 9 + 25 = \lambda^2 + 16 = 0$

$\lambda = \pm 4i$

Take one e-value $\lambda = 4i$

E-vector: $A - \lambda I = \begin{bmatrix} -3-4i & 5 \\ -5 & 3-4i \end{bmatrix}$ $\quad v = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} -3-4i & 5 \\ -5 & 3-4i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-4i & 5 \\ -5 & 3-4i \end{bmatrix} \xrightarrow{(3-4i)R_1} \begin{bmatrix} -(3+4i)(3-4i) & 5\cdot(3-4i) \\ -5 & 3-4i \end{bmatrix} = \begin{bmatrix} -25 & 5(3-4i) \\ -5 & 3-4i \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{5}R_1+R_2]{\frac{1}{5}R_1} \begin{bmatrix} -5 & 3-4i \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow -5a + (3-4i)b = 0$

choose $b = 5 \Rightarrow a = 3-4i \Rightarrow v = \begin{bmatrix} 3-4i \\ 5 \end{bmatrix}$

$\Rightarrow$ One solution is:

$$e^{\lambda t} v = e^{4it} \begin{bmatrix} 3-4i \\ 5 \end{bmatrix} = (\cos(4t) + i\sin(4t)) \begin{bmatrix} 3-4i \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos 4t + 4\sin 4t + i(3\sin 4t - 4\cos 4t) \\ 5\cos 4t + i\, 5\sin 4t \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos 4t + 4\sin 4t \\ 5\cos 4t \end{bmatrix} + i \begin{bmatrix} 3\sin 4t - 4\cos 4t \\ 5\sin(4t) \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 3\cos 4t + 4\sin 4t \\ 5\cos 4t \end{bmatrix}$ and $\begin{bmatrix} 3\sin 4t - 4\cos 4t \\ 5\sin(4t) \end{bmatrix}$

are two linearly independent solution.

The general solution is then given by

$$\underline{X}(t) = C_1 \begin{bmatrix} 3\cos 4t + 4\sin 4t \\ 5\cos 4t \end{bmatrix} + C_2 \begin{bmatrix} 3\sin 4t - 4\cos 4t \\ 5\sin 4t \end{bmatrix}$$

Problem #2: Given:

$$y^{(5)} - 8y^{(4)} + 16y^{(3)} + y'' - 8y' + 16y = x^2 e^{4x}$$

(a) Find the homogeneous solution given that the characteristic equation is $r^5 - 8r^4 + 16r^3 + r^2 - 8r + 16$
$= (r^3+1)(r^2-8r+16)$.

Answer:

r^3+1 has three roots. In order to find them, use the Euler's formula:

$$-1 = e^{i\pi + 2k\pi i}, \quad k \text{ any integer.}$$

$k=0$ gives $(-1)^{\frac{1}{3}} = e^{i\pi/3} = \cos\pi/3 + i\sin\pi/3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=1$ gives $(-1)^{1/3} = \left(e^{3\pi i}\right)^{1/3} = e^{i\pi} = \cos\pi + i\sin\pi = -1$

$k=2$ gives $(-1)^{1/3} = e^{5\pi i/3} = \cos\frac{5\pi}{3} + i\sin 5\pi/3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.

$r_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $\quad e^{r_1 x} = e^{\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right)$

$r_2 = -1$ $\quad e^{r_2 x} = e^{-x}$

$\Rightarrow$ The part of the general solution associated to r_1, r_2, r_3 are

$$c_1 e^{-x} + c_2 e^{x/2}\cos\frac{\sqrt{3}}{2}x + c_3 e^{x/2}\sin\frac{\sqrt{3}}{2}x.$$

On the other hand, $r^2 - 8r + 16 = (r-4)^2$

$r=4$ is a double root $\Rightarrow$ e^{4x} and $x e^{4x}$ are two l.i. solutions

The general solution for the homogeneous problem is then

$$y_c = c_1 e^{-x} + c_2 e^{x/2} \cos \frac{\sqrt{3}}{2} x + c_3 e^{x/2} \sin \frac{\sqrt{3}}{2} x$$

$$+ c_4 e^{4x} + c_5 x e^{4x}$$

(b) Write down the form of the particular solution.

The right hand side ~~involv~~ $x^2 e^{4x}$ involves terms that appear in the homogeneous solution. Following Rule 2 in page 345, the particular solution is of the form:

$$y_p = (Ax^2 + Bx + C) e^{4x} x^2 .$$

Problem #3 Find a general solution to

$$y'' + y' + \frac{1}{4}y = \frac{1}{5}t^{-2}e^{-t/2}, \quad t>0.$$

Answer:

Homogeneous solution:

Characteristic equation: $r^2 + r + \frac{1}{4} = (r+\frac{1}{2})^2 = 0$

$r=-\frac{1}{2}$ is a double root.

$\Rightarrow y_1 = e^{-\frac{1}{2}t}$ $y_2 = te^{-\frac{1}{2}t}$ are two l. i. solutions.

$$\Rightarrow y_c = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t}.$$

Let's now use variation of parameters to find a particular solution.

$$y_p = u_1(t)e^{-\frac{1}{2}t} + u_2(t)\, t e^{-\frac{1}{2}t}$$

Using equation (31), page 349, gives

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(t) = \frac{1}{5} t^{-2} e^{-t/2}. \end{cases}$$

$$\Rightarrow \begin{cases} u_1' e^{-\frac{1}{2}t} + u_2' t e^{-\frac{1}{2}t} = 0 \\ u_1'(-\frac{1}{2}e^{-\frac{1}{2}t}) + u_2'(1-\frac{1}{2}t)e^{-\frac{1}{2}t} = \frac{1}{5}t^{-2}e^{-t/2} \end{cases}$$

$$\Rightarrow \begin{cases} u_1' + t u_2' = 0 \\ -\frac{1}{2}u_1' + (1-\frac{1}{2}t) u_2' = \frac{1}{5}t^{-2} \end{cases}$$

$\Rightarrow u_1' = -t\,u_2'$ and

$-\frac{1}{2}(-\cancel{t}u_2') + (1-\cancel{\tfrac{1}{2}}t)u_2' = \frac{1}{5}t^{-2}$

$\Rightarrow u_2' = \frac{1}{5}t^{-2} \Rightarrow$ Take $u_2 = \frac{1}{5}\frac{t^{-1}}{-1} = -\frac{1}{5t}$

$u_1' = -t\,\frac{1}{5}t^{-2} = -\frac{1}{5t} \Rightarrow$ Take $u_1 = -\frac{1}{5}\ln t$

Therefore $y_p = -\frac{1}{5}\ln t\, e^{-\frac{1}{2}t} - \frac{1}{5}\frac{1}{t}\,t e^{-\frac{1}{2}t}$

$= -\frac{1}{5}(1+\ln t)e^{-\frac{1}{2}t}$.

$\therefore\ y = y_c + y_p = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t} - \frac{1}{5}(1+\ln t)e^{-\frac{1}{2}t}$.

Problem #4 Consider the system of equations:

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) Find the homogeneous solution.

Answer: $A = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{bmatrix}$

Characteristic eqn:

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(9-\lambda)(2-\lambda) = (2-\lambda)^2(9-\lambda)$$

$\Rightarrow \lambda_1 = 2$ is a double eigenvalue.

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -7 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{switch}]{-\frac{1}{7}R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

E-vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad a - b - c = 0 \quad a = b + c$

$$\Rightarrow v = \begin{bmatrix} b+c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent e-vectors with the same e-value $\lambda_1 = 2$.

On the other hand, $\lambda_3 = 9$ is another eigenvalue

$$A - 9I = \begin{bmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{bmatrix} \xrightarrow[R_2 + R_3]{-R_1 + R_2} \begin{bmatrix} -7 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{7}R_2]{-\frac{1}{7}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ If $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow a = 0, \quad c = 0, \quad b$ is arbitrary

$\Rightarrow v = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an e-vector with e-value $\lambda = 9$.

$$\Rightarrow X(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{9t}$$

(b) Find the fundamental matrix and its inverse.

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{2t} & 0 \\ e^{2t} & 0 & e^{9t} \\ 0 & e^{2t} & 0 \end{bmatrix}$$

In fact, we need to compute the inverse at $t=0$.

$$\Phi(0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & +1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Therefore $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

(c) Use the Fundamental matrix and its inverse to find the solution that satisfies the initial condition

$$\mathbf{X}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Answer:

$$\mathbf{X}(t) = \Phi(t)\,\Phi^{-1}(0)\,\mathbf{X}_0 = \begin{bmatrix} e^{2t} & e^{2t} & 0 \\ e^{2t} & 0 & e^{9t} \\ 0 & e^{2t} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{2t} & 0 \\ e^{2t} & 0 & e^{9t} \\ 0 & e^{2t} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{9t} \\ e^{2t} \end{bmatrix}.$$

Problem #5 Let A be the 3×3 matrix given by

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues and corresponding complete set of eigenvectors.

Answer:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2 .$$

Case: $\lambda = 1$

$$A - I = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_2+R_1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{swap}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the e-vector, then

$b = 0,\ c = 0$ a arbitrary $\Rightarrow v = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is e-vector with $\lambda = 1$ as e-value

Case $\lambda_2 = 2$ Double e-value.

$$A - 2I = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an e-vector,

$\Rightarrow -a - 3b + c = 0 \Rightarrow c = a + 3b$

$$\Rightarrow v = \begin{bmatrix} a \\ b \\ a+3b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

are 2 l.i. e-vectors associated to $\lambda = 2$.

Therefore $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an e-vector associated to $\lambda_1 = 1$

$v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ are two linearly independent e-vectors associated to $\lambda_2 = \lambda_3 = 2$.

(b) Diagonalize the matrix A and use it to compute the power A^{500} of the matrix A.

Answer:

Let's form the matrix P that has the eigenvectors as its column vectors

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Let Λ be the diagonal matrix having the e-values as its diagonal elements:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow A = P \Lambda P^{-1}$

Let's compute P^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3+R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -3 & 1 \end{array}\right] \xrightarrow{\text{swap}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow P^{-1} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow A^{500} = (P\Lambda P^{-1})^{500} = \underbrace{P\Lambda \cancel{P^{-1}}\, \cancel{P}\Lambda \cancel{P^{-1}}\, \cancel{P}\Lambda \cancel{P^{-1}} \ldots \cancel{P}\Lambda P^{-1}}_{500 \text{ times}}$$

$$= P\Lambda^{500} P^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{500} & 0 \\ 0 & 0 & 2^{500} \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{500} & 0 \\ 0 & 0 & 2^{500} \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -3\cdot 2^{500} & 2^{500} \\ 0 & 2^{500} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3-3\cdot 2^{500} & 0 \\ 0 & 2^{500} & 0 \\ 0 & \cancel{-3\cdot 2^{500}+3\cdot 2^{500}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3\cdot(1-2^{500}) & -1+2^{500} \\ 0 & 2^{500} & 0 \\ 0 & 0 & 2^{500} \end{bmatrix} = \begin{bmatrix} 1 & 3(1-2^{500}) & -(1-2^{500}) \\ 0 & 2^{500} & 0 \\ 0 & 0 & 2^{500} \end{bmatrix}$$

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Problem #6: Compute the matrix exponential

e^{At}, where:

$$A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}.$$

Answer:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) + 8 = (\lambda-5)(\lambda+1)+8$$

$$= \lambda^2 - 4\lambda - 5 + 8 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$$

Case: $\lambda = 1$.

$$A - I = \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

If $v = \begin{bmatrix} a \\ b \end{bmatrix}$ is an e-vector, then

$a - b = 0$ $\quad b = a \quad \Rightarrow v = \begin{bmatrix} a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Case: $\lambda_2 = 3$

$$A - 3I = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow a - 2b = 0 \quad a = 2b \quad v = \begin{bmatrix} 2b \\ b \end{bmatrix} = b\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$e^{At} = e^{t P\Lambda P^{-1}} = P e^{t\Lambda} P^{-1} = P \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -e^t & 2e^t \\ e^{3t} & -e^{3t} \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{3t} & 2e^t - 2e^{3t} \\ -e^t + e^{3t} & 2e^t - e^{3t} \end{bmatrix}$$