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Homework 10

Problem 1: Consider the Sturm-Liouville problem with boundary conditions of the third kind:

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda\phi = 0 \\ h_1\phi(0) - \dfrac{d\phi}{dx}(0) = 0 \\ h_2\phi(L) + \dfrac{d\phi}{dx}(L) = 0 \end{cases}$$

Show that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem if and only if the parameters h_1, h_2 satisfy the equation of the two-sheeted hyperbola $h_1 + h_2 + Lh_1h_2 = 0$.

Answer: We need to prove the two directions.

Let's assume that $\lambda = 0$ is an eigenvalue

Proof by contradiction.

Assume that $h_1 + h_2 + Lh_1h_2 \neq 0$

Since $\dfrac{d^2\phi}{dx^2} = 0 \Rightarrow \phi = c_1x + c_2$ not trivial

Applying the boundary conditions we get:

$h_1 c_2 - c_1 = 0 \Rightarrow c_1 = h_1 c_2$

$h_2(c_1L + c_2) + c_1 = 0 \Rightarrow (h_2h_1L + h_2 + h_1)c_2 = 0$

$\Rightarrow h_2h_1Lc_2 + h_2c_2 + c_2h_1 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = h_1c_2 = 0$

Since $h_2h_1L + h_2 + h_1 \neq 0$

$\Rightarrow \phi = 0$ is a trivial solution, which is a contradiction.

Let's now prove the other direction.

Assume $h_1 + h_2 + Lh_1h_2 = 0$

We then need to find a nontrivial solution to

$$\begin{cases} \frac{d^2\phi}{dx^2} = 0 \\ h_1\phi(0) - \frac{d\phi}{dx}(0) = 0 \\ h_2\phi(L) + \frac{d\phi}{dx}(L) = 0 \end{cases}$$

$\phi(x) = c_1 x + c_2$ and the boundary conditions are:

$h_1 c_2 - c_1 = 0 \Rightarrow c_1 = h_1 c_2$

$h_2(c_1 L + c_2) + c_1 = (h_1 h_2 L + h_1 + h_2)c_2 = 0$ since $h_1 h_2 L + h_1 + h_2 = 0$

$\Rightarrow$ The second b.c. is trivial

$\Rightarrow c_1 = h_1 c_2$ is the only restriction

$\Rightarrow \phi(x) = c_1 x + c_2 = h_1 c_2 x + c_2 = c_2(h_1 x + 1)$

$\Rightarrow \phi(x) = 1 + h_1 x$ is an eigenfunction with eigenvalue $\lambda = 0$,

and any other eigenfunction is an scalar multiple.

Problem 2: Consider

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + \alpha u$$

where c, ρ, K_0, α are functions of x, subject to:

$u(0,t) = 0$
$u(L,t) = 0$
$u(x,0) = f(x)$

Assume that the ~~eige~~ appropriate eigenfunctions are known.

(a) Show that the eigenvalues are positive if $\alpha < 0$.

Answer:

The corresponding Sturm-Liouville problem is:

$$\frac{d}{dx}\left(K_0(x)\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c(x)\rho(x)\phi = 0$$

$p = K_0(x)$, $q = \alpha$, $\sigma = c(x)\rho(x)$

For Dirichlet boundary conditions, the Rayleigh quotient reads:

$$\lambda = \frac{\cancelto{0}{-K_0\phi\frac{d\phi}{dx}\Big|_0^L} + \int_0^L\left[K_0(x)\left(\frac{d\phi}{dx}\right)^2 - \alpha\phi^2\right]dx}{\int_0^L \phi^2\, c(x)\rho(x)\,dx}$$

$$= \frac{\int_0^L\left[K_0(x)\left(\frac{d\phi}{dx}\right)^2 - \alpha\phi^2\right]dx}{\int_0^L \phi^2(x)\, c(x)\rho(x)\,dx}$$

$\int_0^L K_0(x)\left(\frac{d\phi}{dx}\right)^2 dx \geq 0$ if $K_0 > 0$.

and since $\phi \neq 0$

if $\alpha < 0 \Rightarrow -\alpha\phi^2 \geq 0$

$\Rightarrow \int_0^L -\alpha\phi^2\,dx > 0$

$\Rightarrow \lambda > 0$.

(b) Solve the initial value problem.

Answer:

We are assuming we know the eigenfunctions ϕ_n, and we know $\lambda_n > 0$.

The time dependent problem is

$$\frac{1}{h}\frac{dh}{dt} = -\lambda_n \Rightarrow h(t) = c\, e^{-\lambda_n t}$$

$\Rightarrow$ ~~ex~~ Using the principle of superposition we get:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}, \quad \lambda_n = \frac{\int_0^l \left[K_0(x)\left[\frac{d\phi_n}{dx}\right]^2 - \alpha \phi_n^2\right] dx}{\int_0^l \phi_n^2\, c(x)\rho(x)\, dx}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

where

$$a_n = \frac{\int_0^l f(x)\,\phi_n(x)\, c(x)\rho(x)\,dx}{\int_0^l \phi_n^2\, c(x)\rho(x)\,dx}.$$

(c) Briefly discuss $\lim_{t\to\infty} u(x,t)$

Here we are still assuming $\alpha < 0$

$$e^{-\lambda_n t} \to 0 \text{ as } t \to \infty$$

$\Rightarrow \lim_{t\to\infty} u(x,t) \equiv 0$, which is the equilibrium solution satisfying the Dirichlet boundary conditions.

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Problem 3: Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{subject to}$$

$$\frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0$$

Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).

Answer:

$p = 1, \quad q = -x^2, \quad \sigma = 1$

The Rayleigh quotient implies:

$$\lambda = \frac{\cancel{-p\phi\frac{d\phi}{dx}\Big|_0^1}^{\,0} + \int_0^1 \left[\left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2\right]dx}{\int_0^1 \phi^2(x)\,dx}$$

$$= \frac{\int_0^1 \left[\left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2(x)\right]dx}{\int_0^1 \phi^2(x)\,dx}$$

$\int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx \geq 0,$

Since $\phi \neq 0 \Rightarrow \int_0^1 x^2\phi^2(x)\,dx > 0$, $\int_0^1 \phi^2(x)\,dx > 0$

They cannot be zero because otherwise $x^2\phi^2(x) = 0 \Rightarrow \phi^2 \equiv 0$ by continuity

($\lambda = 0$ is not an eigenvalue).

$\Rightarrow \lambda > 0$

Problem 4: Determine an upper and a (non-zero) lower bound for the lowest frequency of vibration of a non-uniform string fixed at $x=0$, and $x=1$ with $c^2 = 1+4\alpha^2(x-\frac{1}{2})^2$.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \\ u(0,t)=0 \\ u(L,t)=0. \end{cases}$$

Answer:

The corresponding Sturm-Liouville problem is:

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda \frac{1}{c^2(x)} \phi = 0 \\ \phi(0)=0 \\ \phi(L)=0 \end{cases}$$

$\Rightarrow p=1,\ q=0,\ \sigma = \frac{1}{c^2(x)}.$

By the minimization principle, we know

$$\lambda_1 = \min_{u \text{ trial}} \frac{\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\int_0^1 u^2 \frac{1}{c^2(x)} dx}$$

Assume $C_{min} \le c(x) \le C_{max}$

$$\Rightarrow \frac{1}{C_{max}^2} \le \frac{1}{c^2(x)} \le \frac{1}{C_{min}^2} \Rightarrow \int_0^1 u^2 \frac{1}{C_{max}^2} dx \le \int_0^1 u^2 \frac{1}{c^2(x)} dx \le \int_0^1 u^2 \frac{1}{C_{min}^2} dx$$

$$\Rightarrow \frac{\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\frac{1}{C_{min}^2}\int_0^1 u^2 dx} \le \frac{\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\int_0^1 u^2 \frac{1}{c^2(x)} dx} \le \frac{\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\frac{1}{C_{max}^2}\int_0^1 u^2 dx}$$

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Taking the ~~max~~ minimum over trial functions satisfying the boundary conditions

$u(0)=0$
$u(1)=0$

we get

$$C_{min}^2 \min_{u \text{ trial}} \frac{\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\int_0^1 u^2 dx} \leq \lambda_1 \leq C_{max}^2 \min_{u \text{ trial}} \frac{\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\int_0^1 u^2 dx}$$

However, we know that the minimum

$$\min_{u \text{ trial}} \frac{\int_0^1 \left(\frac{du}{dx}\right)^2 dx}{\int_0^1 u^2 dx}$$

is the first eigenvalue associated to the same problem for $c=1$, which is $\left(\frac{\pi}{L}\right)^2 = \pi^2$

$$\Rightarrow \quad C_{min}^2 \pi^2 \leq \lambda_1 \leq C_{max}^2 \pi^2 ,$$

For the present problem, $c^2(x) = 1 + 4\alpha^2 \left(x - \frac{1}{2}\right)^2$

We need to compute the min and max

$$\frac{dc^2}{dx} = 8\alpha^2 \left(x - \frac{1}{2}\right) \Rightarrow x = \frac{1}{2}$$ is the only critical point inside the interval $[0,1]$

$\Rightarrow$ The min/max may occur at $x=\frac{1}{2}$, or at the end points $x=0,1$

$c^2(0) = 1+\alpha^2, \quad c^2(1) = 1+\alpha^2, \quad c^2\left(\frac{1}{2}\right) = 1$

Then $C_{min}^2 = 1$, $C_{max}^2 = 1+\alpha^2$

$\Rightarrow \quad \pi^2 \leq \lambda_1 \leq (1+\alpha^2)\,\pi^2$.