Hw12.1

Homework 12

Problem 1: Consider for $\lambda >> 1$:

$$\frac{d^2\phi}{dx^2} + [\lambda\sigma(x) + q(x)]\phi = 0.$$

(a) Subsitute $\phi = A(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right]$

Determine a differential equatron for $A(x)$.

Answer:

$$\phi' = A'\exp[\ \ ] + A\,i\lambda^{1/2}\sigma^{1/2}\exp[\ \ ]$$

$$\phi'' = A''\exp[\ \ ] + A'i\lambda^{1/2}\sigma^{1/2}\exp[\ \ ] + A i^2\lambda\sigma\exp[\ \ ]$$

$$+ i\lambda^{1/2}A'\sigma^{1/2}\exp[\ \ ] + i\lambda^{1/2}A\frac{1}{2}\sigma^{-1/2}\sigma'\exp[\ \ ]$$

$$= \left[A'' + 2i\lambda^{1/2}A'\sigma^{1/2} + \frac{i}{2}\lambda^{1/2}A\sigma^{-1/2}\sigma' - A\lambda\sigma\right]\exp\left[i\lambda^{1/2}\int_0^x\sigma^{1/2}(z)\,dz\right].$$

$$\Rightarrow 0 = \left[A'' + 2i\lambda^{1/2}A'\sigma^{1/2} + \frac{i}{2}\lambda^{1/2}A\sigma^{-1/2}\sigma' \cancel{- A\lambda\sigma} \cancel{+ A\lambda\sigma} + qA\right]\exp\left[i\lambda^{1/2}\int_0^x\sigma^{1/2}(z)\,dz\right]$$

Therefore, the differential equatron for A is:

$$A'' + 2i\lambda^{1/2}A'\sigma^{1/2} + \frac{i}{2}\lambda^{1/2}A\sigma^{-1/2}\sigma' + qA = 0.$$

(b) Let $A(x) = A_0(x) + \lambda^{-1/2} A_1(x) + \dots$ Solve for $A_0(x)$ and $A_1(x)$. Verify equation (5.9.8) from the book.

Answer:

Subsituting this (asymptotic) ansatz in the differential equation for A above, we get:

$$
\begin{aligned}
0 = {} & A_0'' + \lambda^{-1/2} A_1'' + \lambda^{-1} A_2'' + \dots \\
& + 2i\lambda^{1/2}\sigma^{1/2}\left(A_0' + \lambda^{-1/2} A_1' + \lambda^{-1} A_2' + \dots\right) \\
& + \frac{i}{2}\lambda^{1/2}\sigma^{-1/2}\sigma'\left(A_0 + \lambda^{-1/2} A_1 + \lambda^{-1} A_2 + \dots\right) \\
& + q\left(A_0 + \lambda^{-1/2} A_1 + \lambda^{-1} A_2 + \dots\right)
\end{aligned}
$$

Collecting terms in λ with the same power, we get:

$$
\begin{aligned}
0 = {} & \lambda^{+1/2}\left[2i\sigma^{1/2} A_0' + \frac{i}{2}\sigma^{-1/2}\sigma' A_0\right] \\
& + \lambda^{0}\left[A_0'' + 2i\sigma^{1/2} A_1' + \frac{i}{2}\sigma^{-1/2}\sigma' A_1 + qA_0\right] \\
& + \lambda^{-1/2}\left[A_1'' + 2i\sigma^{1/2} A_2' + \frac{i}{2}\sigma^{-1/2}\sigma' A_2 + qA_1\right] \\
& + \vdots \\
& + \lambda^{-k/2}\left[A_k'' + 2i\sigma^{1/2} A_{k+1}' + \frac{i}{2}\sigma^{-1/2}\sigma' A_{k+1} + qA_k\right] \\
& + \vdots
\end{aligned}
$$

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We want to get a zero in that equatron.
We can proceed in an asymptotic way by first killing the leading order term. This is done if

$$2i\sigma^{1/2}A_0' + \frac{i}{2}\sigma^{-1/2}\sigma' A_0 = 0$$

$$\Leftrightarrow \quad A_0' = -\frac{1}{4}\sigma^{-1}\sigma' A_0 \Leftrightarrow \frac{A_0'}{A_0} = -\frac{1}{4}\frac{\sigma'}{\sigma}$$

$$\Leftrightarrow (\log A_0)' = -\frac{1}{4}(\log\sigma)' \Leftrightarrow \log A_0 = -\frac{1}{4}\log\sigma + \text{const}.$$

Assuming the constant is zero (otherwise it will change the e-function by a multiple constant) we get

$$A_0 = e^{-\frac{1}{4}\log\sigma} = \sigma^{-1/4}$$

This verifies equation (5.9.8) for $p=1$.

With is choice of A_0, the right hand side will be of order $O(1)$ at most.

We can kill that term by equating:

$$A_0'' + 2i\sigma^{1/2}A_1' + \frac{i}{2}\sigma^{-1/2}\sigma' A_1 + q A_0 = 0$$

$$\Rightarrow \quad A_1' + \frac{1}{4}\sigma^{-1}\sigma' A_1 = \frac{i}{2}\sigma^{-1/2}\left[A_0'' + q A_0\right]$$

where $A_0 = \sigma^{-1/4}$ is now known.

This is an equation for A_1 only!

We can use integrating factors to solve it:

$$\frac{d}{dx}(e^f A) = e^f [A' + f'A] \Rightarrow f' = \frac{1}{4}\sigma^{-1}\sigma' = \frac{1}{4}(\log\sigma)'$$

$\Rightarrow$ choose $f = \frac{1}{4}\log\sigma$

$$\Rightarrow \frac{d}{dx}\left(e^{\frac{1}{4}\log\sigma} A_1\right) = e^{\frac{1}{4}\log\sigma}\left(A_1' + \frac{1}{4}(\log\sigma)' A_1\right)$$

$$= \sigma^{1/4}\,\frac{i}{2}\,\sigma^{-\frac{1}{2}}\left[A_0'' + q A_0\right]$$

$$\Rightarrow \sigma^{1/4}(x) A_1(x) = \frac{i}{2}\int_0^x \sigma^{-1/4}(z)\left[A_0''(z) + q(z)A_0(z)\right]dz + \text{const}.$$

The constant of integration above can be determined using the boundary conditions.

(c) Suppose that $\phi(0) = 0$. Use A_1 to improve (5.9.9)

Answer:

With this boundary condition, the constant above is zero.

$$\Rightarrow A_1(x) = \frac{i}{2}\sigma^{-1/4}(x)\int_0^x \sigma^{-1/4}(z)\left[A_0''(z) + q(z)A_0(z)\right]dz.$$

Let's define $B_1(x) = \frac{1}{2}\sigma^{-1/4}(x)\int_0^x \sigma^{-1/4}(z)\left[A_0''(z) + q(z)A_0(z)\right]dz$

$\Rightarrow$ The eigenfunction $\phi(x)$ is now better approximated as:

$$\cancel{\phi(x)} \left(A_0(x) + i B_1(x)\lambda^{-1/2}\right)\exp\left[i\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right]$$

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Take linear combinations of real and imaginary parts:

$$\Rightarrow \phi(x) = c_1\left(A_0(x)\cos\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right) - B_1(x)\lambda^{-1/2}\sin\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right)\right)$$

$$+ c_2\left(A_0(x)\sin\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right) + B_1(x)\lambda^{-1/2}\cos\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right)\right)$$

$$\Rightarrow 0 = \phi(0) = c_1 A_0(0) + c_2\cdot \cancelto{0}{B_1(0)}\,\lambda^{-1/2} = c_1\sigma^{-1/4}(0)$$

$$\Rightarrow c_1 = 0$$

Assume $c_2 = 1$ (multiplicative constants)

$$\Rightarrow \phi(x) = A_0(x)\sin\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right) + B_1(x)\lambda^{-1/2}\cos\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right)$$

$$= \sigma^{-1/4}(x)\sin\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right)$$

$$+ \lambda^{-1/2}\cos\left(\lambda^{1/2}\int_0^x \sigma^{1/2}(z)\,dz\right)\frac{1}{2}\sigma^{-1/4}(x)\int_0^x \sigma^{-1/4}\left[A_0''(z) + q(z)A_0(z)\right]dz$$

The first term coincides with equation (5.9.9) for the case $p=1$. The second terms ~~the~~ improves the approximation by including this next order correction term.

We now know that the error is $O(\lambda^{-1})$ which decays to zero as $\lambda \to \infty$.

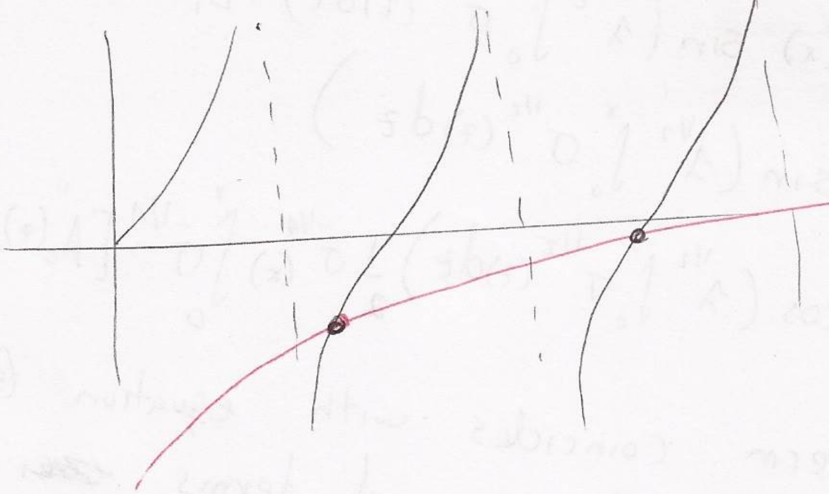
(d) Use part (c) to improve (5.9.10) if $\phi(L)=0$.

Answer:

$$0=\phi(L)=\sigma^{-1/4}(L)\sin\left(\lambda^{1/2}\int_0^L\sigma^{1/2}(z)dz\right)$$
$$+B_1(L)\lambda^{-1/2}\cos\left(\lambda^{1/2}\int_0^L\sigma^{1/2}(z)dz\right)$$

$$\Rightarrow \tan\left(\lambda^{1/2}\int_0^L\sigma^{1/2}(z)dz\right)=-B_1(L)\sigma^{1/4}(L)\lambda^{-1/2}$$

We need to solve for λ in this equation to improve λ. We can do it numerically.



(e) Obtain a recusion formula for $A_n(x)$

Answer: We can observe the pattern in part (b) and get: $2i\sigma^{1/2}A_n' + \frac{i}{2}\sigma^{-1/2}\sigma' A_n + A_{n-1}'' + q A_0 = 0$

whose solution is:

$$\sigma^{1/4}(x)A_n(x)=\frac{i}{2}\int_0^x\sigma^{-1/4}(z)\left[A_{n-1}''(z)+q(z)A_{n-1}(z)\right]dz + \text{const}$$

if $\phi(0)=\phi(L)=0 \Rightarrow$ we get: const $=0$ and

$$A_n(x)=\frac{i}{2}\sigma^{-1/4}(x)\int_0^x\sigma^{-1/4}(z)\left[A_{n-1}''(z)+q(z)A_{n-1}(z)\right]dz.$$

HW12.4

Problem 2: Suppose that we didn't know equation (6.2.15) in the textbook, but thought it possible to approximate $\frac{d^2f}{dx^2}$ by an unknown linear combination of three function values, $f(x_0-\Delta x)$, $f(x_0)$ and $f(x_0+\Delta x)$:

$$\frac{d^2f}{dx^2}(x_0) \approx a\, f(x_0-\Delta x) + b\, f(x_0) + c\, f(x_0+\Delta x).$$

Determine a, b, and c by expanding the right hand side in a Taylor series around x_0 and equating coefficients through $\frac{d^2f}{dx^2}$.

Answer:

$$\frac{d^2f}{dx^2}(x_0) \approx a\left(f(x_0) - \Delta x\, f'(x_0) + \frac{\Delta x^2}{2} f''(x_0) + O(\Delta x^3)\right)$$

$$+ b\, f(x_0)$$

$$+ c\left(f(x_0) + f'(x_0)\Delta x + \frac{1}{2} f''(x_0)\Delta x^2 + O(\Delta x^3)\right)$$

$$= (a+b+c)\, f(x_0) + f'(x_0)\Delta x\,(c-a) + f''(x_0)\Delta x^2\left(\frac{a+c}{2}\right) + O(\Delta x^3)$$

Matching the terms in the Taylor expansion we get:

$a+b+c=0$

$c-a=0$

$\Delta x^2\left(\frac{a+c}{2}\right) = 1$

$\Rightarrow c = a$, and $1 = \Delta x^2 \cdot a \Rightarrow a = c = \frac{1}{\Delta x^2}$

$b = -a - c = -\frac{2}{\Delta x^2}$

Therefore, $a = \frac{1}{\Delta x^2}$

$b = -\frac{2}{\Delta x^2}$

$c = \frac{1}{\Delta x^2}$

and $\frac{d^2 f}{dx^2}(x_0) \approx \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{\Delta x^2}$

Problem 3:

(a) Show that the truncation error for our numerical scheme (6.3.3), becomes much smaller if $\frac{k\Delta t}{\Delta x^2} = \frac{1}{6}$.

Hint: u satisfies the partial differential equation in (6.3.1).

Answer: ~~False~~ Assume u is an exact sol. to the heat eqn. The truncation error in our numerical scheme is:

$$E = \frac{u(x_0, t_0 + \Delta t) - u(x_0, t_0)}{\Delta t} - k\frac{u(x_0 + \Delta x, t_0) - 2u(x_0, t_0) + u(x_0 - \Delta x, t_0)}{\Delta x^2}$$

Taking the Taylor expansion and using the notation

$u = u(x_0, t_0)$, $u_t = \frac{\partial}{\partial t} u(x_0, t_0)$, $u_{tt} = \frac{\partial^2}{\partial t^2} u(x_0, t_0)$, $u_x = \frac{\partial}{\partial x} u(x_0, t_0)$, etc.:

we get:

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$$E = \frac{1}{\Delta t}\left[\cancel{u^{(x_0,t_0)}} + u_t \Delta t + \frac{1}{2} u_{tt} \Delta t^2 + O(\Delta t^3) - \cancel{u(x_0,t_0)}\right]$$

$$-\frac{k}{\Delta x^2}\left[\cancel{u(x_0,t_0)} + \cancel{\Delta x u_x} + u_{xx}\frac{\Delta x^2}{2} + \cancel{u_{xxx}\frac{\Delta x^3}{6}} + u_{xxxx}\frac{\Delta x^4}{24} + O(\Delta x^5)\right.$$

$$-2\cancel{u(x_0,t_0)}$$

$$\left.+\cancel{u(x_0,t_0)} - \cancel{\Delta x u_x} + u_{xx}\frac{\Delta x^2}{2} - \cancel{u_{xxx}\frac{\Delta x^3}{6}} + u_{xxxx}\frac{\Delta x^4}{24} - O(\Delta x^5)\right]$$

$$= \cancel{u_t} + \frac{1}{2}\Delta t\, u_{tt} + O(\Delta t^2)$$

$$- k\cancel{u_{xx}} - k\Delta x^2 \frac{u_{xxxx}}{16} + O(\Delta x^4)$$

Since $u_t = k u_{xx}$, then $u_{tt} = k\partial_x^2 u_t = k^2 u_{xxxx}$ and

$$E = \frac{k^2}{2}\Delta t\, u_{xxxx} + O(\Delta t^2) - \frac{k\Delta x^2}{16} u_{xxxx} + O(\Delta x^4)$$

If $k\dfrac{\Delta t}{\Delta x^2} = \dfrac{1}{6} \Rightarrow \dfrac{k^2}{2}\dfrac{\Delta t}{\Delta x^2} = \dfrac{k}{16}$

$$\Rightarrow E = \cancel{\frac{k}{16} u_{xxxx}} + O(\Delta t^2) - \cancel{\frac{k\Delta x^2}{16} u_{xxxx}} + O(\Delta x^4)$$

$$= O(\Delta t^2) + O(\Delta x^4)$$

(b) If $\dfrac{k\Delta t}{\Delta x^2} = 1/6$, determine the order of magnitude of the truncation error.

Answer: Second order in time and fourth order in space.

Problem 4: Numerically compute solutions to the heat equation with the temperature initially given in Fig. 6.3.4. Use (6.3.16) - (6.3.18) with $N=10$. Do for various s (discuss stability):

$s=0.49$, $s=0.50$, $s=0.51$, $s=0.52$.

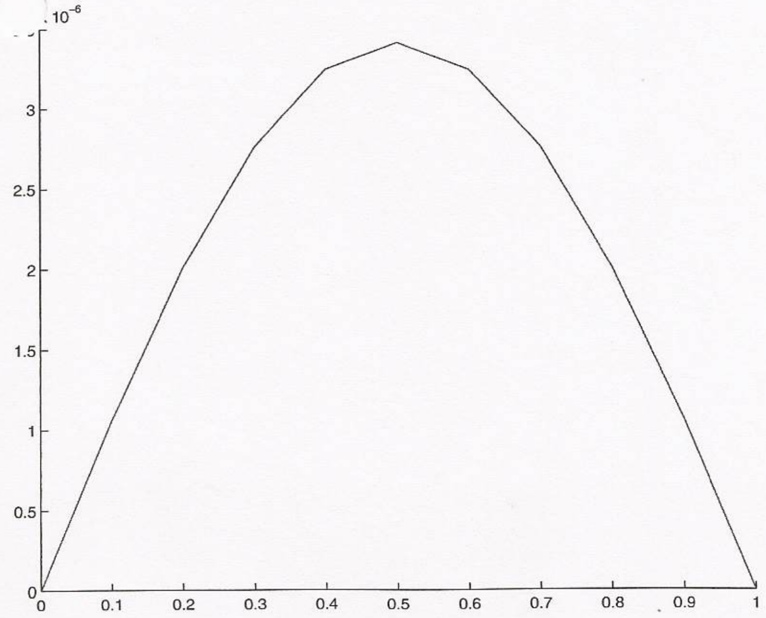
Answer:

You can see the numerical results in the next page for $k=1$, $L=1$, $N=10$ and 0.25 as the height of the triangle.

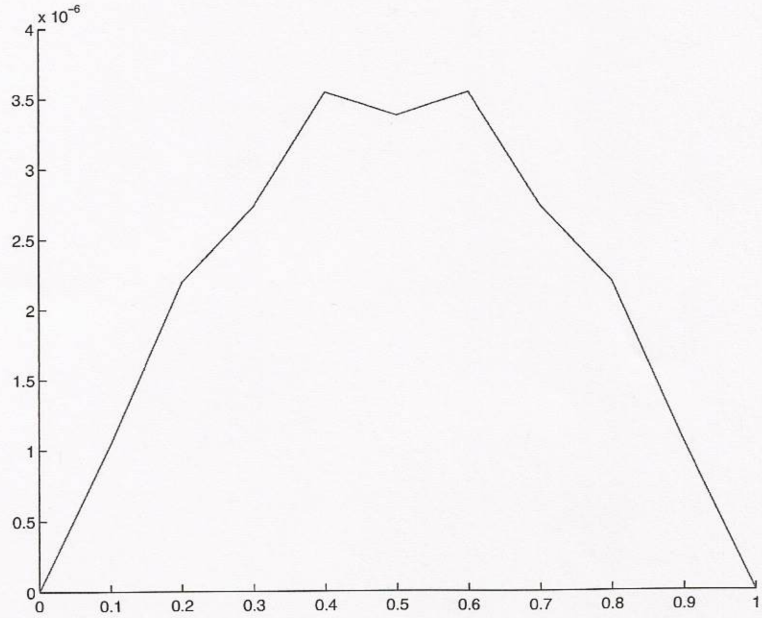
if we pay attention to the scale, we see clear instabilities for $s=0.51$ and $s=0.52$.

Hw12.6

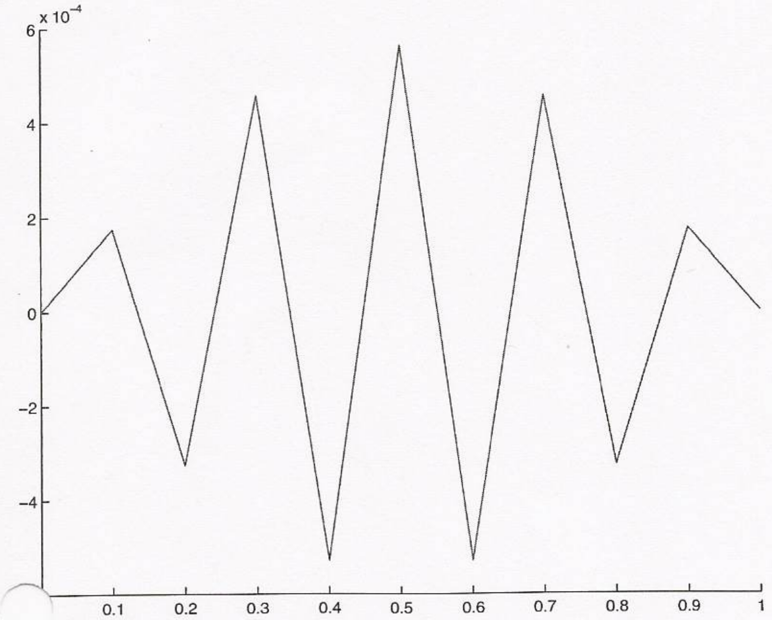
s=0.49



s=0.50



s=0.51



s=0.52