MATH 322 - SEC 001, SPRING 2013. HOMEWORK 2

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Due : Friday, February 15

Please show all your work and/or justify your answers for full credit.

Problem 1: (*Textbook problem 1.5.8*) If the Laplace's equation is satisfied in three dimensions, show that

$$\oiint \nabla u \cdot \hat{\mathbf{n}} dS = 0$$

for any closed surface. (*Hint:*Use the divergence theorem). Give a physical interpretation of this results (in the context of heat flow).

Problem 2: (*Textbook problem 1.5.5*) Assume that the temperature is circularly symmetric: $u = u(r,t)$, where $r^2 = x^2 + y^2$. We will derive the heat equation for this problem. Consider any circular annulus $a \leq r \leq b$.

(a) Show that the total heat energy is $2\pi \int_a^b c\rho ur \; dr$

(b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is

$$-2\pi b K_0 \frac{\partial u}{\partial r}\Big|_{r=b}.$$

A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the circularly symmetric heat equation without sources

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Problem 3: (*Textbook problem 1.5.9*) Determine the equilibrium (*circularly symmetric*) temperature distribution inside a circular annulus ($r_1 \leq r \leq r_2$)

(a) if the outer radius is at temperature T_2 and the inner at T_1

(b) if the outer radius is insulated and the inner radius is at temperature T_1.

Problem 4: (*Textbook problem 1.5.11*) Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), a \leq r \leq b$$

subject to

$$u(r,0) = f(r), \; \frac{\partial u}{\partial r}(a,t) = \beta, \; \text{ and } \; \frac{\partial u}{\partial r}(b,t) = 1.$$

Using physical reasoning for what value(s) of β does an equilibrium temperature exist?

Problem 5: Find the separated equations satisfied by $a(x)$ and $b(y)$ for the following partial differential equations

(a) $u_{xx} - 2u_{yy} = 0$

(b) $u_{xx} + u_{yy} + 2u_x = 0$

(c) $x^2 u_{xx} - 2yu_y = 0$

(d) $u_{xx} + u_x + u_y - u = 0$

Problem 6: Which of the following are solutions of the Laplace's equation?

(a) $u(x, y) = e^x \cos 2y$
(b) $u(x, y) = e^x \cos y + e^y \cos x$
(c) $u(x, y) = e^x e^y$
(d) $u(x, y) = (3x + 2)e^y$

Problem 7: Find the separated solution $u(x, y)$ of Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in the region $0 < x < L, y > 0$ that satisfies the boundary conditions

$$\begin{cases} u(0, y) = 0 \\ u(L, y) = 0, \end{cases}$$

and the boundedness condition

$$|u(x, y)| \leq M$$

for $y > 0$, where M is a constant independent of (x, y).

Problem 8: Find the separated solutions $u(x, t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 \leq x \leq L, t > 0$, that satisfy the boundary conditions $u(0, t) = 0$, $\frac{\partial u}{\partial x}(L, t) = 0$.

Problem 9: (*Textbook problem 2.2.2*)

(a) Show that

$$L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$$

is a linear operator.

(b) Show that usually

$$L(u) = \frac{\partial}{\partial x}\left[K_o(x, u)\frac{\partial u}{\partial x}\right]$$

is not a linear operator.