HW 3.1
2013

Problem 1: Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and the corresponding eigenfunctions), if ϕ satisfies the following boundary conditions:

(f) $\phi(a) = 0$, $\phi(b) = 0$ (You may assume $\lambda > 0$).

Answer:

If ϕ satisfies the equation above for $a \le x \le b$

$\Rightarrow \varphi(x) = \phi(x+a)$ satisfies the same equation

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

with b.c.:

$\varphi(0) = \phi(a) = 0$

$\varphi(L) = \varphi(b-a) = \phi(b) = 0$, $\quad L = b-a$

$\Rightarrow$ If $\lambda = \alpha^2 \Rightarrow \varphi(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$

$0 = \varphi(0) = c_1 \Rightarrow c_1 = 0 \Rightarrow \varphi(x) = c_2 \sin\alpha x$

$0 = c_2 \sin\alpha L$ $\quad \alpha L = n\pi \Rightarrow \alpha = \frac{n\pi}{L} = \frac{n\pi}{b-a}$

$\Rightarrow \varphi(x) = c_2 \sin\frac{n\pi x}{b-a}$

$\Rightarrow \phi(x) = \varphi(x-a) = c_2 \sin\frac{n\pi(x-a)}{b-a}$, $\lambda = \left(\frac{n\pi}{b-a}\right)^2$.

(g) $\phi(0) = 0$, $\frac{d\phi}{dx}(L) + \phi(L) = 0$.

Answer:

Case: $\lambda = \alpha^2 > 0 \Rightarrow \phi(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$

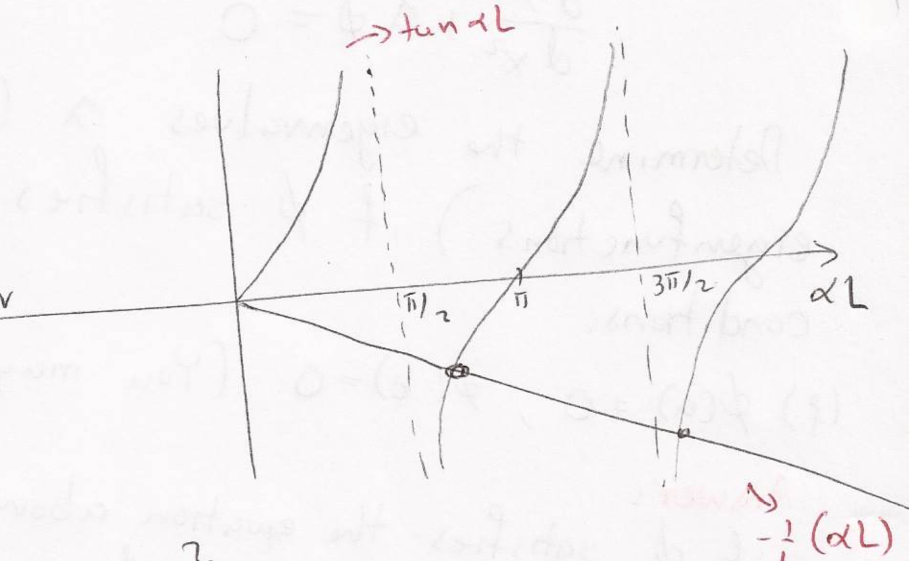
$0 = \phi(0) = c_1 \Rightarrow c_1 = 0 \quad \Rightarrow \phi(x) = c_2 \sin\alpha x \quad \frac{d\phi}{dx} = +c_2\alpha\cos\alpha x$

$0 = c_2\alpha\cos\alpha L + c_2\sin\alpha L \Rightarrow \alpha\cos\alpha L + \sin\alpha L = 0$

$\cos \alpha L \neq 0$ because if it was zero, then from the previous equation, we get $\sin \alpha L = 0$ ↯.

$\Rightarrow \alpha = -\tan \alpha L$

$\tan \alpha L = -\frac{1}{L}(\alpha L)$



⇒ we get an infinite number of α_n such that

$\tan \alpha_n L = -\alpha_n$.

$\alpha_1 < \alpha_2 < \cdots$

$\Rightarrow \quad \phi(x) = c_2 \sin \alpha_n x \quad , \quad \lambda = \alpha_n^2$

where $\quad \tan \alpha_n L = -\alpha_n$.

Case: $\lambda = 0 \quad \Rightarrow \phi(x) = c_1 x + c_2$

$0 = \phi(0) = c_2 \quad \Rightarrow c_2 = 0 \quad \Rightarrow \phi = c_1 x \quad \frac{d\phi}{dx} = c_1$

$\Rightarrow 0 = \frac{d\phi}{dx}(L) + \phi(L) = c_1 + c_1 L = c_1(1+L) \quad L > 0$

$\Rightarrow c_1 = 0 \quad \Rightarrow \lambda = 0$ is not an e-value

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Case: $\lambda = -\alpha^2 < 0 \quad \Rightarrow \phi(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$0 = \phi(0) = c_1 + c_2 \Rightarrow c_2 = -c_1 \quad \Rightarrow \phi(x) = c_1(e^{\alpha x} - e^{-\alpha x})$

$\frac{d\phi}{dx} = c_1(\alpha e^{\alpha x} + \alpha e^{-\alpha x}) = c_1 \alpha (e^{\alpha x} + e^{-\alpha x})$

$\Rightarrow 0 = \frac{d\phi}{dx}(L) + \phi(L) = c_1 \alpha (e^{\alpha L} + e^{-\alpha L}) + c_1(e^{\alpha L} - e^{-\alpha L})$

If $c_1 \neq 0 \Rightarrow \quad \alpha e^{\alpha L} + e^{\alpha L} = e^{-\alpha L} - \alpha e^{-\alpha L} \quad \Rightarrow (1+\alpha) e^{\alpha L} = (1-\alpha) e^{-\alpha L}$

$\Rightarrow \frac{1+\alpha}{1-\alpha} = e^{-2\alpha L}$ since $\alpha > 0 \Rightarrow 0 < e^{-2\alpha L} < 1$

if $0 < \alpha < 1 \Rightarrow \frac{1+\alpha}{1-\alpha} < 0$, and if $\alpha > 1 \Rightarrow \frac{1+\alpha}{1-\alpha} > 1$ since $1+\alpha > 1-\alpha$

$\Rightarrow e^{-2\alpha L} \in (0,1)$ and $\frac{1+\alpha}{1-\alpha} \in (-\infty, 0) \cup (1, \infty)$

⇒ A solution for α doesn't exist ⇒ $\lambda < 0$ is not an e-value.

Problem 2: Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$ subject to the boundary condition

$u(0,t)=0, \quad u(L,t)=0.$

Solve the initial value problem if the temperature is initially

(b) $u(x,0) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$

Answer:

$$u(x,t) = 3\sin\left(\frac{\pi x}{L}\right)e^{-k\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi x}{L}\right)e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

(c) $u(x,0) = 2\cos\frac{3\pi x}{L}$

Answer:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where

$$B_n = \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

Using the trigonometric identity $2\sin a\cos b = \sin(a+b)+\sin(a-b)$

we get:

$$B_n = \frac{2}{L}\int_0^L \left[\sin\left(\frac{3\pi+n\pi}{L}x\right) + \sin\left(\frac{-3\pi+n\pi}{L}x\right)\right]dx$$

if $n\neq 3$

$$= -\frac{2}{L}\left[\frac{L}{(n+3)\pi}\cos\left(\frac{(3+n)\pi}{L}x\right)\Big|_0^L + \frac{L}{(n-3)\pi}\cos\left(\frac{(n-3)\pi}{L}x\right)\Big|_0^L\right]$$

$$= -\frac{2}{(n+3)\pi}\left(\cos((n+3)\pi)-1\right) - \frac{2}{(n-3)\pi}\left(\cos((n-3)\pi)-1\right)$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{4}{(n+3)\pi} + \frac{4}{(n-3)\pi} & \text{if } n \text{ is even.} \end{cases}$$

(d) $u(x,0) = \begin{cases} 1, & 0 < x \le \frac{L}{2} \\ 2, & \frac{L}{2} < x < L \end{cases}$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^{\frac{L}{2}} \sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} 2\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^{\frac{L}{2}} + \frac{4}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_{\frac{L}{2}}^{L}$$

$$= -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right) - \frac{4}{n\pi}\left(\cos n\pi - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \begin{cases} 0 & \text{if } n = 4k, \ k \text{ integer} \\ \frac{6}{n\pi} & \text{if } n = 4k+1, \ k \text{ integer} \\ -\frac{4}{n\pi} & \text{if } n = 4k+2, \ k \text{ integer} \\ \frac{6}{n\pi} & \text{if } n = 4k+3, \ k \text{ integer.} \end{cases}$$

Problem 3: Evaluate $\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$ for $n \ge 0, m \ge 0$, using the trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

$$\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left[\cos\left(\frac{(n+m)\pi x}{L}\right) \cancel{\cos} + \cos\left(\frac{(n-m)\pi x}{L}\right)\right]$$

Assume $m \ne n \Rightarrow m+n > 0, \ n-m \ne 0$

$$\Rightarrow \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\Big|_0^L + \frac{1}{2}\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)$$

$$= \frac{L}{2(n+m)\pi}\sin((n+m)\pi) + \frac{L}{2(n-m)\pi}\sin((n-m)\pi) = 0.$$

If $n = m \ne 0 \Rightarrow \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^L \cos\frac{2n\pi x}{L}\,dx + \frac{1}{2}L$

$$= \frac{1}{2}\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\Big|_0^L + \frac{1}{2}L = \frac{1}{2}L$$

Assume $n = m = 0 \Rightarrow \int_0^L \cos\left(\frac{0x}{L}\right)\cos\left(\frac{0\pi x}{L}\right)dx = L \Rightarrow \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$

$$= \begin{cases} 0 & n \ne m \\ \frac{L}{2} & n = m \ne 0 \\ L & n = m = 0. \end{cases}$$

HW 3.3

Problem 4 Consider $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u$.

Suppose that the boundary conditions are:

$u(0,t)=0, \quad u(L,t)=0$.

(a) what are the possible equilibrium temperature distributions if $\alpha > 0$?

Answer:

$$\frac{\partial u}{\partial t} = 0 \Rightarrow k\frac{\partial^2 u}{\partial x^2} - \alpha u = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} - \frac{\alpha}{k} u = 0.$$

$$\Rightarrow u(x) = c_1 e^{\sqrt{\frac{\alpha}{k}}x} + c_2 e^{-\sqrt{\frac{\alpha}{k}}x}$$ since $\alpha > 0$.

$0 = u(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$

$0 = u(L) = c_1\left(e^{\sqrt{\frac{\alpha}{k}}L} - e^{-\sqrt{\frac{\alpha}{k}}L}\right)$

$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0$

$\therefore u(x) = 0$ is the only one.

(b) Solve the time-dependent problem $[u(x,0)=f(x)]$ if $\alpha > 0$. Analyze the temperature for large time $(t \to \infty)$ and compare to

Answer: Let's find the separated solutions.

$$u(x,t) = G(t)\phi(x)$$

$$G'(t)\phi(x) = kG(t)\frac{d^2\phi}{dx^2} - \alpha G(t)\phi(x)$$

$$\Rightarrow \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} - \frac{\alpha}{k}$$

$$\Rightarrow \frac{1}{kG}\frac{dG}{dt} + \frac{\alpha}{k} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

$$\begin{cases} \frac{1}{kG}\frac{dG}{dt} + \frac{\alpha}{k} = -\lambda \\ \frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \phi(0)=0, \quad \phi(L)=0 \end{cases}$$

$\Rightarrow$ We know that this implies

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad \phi(x) = c_1 \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$

Solving for $G(t)$:

$$\frac{1}{kG}\frac{dG}{dt} = -\lambda - \frac{\alpha}{k} \qquad \frac{1}{G}\frac{dG}{dt} = -\lambda k - \alpha$$

$$\Rightarrow G(t) = c\, e^{(-k\lambda-\alpha)t}$$

$$\Rightarrow u(x,t) = e^{-\alpha t} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}, \quad n=1,2,\dots$$

So, in general the solution is

$$u(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Since $\alpha>0$ $e^{-\alpha t} \longrightarrow 0$ as $t\to\infty$, and $e^{-k\left(\frac{n\pi}{L}\right)^2 t} \xrightarrow[t\to\infty]{} 0$

$$\Rightarrow \lim_{t\to\infty} u(x,t) = 0$$

and $u(x)=0$ is the only equilibrium solution.

Problem 5: Explicitly show that there are no negative eigenvalues for

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \quad \frac{d\phi}{dx}(0)=0, \quad \frac{d\phi}{dx}(L)=0.$$

$\lambda = -\alpha^2 < 0$

$$\Rightarrow \phi(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} \Rightarrow \frac{d\phi}{dx} = c_1\alpha e^{\alpha x} - c_2\alpha e^{-\alpha x}$$

$0 = c_1\alpha - c_2\alpha \Rightarrow c_1 = c_2$

$0 = c_1\alpha e^{\alpha L} - c_2\alpha e^{-\alpha L} \Rightarrow c_1\alpha e^{\alpha L} = c_2\alpha e^{-\alpha L} = c_1\alpha e^{-\alpha L}$

If $c_1 \neq 0 \Rightarrow e^{-\alpha L} = e^{\alpha L} \Rightarrow -\alpha L = \alpha L \Rightarrow \alpha = 0$, a contradiction

therefore, $\lambda < 0$ is not an eigenvalue.

Problem 6:

(a) Using the divergence theorem, determine an alternative expression for

$$\iiint u \nabla^2 u \, dxdydz$$

Answer:

Note that:

$$\nabla\cdot(u\nabla u) = \nabla\cdot\left(\left(u\frac{\partial u}{\partial x}, u\frac{\partial u}{\partial y}, u\frac{\partial u}{\partial z}\right)\right)$$

$$= \partial_x\left(u\frac{\partial u}{\partial x}\right) + \partial_y\left(u\frac{\partial u}{\partial y}\right) + \partial_z\left(u\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial z}\right)^2 + u\frac{\partial^2 u}{\partial z^2}$$

$$= |\nabla u|^2 + u\cdot\nabla^2 u.$$

$$\therefore\ u\nabla^2 u = \nabla\cdot(u\nabla u) - |\nabla u|^2$$

Using the divergence theorem we get:

$$\iiint_R u\nabla^2 u \, dv = \iiint_R \nabla\cdot(u\nabla u)\, dv - \iiint_R |\nabla u|^2 dv$$

$$= \oiint_{\partial R} u\nabla u\cdot\hat{n}\, ds - \iiint_R |\nabla u|^2 dv$$

(b) Using part (a), prove that the solution of Laplace's equation $\nabla^2 u = 0$ (with a given u on the boundary) is unique.

Answer:

Assume u_1, u_2 are two solutions with the same boundary conditions

$$\begin{cases}\nabla^2 u_1 = 0 & \text{on } R \\ u_1 = f & \text{on } \partial R\end{cases} \qquad \begin{matrix}\nabla^2 u_2 = 0 & \text{on } R \\ u_2 = f & \text{on } \partial R\end{matrix}$$

Define $u = u_1 - u_2$

$$\Rightarrow \begin{cases} \nabla^2 u = 0 & \text{on } R \\ u = 0 & \text{on } \partial R \end{cases}$$

Apply the formula in part (a) to $u = u_1 - u_2$

$$\Rightarrow \iiint_R u \cancel{\nabla^2 u}^{\,0} \, dv = \oiint_{\partial R} u \nabla u \cdot \hat{n} \, ds - \iiint_R |\nabla u|^2 \, dV$$

satisfied Laplace's eqn.

$u = 0$ on ∂R

$$\Rightarrow \iiint_R |\nabla u|^2 \, dv = 0 \quad \text{and} \quad |\nabla u|^2 \geq 0$$

$\Rightarrow \nabla u = 0 \quad \Rightarrow u = \text{constant}$

but $u = 0$ on $\partial R \Rightarrow u \equiv 0$ on R

Therefore $u_1 - u_2 \equiv 0$ on R

$\Rightarrow u_1 = u_2$ and we showed uniqueness.

Problem 7: Show that the "backward" heat equation $\frac{\partial u}{\partial t} = -k \frac{\partial^2 u}{\partial x^2}$ subject to $u(0,t) \equiv 0 = u(L,t)$, $u(x,0) = f(x)$ is not well posed

Answer:

Let's find separated solutions $u(x,t) = G(t)\phi(x)$

$$G'(t)\phi(x) = -k \frac{d^2\phi}{dx^2} G(t)$$

$$\frac{1}{k}\frac{1}{G}\frac{dG}{dt} = -\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

Since now the equation for ϕ is

$\frac{d^2\phi}{dx^2} - \lambda\phi = 0$, we know that ~~negative~~ λ has to be negative since $\phi(0) = \phi(L) = 0$

$\lambda = -\left(\frac{n\pi}{L}\right)^2$

$\phi(x) = \cancel{c_1 \cos}\ c_1 \sin\left(\frac{n\pi x}{L}\right)$

Solving for $G(t)$:

$\frac{1}{G}\frac{dG}{dt} = -\lambda k \Rightarrow G(t) = ce^{-\lambda k t} = ce^{k\left(\frac{n\pi}{L}\right)^2 t}$

$\Rightarrow$ The separated solutions are:

$u(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right) e^{k\left(\frac{n\pi}{L}\right)^2 t}$

Suppose the initial condition is

$u(x,0) = f(x) = \sin\frac{n\pi x}{L}$

$\Rightarrow$ The solution is $u(x,t) = \sin\frac{n\pi x}{L} e^{k\left(\frac{n\pi}{L}\right)^2 t}$

Assume $n \gg 0$ is Large, and let's change the initial data by an small amount:

$f_2(x) = \sin\frac{n\pi x}{L} + \frac{1}{n}\sin\frac{n\pi x}{L} = \left(1+\frac{1}{n}\right)\sin\frac{n\pi x}{L}$

$\Rightarrow$ The new solution is:

$u_2(x,t) = \left(1+\frac{1}{n}\right)\sin\frac{n\pi x}{L} e^{k\left(\frac{n\pi}{L}\right)^2 t} = u_1(x,t) + \frac{1}{n}\sin\frac{n\pi x}{L} e^{k\left(\frac{n\pi}{L}\right)^2 t}$

The difference is $\frac{1}{n}\sin\frac{n\pi x}{L} e^{k\left(\frac{n\pi}{L}\right)^2 t}$, and $e^{k\left(\frac{n\pi}{L}\right)^2 t}$ grows exponentiall

$\Rightarrow$ For large t, the difference can be arbitrarily large if $\sin\frac{n\pi x}{L} \neq 0$. for some x.