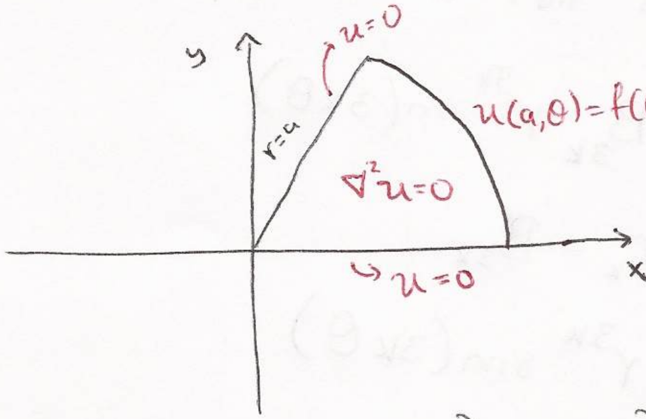
Problem 1: Solve the Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions

$$u(r,0)=0, \quad u(r,\tfrac{\pi}{3})=0, \quad u(a,\theta)=f(\theta)$$



Answer:

$$\begin{cases} \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 & \\ u(r,0)=0 & 0 \le r \le a \\ u(r,\pi/3)=0 & \\ u(a,\theta)=f(\theta) & 0 \le \theta \le \pi/3 \end{cases}$$

We assume u is finite at the origin.

$\Rightarrow$ In class we derived the general solution:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right)$$

Apply boundary conditions:

$$0 = u(r,0) = A_0 + \sum_{n=1}^{\infty} A_n r^n = 0$$

$\Rightarrow A_n = 0$ for all $n = 0,1,2,\ldots$

$$\Rightarrow u(r,\theta) = \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

Also, $0 = u(r,\pi/3) = \sum_{n=1}^{\infty} B_n r^n \left(\sin\left(\frac{n\pi}{3}\right)\right)$

$\Rightarrow B_n \sin\left(n\frac{\pi}{3}\right) = 0 \quad \forall n = 1, 2, \ldots$

$\sin\left(n\frac{\pi}{3}\right)$ is zero only when n is a multiple of 3,
$n = 3k$, k integer.

$\Rightarrow B_n = 0$ if n is not a multiple of 3

$$\Rightarrow u(r,\theta) = \sum_{k=1}^{\infty} B_{3k} r^{3k} \sin(3k\theta)$$

Let's redefine $C_k = B_{3k}$

$$\Rightarrow u(r,\theta) = \sum_{k=1}^{\infty} C_k r^{3k} \sin(3k\theta)$$

Apply the initial conditions:

$$f(\theta) = u(a,\theta) = \sum_{k=1}^{\infty} C_k a^{3k} \sin(3k\theta)$$

How to find C_k in this case?

f is defined for $0 \le \theta \le \pi/3$.

$$\int_0^{\pi/3} f(\theta)\sin(3m\theta)\, d\theta = \sum_{k=1}^{\infty} C_k a^{3k} \int_0^{\pi/3} \sin(3k\theta)\sin(3m\theta)\, d\theta$$

Notice that by a change of variables $\alpha = 3\theta$

$$\int_0^{\pi/3} \sin(3k\theta)\sin(3m\theta)\, d\theta = \int_0^{\pi} \sin(k\alpha)\sin(m\alpha)\,\frac{1}{3}\, d\alpha$$

$$= \begin{cases} 0 & \text{if } m \neq k \\ \frac{1}{3}\frac{\pi}{2} & \text{if } m = k \end{cases}$$

$\Rightarrow$

HW 4.2

$$\Rightarrow \int_0^{\pi/3} \sin(3k\theta) f(\theta)\, d\theta = c_k a^{3k} \frac{\pi}{6}$$

$$\therefore u(r,\theta) = \sum_{k=1}^{\infty} c_k r^{3k} \sin(3k\theta)$$

where $c_k = \frac{6}{\pi} a^{-3k} \int_0^{\pi/3} f(\theta)\sin(3k\theta)\, d\theta$.

Problem 2: Consider the velocity u_θ at the cylinder. If the circulation is negative, show that the velocity will be larger above the cylinder than below.

Answer:

We need to consider $|\vec{u}|^2 = u^2 + v^2$ at $r=a$.

We can compute this using the angular and radial component as follows:

$$u_\theta = -\frac{\partial \psi}{\partial r} = -\cos\theta \frac{\partial \psi}{\partial x} - \sin\theta \frac{\partial \psi}{\partial y} = \cos\theta\, v - \sin\theta\, u$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\sin\theta \frac{\partial \psi}{\partial x} + \cos\theta \frac{\partial \psi}{\partial y} = \sin\theta\, v + \cos\theta\, u$$

$$\Rightarrow u_\theta^2 + u_r^2 = (\cos\theta\, v - \sin\theta\, u)^2 + (\sin\theta\, v + \cos\theta\, u)^2$$

$$= \cos^2\theta\, v^2 \cancel{- 2\cos\theta\sin\theta\, uv} + \sin^2\theta\, u^2 + \sin^2\theta\, v^2 \cancel{+ 2\cos\theta\sin\theta\, uv} + \cos^2\theta\, u^2$$

$$= v^2 + u^2$$

$$\Rightarrow |\vec{u}|^2 = u_\theta^2 + u_r^2$$

But $u_r = 0$ at the cylinder

$$\Rightarrow |\vec{u}|^2 = u_\theta^2.$$

Since $u_\theta = -\frac{c_1}{r} - \bar{U}\left(1+\frac{a^2}{r^2}\right)\sin\theta$

$= -\frac{c_1}{a} - \bar{U}(2)\sin\theta$ at $r=a$

$\Rightarrow |\vec{u}|^2 = \left(\frac{c_1}{a} + 2\bar{U}\sin\theta\right)^2$

A point $(x,y) = r(\cos\theta, \sin\theta)$ is above the cylinder if $\sin\theta > 0$, which happens when $\theta \in (0,\pi)$.

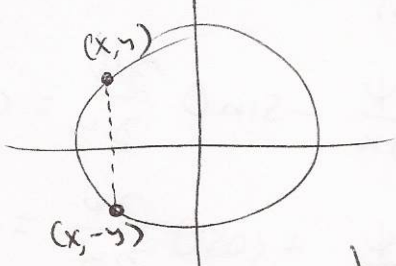
and $(x,-y) = (r\cos\theta, -r\sin\theta) = r(\cos(-\theta), \sin(-\theta))$ is below.

$|\vec{u}|^2(x,y) = \frac{c_1^2}{a^2} + 4\bar{U}^2\sin^2\theta + 4\frac{c_1}{a}\sin\theta$

$|\vec{u}|^2(x,-y) = \frac{c_1^2}{a^2} + 4\bar{U}^2\sin^2\theta - 4\frac{c_1}{a}\sin\theta$, $\sin\theta > 0$

if $\Gamma = -2\pi c_1$ is negative $\Rightarrow c_1 > 0$

$\Rightarrow |\vec{u}|^2(x,-y) \le |\vec{u}|^2(x,y)$ when (x,y) is above $(x,-y)$ is below.



Problem 3: A stagnation point is a place where $\vec{u} = 0$. For what values of the circulation does a stagnation point exist on the cylinder?

Answer: At the cylinder: $u_r = 0$, $u_\theta = -\frac{c_1}{a} - 2\bar{U}\sin\theta$

if $u_\theta = 0 \Rightarrow c_1 = -2a\bar{U}\sin\theta \Rightarrow \Gamma = -2\pi c_1 = 4\pi a\bar{U}\sin\theta$

Since $-1 \le \sin\theta \le 1 \Rightarrow -4\pi a\bar{U} \le \Gamma \le 4\pi a\bar{U}$

$\Rightarrow$ For $-4\pi a\bar{U} \le \Gamma \le 4\pi a\bar{U}$, a stagnation point exists on the cylinder. Namely at $\theta = \arcsin\left[\frac{\Gamma}{4\pi a\bar{U}}\right]$.