HW6.1
2013

Homework 6

Problem 1:

(a) Set $x = \frac{\pi}{2}$ in the Fourier series of $f(x) = x,\ -\pi < x < \pi$, to obtain the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Answer:

~~abc~~ $f(x) = x$ is an odd function $\Rightarrow a_0 = 0,\ a_n = 0$

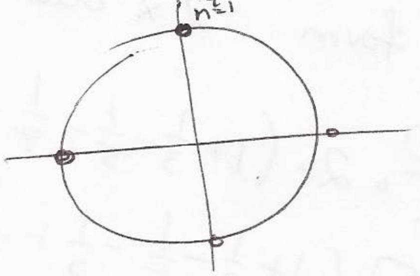
$$b_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[x \cdot \frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L 1 \cdot \frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx \right]$$

$$= \frac{-2L}{n\pi}(-1)^n + \frac{2}{n\pi}\cancel{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L} \to 0 \qquad L = \pi.$$

$$= \frac{2\pi}{n\pi}(-1)^{n+1} = \frac{2}{n}(-1)^{n+1}$$

$$x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

Since the 2π-periodic extension of f is continuous at $x = \frac{\pi}{2}$, then



$$\frac{\pi}{2} = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin\left(n\frac{\pi}{2}\right)$$

$$(-1)^{n+1}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & n = 4k+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 \\ 0 & n = 4k \end{cases}$$

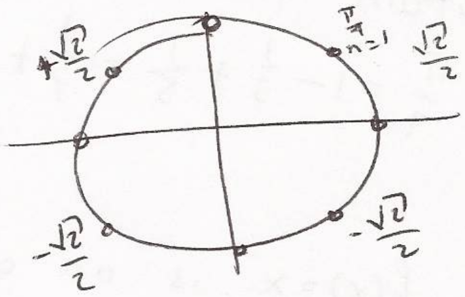
$\Rightarrow$ all even n have zero coefficients, and the odd numbers have alternating signs

$$\Rightarrow \frac{\pi}{2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

(b) Set $x = \frac{\pi}{4}$ in the series of part (a) to obtain:

$$\frac{\pi}{4} = \sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\dots\right)-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)$$

Answer: The 2π-periodic extension of f is continuous at $x = \frac{\pi}{4}$



$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin\left(n\frac{\pi}{4}\right)$$

$$\sin\left(\frac{n\pi}{4}\right) = \begin{cases} \frac{\sqrt{2}}{2} & n=8k+1 \\ -1 & n=8k+2 \\ +\frac{\sqrt{2}}{2} & n=8k+3 \\ 0 & n=8k+4 \\ -\frac{\sqrt{2}}{2} & n=8k+5 \\ +1 & n=8k+6 \\ -\frac{\sqrt{2}}{2} & n=8k+7 \\ 0 & n=8k \end{cases}$$

As we can see, the coefficients involving $\frac{\sqrt{2}}{2}$ correspond to odd numbers, with alternating sign every two of them. Also, the coefficients involving ± 1 are those ~~with~~ of the form $2 \times$ odd, with alternating signs, starting with -1.

$$\Rightarrow \frac{\pi}{4} = \frac{\sqrt{2}}{2}\cdot 2\cdot\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\dots\right) - 2\cdot\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)$$

$$= \sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\dots\right)-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right).$$

(c) Conclude from part (b) that

$$\frac{\pi}{2\sqrt{2}} = 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\dots$$

Answer: Form the equation above we have:

$$\frac{\pi}{4} = \sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\dots\right)-\frac{\pi}{4} \Rightarrow \frac{\pi}{2\sqrt{2}} = 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\dots$$

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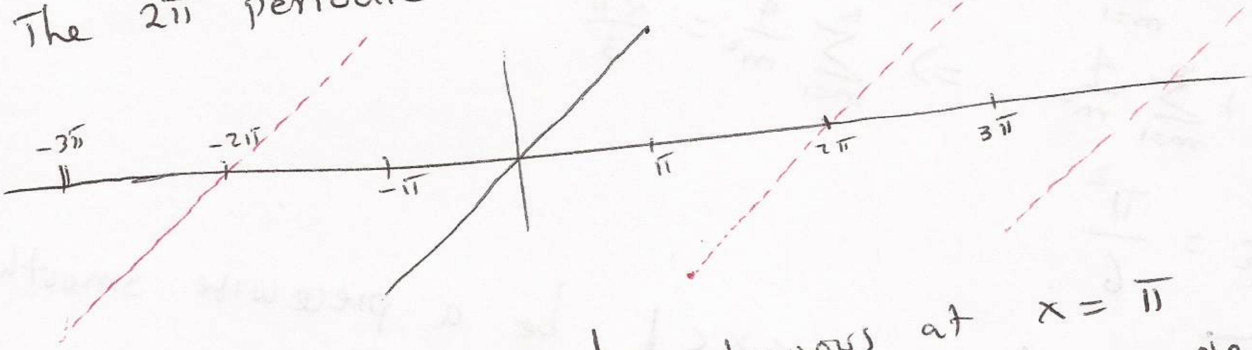
(d) If we set $x=\pi$ in the series in part (a), we find that the series sums to zero. Why doesn't it contradict $f(x)=x$?

Answer:

$f(x)=x \quad , -\pi < x < \pi$

The 2π periodic extension looks like:



$\Rightarrow$ The extension is discontinuous at $x=\pi$

$\Rightarrow$ The Fourier series converges to the average of the left and right limit:

$$\frac{f^+(\pi)+f^-(\pi)}{2} = \frac{-\pi+\pi}{2} = 0$$, which doesn't contradict anything.

Problem 2: From homework 5 (problem 1) we know that:

$$x^2 \sim \frac{\pi^2}{6} - 4\cos x + 2\cos x - \frac{4}{9}\cos 3x + \dots + (-1)^m \frac{4}{m^2}\cos(mx) + \dots$$

for $-\pi \le x \le \pi$

(a) Setting $x=0$, find the sum $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\dots = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{m^2}$.

Answer: Since x^2 is continuous at $x=0$, we substitute $x=0$ in the above Fourier series to get:

$$0 = \frac{\pi^2}{6} + \sum_{m=1}^{\infty}(-1)^m \frac{4}{m^2} \Rightarrow \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{m^2} = \frac{\pi^2}{6}.$$

(b) What is $\sum_{m=1}^{\infty} \frac{1}{m^2}$?

Answer:

The Fourier cosine series of x^2 is continuous everywhere. Substitute $x = \pi$ to get:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{m=1}^{\infty} (-1)^m \frac{4}{m^2} \cos(m\pi) = \frac{\pi^2}{3} + \sum_{m=1}^{\infty} (-1)^m \frac{4}{m^2} (-1)^m$$

$$= \frac{\pi^2}{3} + \sum_{m=1}^{\infty} \frac{4}{m^2} \quad \Rightarrow \sum_{m=1}^{\infty} \frac{4}{m^2} = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

Problem 3: Let $f(x)$, $-L < x < L$ be a piecewise smooth function with Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}.$$

Show that $a_n = O(\frac{1}{n})$, and $b_n = O(\frac{1}{n})$ are both of order $\frac{1}{n}$ when $n \to \infty$.

Answer:

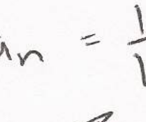
$$n a_n = \frac{1}{L} \int_{-L}^{L} n f(x) \cos \frac{n\pi x}{L} \, dx$$

$$n b_n = \frac{1}{L} \int_{-L}^{L} n f(x) \sin \frac{n\pi x}{L} \, dx$$

We can assume without loss of generality that $f(x)$ is smooth everywhere on $(-L, L)$. Otherwise we can break the interval into pieces where f is smooth and apply the analysis on each subinterval.

Applying integration by parts we get:

$$n a_n = \frac{1}{L} n \cancelto{0}{f(x) \frac{L}{n\pi} \sin \frac{n\pi x}{L} \Big|_{-L}^{L}} - \frac{1}{L} \int_{-L}^{L} f'(x)\, n \frac{L}{n\pi} \sin \frac{n\pi x}{L} \, dx$$



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$$= -\frac{1}{\pi}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow |n a_n| = \frac{1}{\pi}\left|\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx\right|$$

$$\le \frac{1}{\pi}\int_{-L}^{L} |f'(x)|\left|\sin\frac{n\pi x}{L}\right|dx \quad \leftarrow \text{triangle inequality}$$

Since $f'(x)$ is smooth on $(-L,L)$, it is bounded:

$\exists M$ such that $|f'(x)| \le M \quad \forall x \in (-L,L)$

$$\Rightarrow |n a_n| \le \cancel{\tfrac{1}{\pi}2L}\ \frac{1}{\pi}\int_{-L}^{L} M\,dx = \frac{2LM}{\pi}$$

$\Rightarrow n\,a_n$ is bounded $\Rightarrow a_n = O(\frac{1}{n})$ as $n\to\infty$.

$$n b_n = \frac{1}{L}\int_{-L}^{L} n f(x)\sin\frac{n\pi x}{L}\,dx = \frac{n}{L} f(x)\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{1}{L}\int_{-L}^{L} n f'(x)\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{1}{\pi}(-1)^n\left(f(L)-f(-L)\right) + \frac{1}{\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx$$

$$|n b_n| \le \frac{1}{\pi}|f(L)-f(-L)| + \frac{1}{\pi}\int_{-L}^{L}|f'(x)|\left|\cos\frac{n\pi x}{L}\right|dx$$

$$\le \frac{1}{\pi}|f(L)-f(-L)| + \frac{M2L}{\pi} \quad \leftarrow \text{bounded (not dependent on } n)$$

$\Rightarrow b_n = O(\frac{1}{n})$ as $n\to\infty$.

Problem 5: Let $f(x) = x(\pi - x)$, $0 \le x \le \pi$

(a) Compute the Fourier sine series of f.

Answer: $b_n = \frac{2}{L}\int_0^L x(\pi - x)\sin\frac{n\pi x}{L}\,dx$

Need to compute the following integrals:

$$\frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx = \frac{2L}{n\pi}(-1)^{n+1} \text{ (from problem 1)}$$

$$\frac{2}{L}\int_0^L x^2\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}x^2\cdot\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L - \frac{2}{L}\int_0^L 2x\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}L^2\frac{-L}{n\pi}(-1)^n + \frac{4}{n\pi}\int_0^L x\cos\frac{n\pi x}{L}\,dx$$

$$\int_0^L x\cos\frac{n\pi x}{L}\,dx = \underbrace{x\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L}_{\to 0} - \int_0^L 1\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{L}{n\pi}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L = \frac{L^2}{n^2\pi^2}\left((-1)^n-1\right)$$

$$\Rightarrow \frac{2}{L}\int_0^L x^2\sin\frac{n\pi x}{L}\,dx = -L^2\frac{2}{n\pi}(-1)^n + \frac{4}{n\pi}\frac{L^2}{n^2\pi^2}\left((-1)^n-1\right), \quad L=\pi$$

$$\Rightarrow b_n = \cancel{\pi\cdot\frac{2L}{n\pi}(-1)^{n+1}} + \cancel{\frac{2\pi^2}{n\pi}(-1)^n} - \frac{4\pi^2}{n^3\pi^3}\left((-1)^n-1\right)$$

$$= \frac{4}{\pi n^3}\left(1-(-1)^n\right) = \begin{cases}\frac{8}{\pi n^3} & n \text{ odd}\\ 0 & n \text{ even}\end{cases}$$

(b) Compute the Fourier cosine series of f.

Answer: $$a_0 = \frac{1}{L}\int_0^L x\cdot(\pi-x)\,dx = \frac{1}{L}\left[\pi\frac{x^2}{2}-\frac{x^3}{3}\right]_0^\pi$$

$$= \frac{1}{\pi}\frac{\pi^3}{2} - \frac{1}{\pi}\frac{\pi^3}{3} = \pi^2\frac{3-2}{6} = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{L}\int_0^L x(\pi-x)\cos\frac{n\pi x}{L}\,dx = \pi\cdot\frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx - \frac{2}{L}\int_0^\pi x^2\cos\frac{n\pi x}{L}\,dx$$

$$= \cancel{\pi}\frac{2}{\cancel{\pi}}\frac{1}{n^2}\left((-1)^n-1\right) - (-1)^n\frac{4}{n^2} \quad \leftarrow \text{ from problem 2}$$

$$= \frac{1}{n^2}\left(-2-2(-1)^n\right) = \frac{-2}{n^2}\left(1+(-1)^n\right) = \begin{cases}0 & n \text{ odd}\\ -\frac{4}{n^2} & n \text{ even.}\end{cases}$$

HW6.4

(c) Find the mean square error incurred by using N terms of each series and find asymptotic estimates when $N \to \infty$.

<u>Answer</u>: Define: $f_N(x) = \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L}$

$$\Rightarrow \varepsilon_N^2 = \|f - f_N\|_{L^2}^2 = \frac{1}{2}\sum_{n=N+1}^{\infty} b_n^2 = \frac{1}{2}\sum_{\substack{n=N+1\\ n \text{ odd}}}^{\infty} \frac{64}{\pi^2 n^6}$$

$$= \frac{32}{\pi^2}\sum_{\substack{n=N+1\\ n \text{ odd}}}^{\infty} \frac{1}{n^6}$$

$\frac{1}{n^6}$ is of order $\frac{1}{N^6}$, and we are adding from N+1 to ∞

$\Rightarrow \varepsilon_N^2$ is of order $N^{-\alpha}$ for any $0 < \alpha < 5$

Define $g_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{L}$

$$\Rightarrow \gamma_N^2 = \|f - g_N\|_{L^2}^2 = \frac{1}{2}\sum_{\substack{n=N+1\\ n \text{ even}}}^{\infty} \frac{16}{n^4}$$

$\Rightarrow \gamma_N^2$ is of order $N^{-\alpha}$ for any $0 < \alpha < 3$.

(d) Which series gives a better mean square approximation?

<u>Answer</u>: According to the estimates above, the Fourier sine series gives a better approximation.