Hw 7.1
2013

Hw 7

Problem 1 For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients.

(a) $f(x) = \cos\left(\frac{\pi x}{L}\right)$

$$B_n = \frac{2}{L}\int_0^L \cos\frac{\pi x}{L}\,\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{1}{2}\int_0^L \sin\left(\frac{(n+1)\pi x}{L}\right) + \sin\left(\frac{(n-1)\pi x}{L}\right)dx$$

For $n>1$ we get:

$$B_n = \frac{1}{L}\left[-\frac{L}{(n+1)\pi}\cos\left(\frac{(n+1)\pi x}{L}\right) - \frac{L}{(n-1)\pi}\cos\left(\frac{(n-1)\pi x}{L}\right)\right]_0^L$$

$$= -\frac{1}{\pi}\left[\frac{1}{n+1}\left((-1)^{n+1}-1\right) + \frac{1}{n-1}\left((-1)^{n-1}-1\right)\right]$$

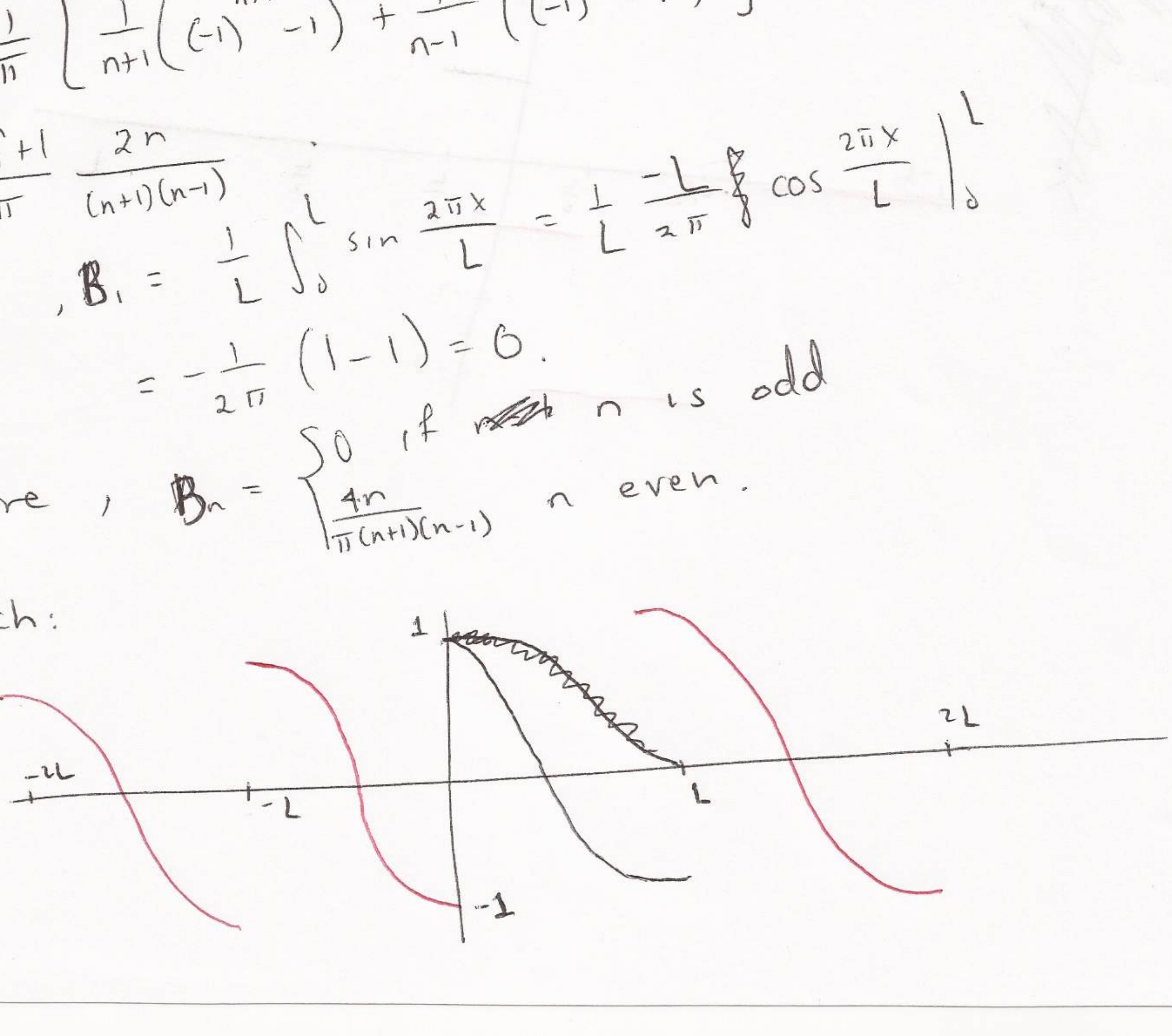
$$= \frac{(-1)^n+1}{\pi}\,\frac{2n}{(n+1)(n-1)}$$

For $n=1$, $B_1 = \frac{1}{L}\int_0^L \sin\frac{2\pi x}{L} = \frac{1}{L}\,\frac{-L}{2\pi}\cos\frac{2\pi x}{L}\Big|_0^L$

$$= -\frac{1}{2\pi}(1-1) = 0.$$

Therefore, $B_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{4n}{\pi(n+1)(n-1)} & n \text{ even.} \end{cases}$

Sketch:



(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

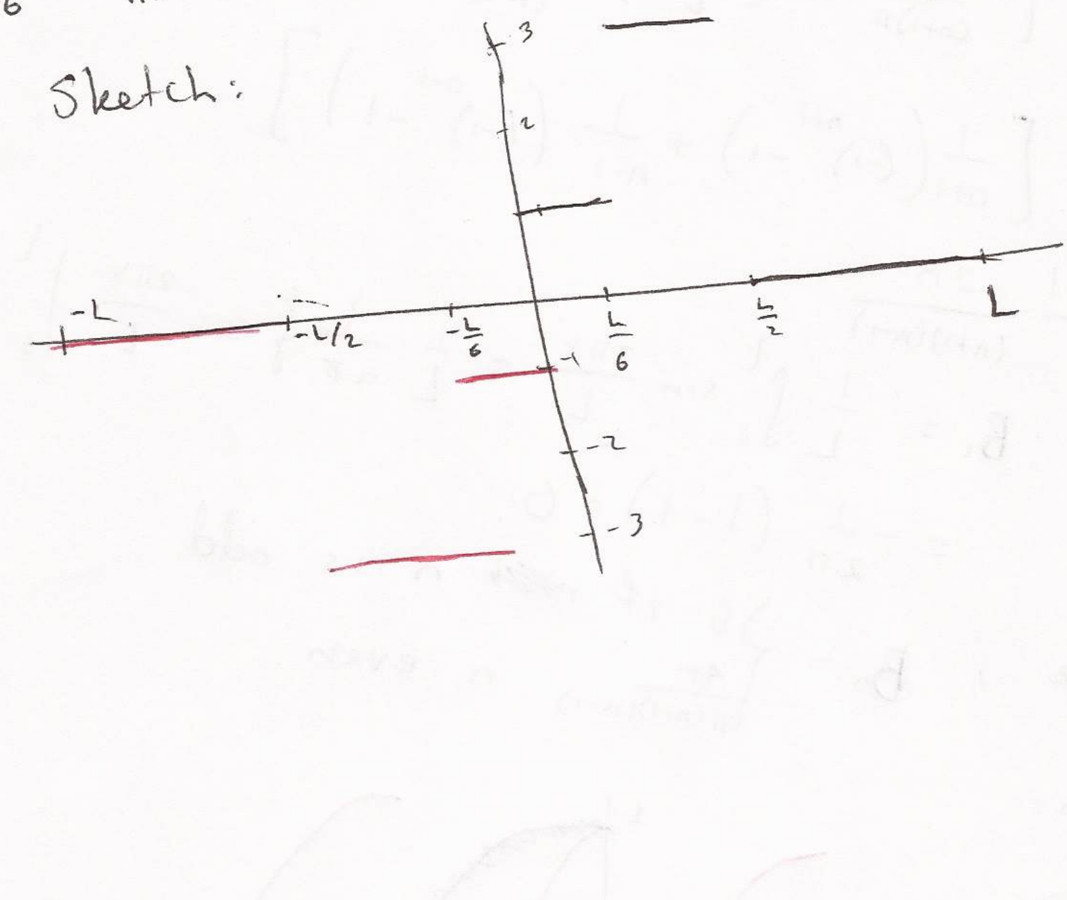
$$B_n = \frac{2}{L}\int_0^{L/6} \sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/6}^{L/2} 3\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\,\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{L/6} + \frac{6}{L}\,\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/6}^{L/2}$$

$$= -\frac{2}{n\pi}\left(\cos\frac{n\pi}{6} - 1\right) - \frac{6}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{6}\right)$$

$$= \frac{4}{n\pi}\cos\frac{n\pi}{6} - \frac{6}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n\pi}$$

Sketch:



Hw 7.2

Problem 2: For the following functions sketch the Fourier cosine series of f(x) and determine its Fourier coefficients.

(a) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

Answer:

$$A_0 = \frac{1}{L}\int_0^{L/6} 1\,dx + \frac{1}{L}\int_{L/6}^{L/2} 3\,dx = \frac{1}{L}\frac{L}{6} + \frac{1}{L}3\cdot\left(\frac{L}{2}-\frac{L}{6}\right)$$

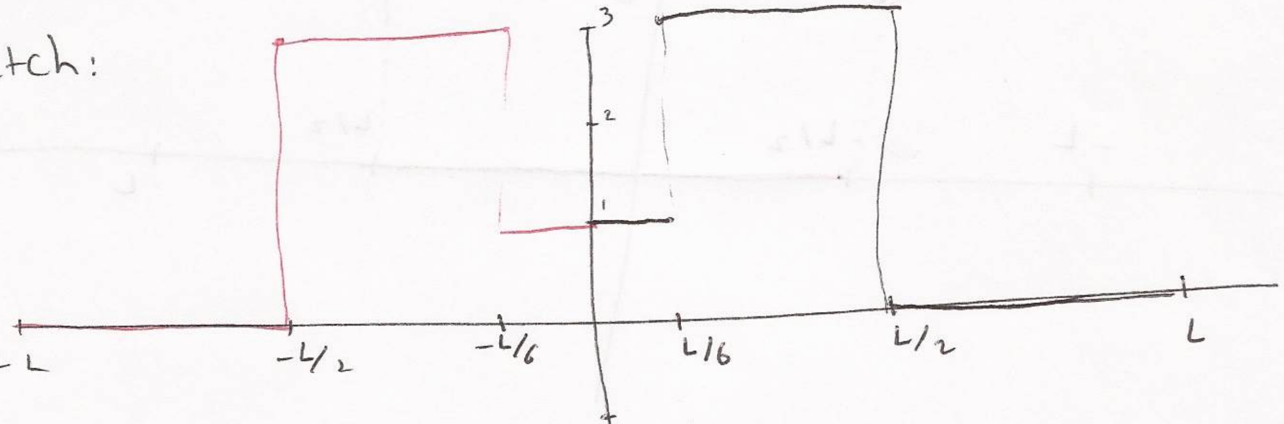
$$= \frac{1}{6} + 3\,\frac{3-1}{6} = \frac{7}{6}$$

$$A_n = \frac{2}{L}\int_0^{L/6} \cos\frac{n\pi x}{L}\,dx + \frac{6}{L}\int_{L/6}^{L/2} \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^{L/6} + \frac{6}{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/6}^{\frac{L}{2}}$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{6} + \frac{6}{n\pi}\left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{6}\right)$$

sketch:



(b) $f(x) = \begin{cases} 0 & x < L/2 \\ x, & x \geq L/2 \end{cases}$

$$A_0 = \frac{1}{L}\int_{L/2}^{L} x\,dx = \frac{1}{L}\left.\frac{x^2}{2}\right|_{\frac{L}{2}}^{L} = \frac{1}{L}\frac{1}{2}\left(L^2 - \frac{L^2}{4}\right)$$

$$= \frac{1}{2}\left(L - \frac{L}{4}\right) = \frac{3}{8}L$$

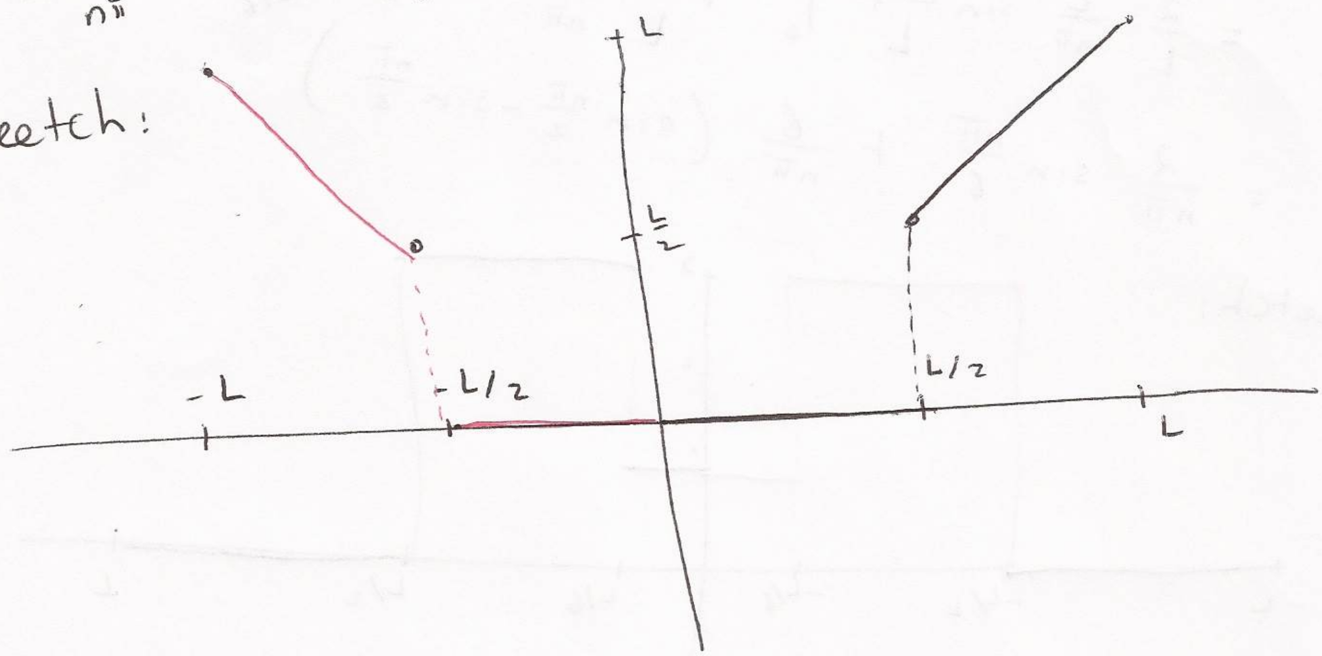
$$A_n = \frac{2}{L}\int_{\frac{L}{2}}^{L} x \cdot \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\, x \cdot \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{\frac{L}{2}}^{L}$$

$$- \frac{2}{L}\int_{\frac{L}{2}}^{L} 1 \cdot \frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{\cancel{L}n\pi}\cdot\left(0 - \frac{L}{2}\cdot\sin\frac{n\pi}{2}\right) - \frac{2}{n\pi}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= -\frac{L}{n\pi}\sin\frac{n\pi}{2} + \frac{2L}{n^2\pi^2}\left((-1)^n - \cos\frac{n\pi}{2}\right)$$

Sketch:



Hw 7.3

Problem 3: What is the sum of the Fourier sine series of $f(x)$ and the Fourier cosine series of $f(x)$? What is the sum of the ~~Fourier~~ even and odd extension of $f(x)$?

Answer:

The Fourier cosine (sine) series converges to the even (odd) extension of $f(x)$ where $f(x)$ is smooth ~~f where f(x) is~~ and to the $\frac{1}{2}$ average of the left and the right limits where there is a discontinuity.

Then, for $0 < x < L$, the sum converges to $2 \times f(x)$ where $f(x)$ is smooth, and to $f^+(x) + f^-(x)$ where f has a discontinuity.

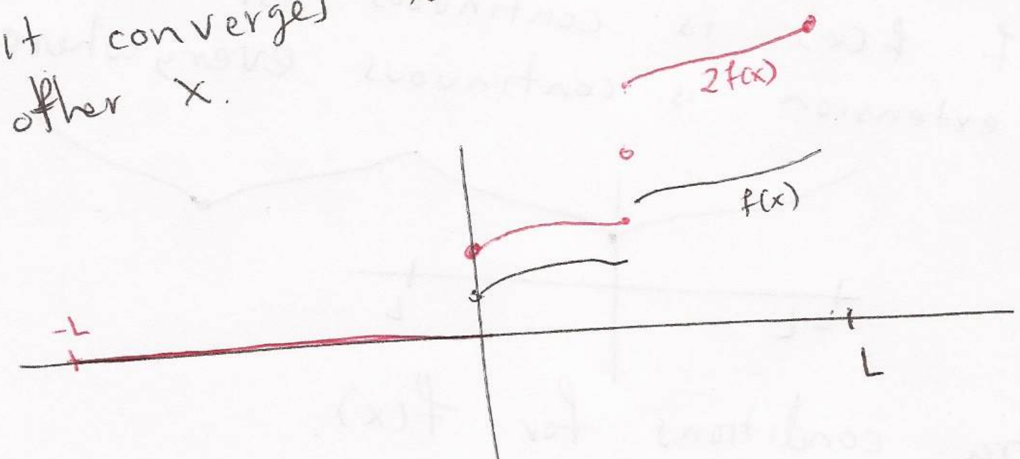
At $x = 0$, it converges to $f^+(0)$

At $x = L$, it converges to $f^-(L)$

For $-L < x < 0$, it converges to zero

At $x = -L$, it converges to zero

And it converges to the $2L$ periodic extension for any other x.



Problem 4: For continuous functions,

(a) Under what conditions does $f(x)$ equal its Fourier series for all x, $-L \le x \le L$?

Answer: The 2L-periodic extension of $f(x)$ may only have discontinuities at $x = L, 3L, \ldots$ $-L, -3L, \ldots$

assuming $f(x)$ is continuous at $(-L, L)$.

$\Rightarrow$ The Fourier series equals $f(x)$ only if

$$f(-L) = f(L).$$

(b) Under what conditions does $f(x)$ equal its Fourier sine series for all x, $0 \le x \le L$?

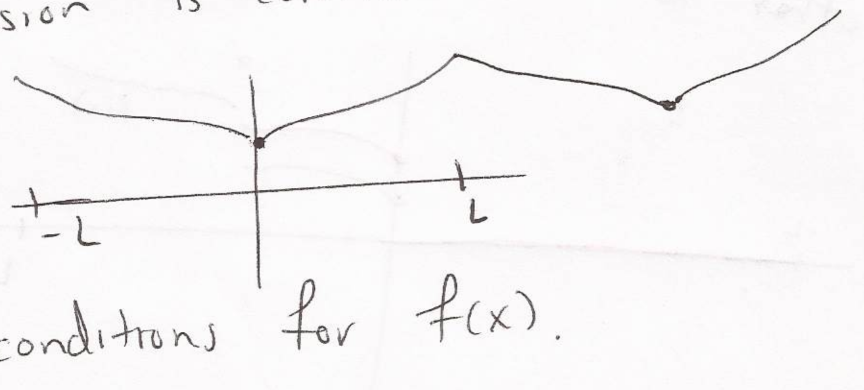
Answer: Assuming $f(x)$ is continuous for $0 < x < L$, the only discontinuities may happen at $x = 0, L, 2L, \ldots$ $-L, -2L, \ldots$

At $x = 0, L$, the Fourier sine series converges to zero

$\Rightarrow$ the condition is $f(0) = 0$, $f(L) = 0$.

(c) Under what conditions does $f(x)$ equal its Fourier cosine series for all x, $0 \le x \le L$?

Answer: if $f(x)$ is continuous for $0 < x < L$, the even extension is continuous everywhere



$\Rightarrow$ No extra conditions for $f(x)$.

HW 7.4

Problem 5: There are some things wrong on the following demonstrations. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of e^x:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

Differentiating yields:

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L}$$

the Fourier sine series of e^x. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos \frac{n\pi x}{L}.$$

$$\Rightarrow A_0 = 0, \quad A_n = \left(\frac{n\pi}{L}\right)^2 A_n \Rightarrow A_n = 0$$

Obviosly wrong

By correcting the mistakes, you should be able to obtain A_0 and A_n without using the typical technique.

Answer:

~~The~~ Since e^x is continuous, the Fourier cosine series can always be differentrated term-by term.

$$\Rightarrow e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad B_n = -\frac{n\pi}{L} A_n$$

However, the Fourier sine series cannot be differentiated term by term in this case. We would need $f(0) = 0 = f(L)$. There is the mistake.

To correct it, I was going to use integration term by term twice. However, Sarah found a shorter way, which I now describe. Thank you Sarah.

For $f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$, and when it cannot be differentrated term by term, we can actually correct the differentratron as

$$f'(x) \sim \frac{1}{L}\left[f(L)-f(0)\right] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L} B_n + \frac{2}{L}\left((-1)^n f(L) - f(0)\right)\right] \cos \frac{n\pi x}{L}.$$

(see page 121 in the textbook).

For $f(x) = e^x$, $B_n = -\frac{n\pi}{L} A_n$, we get

$$e^x \sim \frac{1}{L}\left[e^L - 1\right] + \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2} A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)\right) \cos \frac{n\pi x}{L}$$

$$\sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

$\Rightarrow$ $A_0 = \frac{1}{L}\left[e^L - 1\right]$

and $A_n = -\left(\frac{n\pi}{L}\right)^2 A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$

$$\Rightarrow \quad A_n = \frac{\frac{2}{L}\left((-1)^n e^L - 1\right)}{1 + \left(\frac{n\pi}{L}\right)^2}.$$

HW 7.5

Problem 6:

(a) Using equation (4.2.7) in the textbook, compute the sagged equilibrium position $u_E(x)$ if $Q(x,t) = -g$. The boundary conditions are $u(0) = 0$, $u(L) = 0$

Answer: $\rho_0(x) \dfrac{\partial^2 u}{\partial t^2} = T_0 \dfrac{\partial^2 u}{\partial x^2} + Q(x,t)\rho_0(x)$

$\Rightarrow T_0 \dfrac{\partial^2 u}{\partial x^2} - g\rho_0(x) = 0 \Rightarrow \dfrac{\partial^2 u}{\partial x^2} = g \cdot \dfrac{\rho_0(x)}{T_0} = g c^{-2}$

For simplicity, let's assume c is constant

$\Rightarrow \dfrac{\partial u}{\partial x} = g c^{-2} x + a \Rightarrow u = g c^{-2} \dfrac{x^2}{2} + ax + b$

$0 = u(0) = b \Rightarrow b = 0$

$0 = u(L) = g c^{-2} \dfrac{L^2}{2} + aL \Rightarrow a = -g c^{-2} \dfrac{L}{2}$

$\Rightarrow u_E(x) = g c^{-2}\left(\dfrac{x^2}{2} - \dfrac{L}{2} x\right) = \dfrac{g c^{-2}}{2} x \cdot (x - L)$

(b) Show that $v(x,t) = u(x,t) - u_E(x)$ satisfies the equation $\dfrac{\partial^2 v}{\partial t^2} = c^2 \dfrac{\partial^2 v}{\partial x^2}$.

$\dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} - g$

$c^2 \dfrac{\partial^2 v}{\partial x^2} = c^2 \cdot \left(\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u_E}{\partial x^2}\right) = c^2\left(\dfrac{\partial^2 u}{\partial x^2} - g c^{-2}\right)$

$= c^2 \dfrac{\partial^2 u}{\partial x^2} - g$

Therefore $\dfrac{\partial^2 v}{\partial t^2} = c^2 \dfrac{\partial^2 v}{\partial x^2}$.