Homework 8

Problem 1: Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

(a) Briefly explain why $\beta > 0$

Answer:

If $T_0 = 0 \Rightarrow \rho_0 \frac{\partial^2 u}{\partial t^2} = -\beta \frac{\partial u}{\partial t} \Rightarrow \rho_0 \frac{\partial u}{\partial t} = -\beta u + c$

$\Rightarrow$ u decays if $\beta > 0$, otherwise it would increase exponentially.

(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$u(0,t) = 0, \; u(L,t) = 0,$

and the initial conditions

$u(x,0) = f(x), \; \frac{\partial u}{\partial t}(x,0) = g(x).$

You can assume that the frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

Answer: $u = \phi(x) G(t)$

$$\Rightarrow \rho_0 \phi G'' = T_0 \phi'' G - \beta \phi G' \quad / \phi G$$

$$\Rightarrow \rho_0 \frac{G''}{G} = T_0 \frac{\phi''}{\phi} - \beta \frac{G'}{G}$$

$$\Rightarrow \frac{\rho_0}{T_0} \frac{G''}{G} + \frac{\beta}{T_0} \frac{G'}{G} = \frac{\phi''}{\phi} = -\lambda$$

$$\Rightarrow \phi'' = -\lambda\phi \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\phi(x) = \sin\frac{n\pi x}{L} \quad \text{since} \quad u(0,t)=0,\ u(L,t)=0.$$

$$\frac{\rho_0}{T_0}\frac{G''}{G} + \frac{\beta}{T_0}\frac{G'}{G} = -\lambda \Rightarrow \rho_0 G'' + \beta G' + \lambda T_0 G = 0$$

$$G = e^{rt} \Rightarrow \rho_0 r^2 + \beta r + \lambda T_0 = 0$$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0\lambda T_0}}{2\rho_0} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0}$$

$$\text{Since} \quad \beta^2 < 4\rho_0 T_0\frac{\pi^2}{L^2} \le 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2$$

$$\Rightarrow r = -\frac{\beta}{2\rho_0} \pm \frac{1}{2\rho_0}\sqrt{-\beta^2 + 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}\, i$$

$$\Rightarrow G(t) = e^{-\beta/2\rho_0 t}\left(\cos\frac{1}{2\rho_0}\sqrt{-\beta^2 + 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}\, t + i\sin\frac{1}{2\rho_0}\sqrt{-\beta^2 + 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}\, t\right)$$

$\Rightarrow$ Real and imaginary parts are solution

$\Rightarrow$ The general solution is:

$$G(t) = e^{\frac{-\beta}{2\rho_0}t}\left(a_n\cos\frac{1}{2\rho_0}\sqrt{-\beta^2 + 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}\, t + b_n\sin\frac{1}{2\rho_0}\sqrt{-\beta^2 + 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}\, t\right)$$

$\Rightarrow$ The principle of superposition implies

$$u(x,t) = \sum_{n=1}^{\infty} e^{\frac{-\beta}{2\rho_0}t}\left(a_n\cos\frac{\sqrt{-\beta^2 + 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0}\, t + b_n\sin\frac{\sqrt{-\beta^2 + 4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0}\, t\right).\ \sin\frac{n\pi x}{L}$$

We know need to find the coefficients:

At $t=0$:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{L} \Rightarrow a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

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To find b_n, we need to compute $\frac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} -\frac{\beta}{2\rho_0} e^{-\frac{\beta}{2\rho_0}t}\left(\cancel{\cos}\, a_n \cos \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0} t + b_n \sin \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0} t\right) \cdot \sin\frac{n\pi x}{L}$$

$$+ \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0}t} \cdot \left(-a_n \sin \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0} t \times \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0} + b_n \cos \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0} t \times \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^3}}{2\rho_0}\right) \sin\frac{n\pi x}{L}$$

$$\Rightarrow g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\left(-\frac{\beta}{2\rho_0} a_n + \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0} b_n\right) \sin\frac{n\pi x}{L}$$

$$\Rightarrow -\frac{\beta}{2\rho_0} a_n + \frac{\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}}{2\rho_0} b_n = \frac{1}{L}\int_{-L}^{L} g(x) \sin\frac{n\pi x}{L}\, dx$$

$$\Rightarrow b_n = \frac{2\rho_0}{L\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}} \int_{-L}^{L} g(x) \sin\frac{n\pi x}{L}\, dx + \frac{\beta}{L\sqrt{-\beta^2+4\rho_0 T_0\left(\frac{n\pi}{L}\right)^2}} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx.$$

Problem 2: If a vibrating string satisfying (4.4.1)-(4.4.3) is initially unperturbed, $f(x)=0$, with initial velocity given, show that:

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct} G(y)\,dy,$$

where $G(y)$ is the odd periodic extension of $g(x)$. Hints:

1. For all x, $G(x)=\sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}$
2. $\sin a \sin b = \frac{1}{2}[\cos(a-b)-\cos(a+b)]$

Answer:

The general solution is:

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}.$$

Since $u(x,0)=f(x)=0 \Rightarrow A_n=0 \ \forall n$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty} B_n\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$$

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$$ → assuming differentiation term by term is allowed

$$\Rightarrow g(x)=\frac{\partial u}{\partial t}(x,0)$$

$$=\sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L} = G(x)$$