# On the evaluation of the Tutte polynomial at the points $(1,-1)$ and $(2,-1)$

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**Abstract**

Motivated by the identity $t(K_{n+2};1,-1) = t(K_n;2,-1)$, where $t(G;x,y)$ is the Tutte polynomial of a graph $G$, we search for graphs $G$ having the property that there is a pair of vertices $u,v$ such that $t(G;1,-1) = t(G-\{u,v\};2,-1)$. Our main result gives a sufficient condition for a graph to have this property; moreover, it describes the graphs for which the property still holds when each vertex is replaced by a clique or a coclique of arbitrary order. In particular, we show that the property holds for the class of threshold graphs, a well-studied class of perfect graphs.

**Keywords:** Tutte polynomial, threshold graph, generating function, up-down permutation

## 1 Introduction

The Tutte polynomial is an important polynomial graph invariant that has received much attention in diverse areas of mathematics. For a graph $G=(V,E)$, it is given by

$$t(G;x,y) = \sum_{A\subseteq E} (x-1)^{r(G)-r(A)}(y-1)^{|A|-r(A)},$$

where $r(A)$ is the rank of $A$, defined as $|V|-c(A)$, where $c(A)$ is the number of connected components of the spanning subgraph $(V,A)$ induced by $A$. Although the definition of the Tutte polynomial allows multiple edges and loops, all graphs in this paper are simple.

We refer to [11] for details about the many combinatorial interpretations of evaluations of the Tutte polynomial of a graph at different points of the plane and along several algebraic curves. For example, $t(G;1,1)$ is the number of spanning trees of $G$ when $G$ is connected and $t(G;2,1)$ is the number of spanning forests of $G$. As for curves, the hyperbolae $H_q = \{(x,y) : (x-1)(y-1) = q\}$ play a significant role in the theory of the Tutte polynomial. In particular, for $q\in\mathbb{N}$ the Tutte polynomial specializes on $H_q$ to the partition function of the $q$-state Potts model. Two interpretations especially related to our work are that $t(G;2,0)$ is the number of acyclic orientations of $G$, and that $t(G;1,0)$ is the number of acyclic orientations of $G$ with a unique prescribed source. With this in mind, it follows easily that $t(K_{n+1};1,0) = t(K_n;2,0)$; in fact, $t(G;1,0) = t(G-\{v\};2,0)$ for any graph $G$ with a universal vertex $v$ (a vertex adjacent to all other vertices).

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In this paper we shall be concerned with evaluations of the Tutte polynomial at the points $(1, -1)$ and $(2, -1)$. Merino [7] proved the following identity, which is the starting point for what follows:

$$t(K_{n+2}; 1, -1) = t(K_n; 2, -1).$$

Non-trivial relationships between evaluations of the Tutte polynomial at points on different hyperbolae are uncommon. Here, the point $(2, -1)$ lies on the hyperbola $H_{-2}$ and $(1, -1)$ on the hyperbola $H_0$. The question we are interested in is whether there are other graphs $G$ with the property that there are vertices $u, v$ such that $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$. Merino's proof for complete graphs used generating functions. It is not very difficult to adapt his proof to show that the property holds for complete bipartite graphs and for graphs formed by the join of a clique and a coclique. By a clique we mean a complete graph, and by a coclique a graph with no edges; the join of two graphs is their disjoint union together with edges joining each vertex from the first graph to each vertex of the second.

Our main result (Theorem 1) generalizes these examples by giving sufficient conditions for a graph $G$ to have such a pair of vertices $u, v$ for which $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$; moreover, it describes graphs for which this property still holds when each vertex is replaced by a clique or a coclique of arbitrary order. In particular, we show that the property holds for the class of threshold graphs, a well-studied class of perfect graphs [6].

The paper is organized as follows. First we present the main theorem and discuss its consequences. Section 3 is devoted to its proof, which relies heavily on the use of generating functions for Tutte polynomials, the fundamental example being the formula

$$\sum_{n\geq 0} t(K_n; x, y)\frac{u^n}{n!} = \frac{1}{x-1}\left(\sum_{n\geq 0} y^{\binom{n}{2}}(y-1)^{-n}\frac{u^n}{n!}\right)^{(x-1)(y-1)},$$

obtained by Tutte in an equivalent form [9].

Section 4, which is in fact independent of the previous ones, is devoted to giving a bijective proof of Merino's theorem by using an interpretation of the Tutte polynomial given by Gessel and Sagan [3] in terms of spanning trees and spanning forests. In Section 5 we use the formulas obtained in Section 3 to explore the evaluation at $(2, -1)$ of the Tutte polynomial for complete bipartite graphs and for the join of a clique and a coclique, and we conclude with a related open problem in Section 6.

## 2 Statement of main result

Let $G = (V, E)$ be a simple graph. For a subset of vertices $U \subseteq V$, $G[U]$ denotes the subgraph of $G$ induced by the vertex set $U$. A *clique* in $G$ is a set of pairwise mutually adjacent vertices (inducing a complete graph) and a *coclique* in $G$ is a set of pairwise non-adjacent vertices (inducing the complement of a complete graph). Vertices $u$ and $v$ are *twin vertices* in $G$ if each vertex $w$ distinct from $u$ and $v$ is either adjacent to both $u$ and $v$, or adjacent to neither $u$ nor $v$.

We are now ready to state our main theorem; see Figure 1 for a diagram illustrating the conditions contained in its statement.

**Theorem 1.** *Let $G = (V, E)$ be a simple graph and $i$ and $j$ distinct vertices of $G$ such that $\{i, j\}$ is a vertex cover of $G$, that is, every vertex is adjacent to $i$ or $j$. Let $A = \{v \in V \setminus \{i, j\} : vi \in E, vj \notin E\}$, $B = \{v \in V \setminus \{i, j\} : vi \notin E, vj \in E\}$ and $C = \{v \in V \setminus \{i, j\} : vi \in E, vj \in E\}$.*

*Then $t(G; 1, -1) = t(G - \{i, j\}; 2, -1)$ if the following conditions hold:*

*(i) $G[A]$ and $G[B]$ are cocliques, and $G[C \cup \{i, j\}]$ is a clique (in particular, $ij \in E$);*

*(ii) there is no induced pair of disjoint edges $2P_2$ with endpoints in $A \cup B$ and no induced path of length three $v_1, v_2, v_3, v_4$ with $\{v_2, v_3\} \subseteq C$ and either $\{v_1, v_4\} \subseteq A$ or $\{v_1, v_4\} \subseteq B$;*

*(iii) there is no induced path $P_3$ of length two with one endpoint in $A$ and the other in $B$, nor the complement of such a path.*

*Furthermore, if $G$ satisfies these conditions then so does any graph obtained from $G$ by replacing a vertex of $A \cup B \cup \{i, j\}$ by a coclique of twin vertices, or a vertex of $C \cup \{i, j\}$ by a clique of twin vertices.*

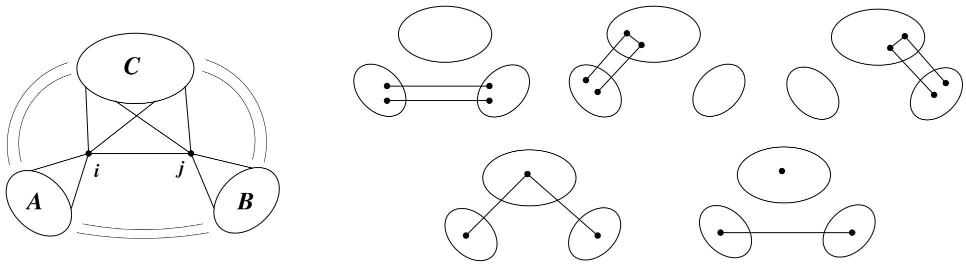


Figure 1: On the left, structure of the graph described in Theorem 1; $A$ and $B$ induce cocliques, and $C \cup \{i, j\}$ induces a clique. On the right, the five forbidden induced subgraphs.

Since $K_2$ satisfies the conditions of the theorem (it is the simplest case $A = B = C = \emptyset$), we recover complete graphs, complete bipartite graphs and the join of a clique and a coclique. If we take $G = K_3$, we have $A = B = \emptyset$ and $|C| = 1$. This means that we cannot replace the three vertices of a $K_3$ by cocliques, but all other possibilities are fine.

The case $B = \emptyset$ gives a much richer class of graphs, namely threshold graphs. These are the graphs whose vertices can be ordered so that each vertex is adjacent to either all or none of the previous vertices. Threshold graphs are also the graphs with no induced $P_4, C_4$ or $2P_2$. (See [6] for a wealth of characterizations and applications.)

**Corollary 2.** *Let $G$ be a connected threshold graph and let $j$ and $i$ be the first and last vertex in an ordering of the vertices of $G$ such that each vertex is adjacent to either all or none of the previous ones. Then $t(G; 1, -1) = t(G - \{i, j\}; 2, -1)$.*

*Proof.* We show that the conditions in Theorem 1 hold. Let $j = u_1, \ldots, u_n = i$ be an ordering of the vertices of $G$ such that each $u_r$ is adjacent to either all or none of the vertices $u_1, \ldots, u_{r-1}$; in the first case we will say that $u_r$ is dominant and in the second, that it is isolated. Since $G$ is connected, the last vertex added must be dominant (that is, adjacent to all other vertices). Then certainly $\{i, j\}$ is a vertex cover of $G$. Consider the sets $A$, $B$ and $C$ as defined in the statement of Theorem 1. Of them, $B$ is clearly empty, since all vertices are adjacent to $i$. The sets $A$ and $C$ are forced to be the sets of isolated and dominant vertices in $\{u_2, \ldots, u_{n-1}\}$, respectively. Thus condition (i) is seen to hold. Condition (ii) is satisfied because a threshold graph contains no induced $P_4$ or $2P_2$. Finally, condition (iii) holds vacuously, since $B$ is empty. □

In Section 3 we shall see that Theorem 1 in fact characterizes those graphs $G$ for which we can replace every one of its vertices by a clique or coclique of arbitrary size and still obtain a graph $G'$ satisfying $t(G'; 1, -1) = t(G' - \{i, j\}; 2, -1)$. (This emerges from Theorem 6.) However, it is not the case that Theorem 1 gives a necessary condition for $G$ to satisfy the property alone that there are two vertices $u, v$ such that $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$. For instance, taking $G$ to be a cycle of length 6 and $u, v$ two vertices at distance two in the cycle yields such a graph. Moreover, if the vertices are, cyclically, $u, x_1, v, x_2, w, x_3$, one can prove that replacing $x_1$ by a clique and $x_2, x_3$ by a coclique, the result satisfies the equation, yet it is not of the form described in Theorem 1 (in particular, $i$ and $j$ are not adjacent, and they do not cover all vertices). Characterizing *all* graphs for which there are two vertices $u, v$ such that $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$ is probably an overly ambitious goal.

## 3 Proof of the main result

We begin with some notation. A clique on $n$ vertices is denoted by $K_n$ and its complement, a coclique, by $\overline{K}_n$. Let $\mathbb{N}$ denote the set of non-negative integers. Given a connected graph $G = (V, E)$, $\mathbf{n} \in \mathbb{N}^V$ and $\mathbf{c} \in \{0, 1\}^V$, define $G(\mathbf{c}; \mathbf{n})$ to be the graph obtained from $G$ by replacing each vertex $k \in V$ by a clique on $n_k$ vertices if $c_k = 1$ or by a coclique on $n_k$ vertices if $c_k = 0$, and for each edge $ij \in E$ join the (co)clique on $n_i$ vertices to the (co)clique on $n_j$ vertices, joining each of the $n_i n_j$ pairs of vertices by an edge in $G(\mathbf{c}; \mathbf{n})$; if $n_k = 0$, the effect is the deletion of vertex $k$. For example, $K_1(1; n) = K_n$, $K_1(0; n) = \overline{K}_n$ and $K_2((0, 0); (m, n)) = K_{m,n}$. Note that $K_r((1, 1, \ldots, 1); (n_1, \ldots, n_r)) = K_1(1; n_1 + \cdots + n_r) = K_{n_1+n_2+\cdots+n_r}$ since the join of two cliques is a clique.

Our goal is to find the conditions that $G, \mathbf{c}$ and vertices $i$ and $j$ must satisfy so that for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$ we have

$$t(G(\mathbf{c}; \mathbf{n}); 1, -1) = t(G(\mathbf{c}; \mathbf{n}'); 2, -1), \tag{1}$$

where $\mathbf{n}'$ is obtained from $\mathbf{n}$ by subtracting 1 from the $i$th and $j$th components. In other words, if $u$ and $v$ are vertices of $G(\mathbf{c}; \mathbf{n})$ taken from the fixed (co)cliques that replace vertices $i$ and $j$ of $G$ in making the graph $G(\mathbf{c}; \mathbf{n})$, then $t(G(\mathbf{c}; \mathbf{n}); 1, -1) = t(G(\mathbf{c}, \mathbf{n}) - \{u, v\}; 2, -1)$.

Our first result (Theorem 6) characterizes pairs $(G, \mathbf{c})$ for which equation (1) holds. The remainder of the section is then devoted to showing how this first result can be rewritten in terms of induced subgraphs, as presented in Theorem 1 above.

We begin by finding the generating function for the Tutte polynomials of the family $G(\mathbf{c}, \mathbf{n})$ and then we express the relationship between the evaluations at $(1, -1)$ and $(2, -1)$ as a differential equation. The statement of the key Theorem 6 is then read from the solutions to this equation.

Let us fix a connected graph $G$ with two distinguished vertices $i, j$ and a $\{0, 1\}$-labelling $\mathbf{c} \in \{0, 1\}^V$. We seek conditions so that $G(\mathbf{c}; \mathbf{n})$ satisfies equation (1) for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$.

A first observation is that $i$ and $j$ together cover $V$, that is, every vertex $k$ is adjacent to either $i$ or $j$. Indeed, suppose that this is not the case for $k$, take a neighbour $l$ of $k$ (which may or may not be adjacent to $i$ or $j$), and consider equation (1) for $n_i = n_j = n_k = n_l = 1$ and all other values set to zero. It is easy to check that this equation does not hold by using the following basic facts about the Tutte polynomial: $t(K_2; x, y) = x$; $t(\overline{K}_n; x, y) = 1$; if $G$ has blocks $G_1, \ldots, G_r$ then $t(G; x, y) = t(G_1; x, y) \cdots t(G_r; x, y)$. So from now on we assume that $i$ and $j$ together cover $V$.

The main tool in the proof are generating functions. More concreteley, let $\mathbf{u} = (u_k : k \in V)$ and define

$$T(x, y; \mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{N}^V} t(G(\mathbf{c}; \mathbf{n}); x, y) \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}, \qquad \mathbf{u}^{\mathbf{n}} = \prod_k u_k^{n_k}, \quad \mathbf{n}! = \prod_k n_k!,$$

taking $t(G(\mathbf{c}; \mathbf{0}); x, y) = t(\emptyset; x, y) = 1$. Observe now that equation (1) holds for all $\mathbf{n} \in N^V$ with $n_i, n_j \geq 1$ if and only if

$$\frac{\partial^2 T(1, -1; \mathbf{u})}{\partial u_i \partial u_j} = T(2, -1; \mathbf{u}). \tag{2}$$

**Lemma 3.** *Let $G = (V, E)$ be a connected graph containing vertices $i$ and $j$ such that $ki \in E$ or $kj \in E$ for every $k \in V \setminus \{i, j\}$. Define*

$$S(z, w; \mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{N}^V} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \sum_{A \subseteq E(G(\mathbf{c}; \mathbf{n}))} z^{|A|} w^{c(A)}.$$

*Then*

$$\frac{\partial^2 T(x, y; \mathbf{u})}{\partial u_i \partial u_j} = \frac{1}{x-1} \frac{\partial^2 S(y-1, (x-1)(y-1); \frac{\mathbf{u}}{y-1})}{\partial u_i \partial u_j}, \tag{3}$$

*and*

$$T(2, y; \mathbf{u}) = S(y-1, y-1; \frac{\mathbf{u}}{y-1}). \tag{4}$$

*Proof.* Letting $|\mathbf{n}| = \sum_k n_k$ denote the number of vertices of $G(\mathbf{c};\mathbf{n})$,

$$\begin{aligned} t(G(\mathbf{c};\mathbf{n});x,y) &= \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} (x-1)^{c(A)-c(G(\mathbf{c};\mathbf{n}))}(y-1)^{|A|-|\mathbf{n}|+c(A)} \\ &= (x-1)^{-c(G(\mathbf{c};\mathbf{n}))} \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} [(x-1)(y-1)]^{c(A)}(y-1)^{|A|-|\mathbf{n}|}. \end{aligned}$$

Hence

$$\begin{aligned} T(x,y;\mathbf{u}) &= \sum_{\mathbf{n}\in\mathbb{N}^V} t(G(\mathbf{c};\mathbf{n});x,y)\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!} \\ &= \frac{1}{x-1} S(y-1,(x-1)(y-1);\frac{\mathbf{u}}{y-1}) \\ &+ \sum_{\substack{\mathbf{n}\in\mathbb{N}^V \\ c(G(\mathbf{c};\mathbf{n}))\neq 1}} \left(\frac{1}{(x-1)^{c(G(\mathbf{c};\mathbf{n}))}} - \frac{1}{x-1}\right)\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!} \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} [(x-1)(y-1)]^{c(A)}(y-1)^{|A|-|\mathbf{n}|}. \end{aligned}$$

Recall that the graph $G(\mathbf{c};\mathbf{n})$ is connected if $n_i \geq 1$ and $n_j \geq 1$. It follows that the second summand on the right-hand side of the above equation for $T(x,y;\mathbf{u})$ is non-zero only if $n_i = 0$ or $n_j = 0$ and hence vanishes upon differentiating with respect to $u_i$ and $u_j$. This second term also vanishes when $x = 2$ because in this case $(x-1)^{-c} = 1 = (x-1)^{-1}$ for any $c$. □

With a view to finding an alternative expression for $S(z,w;\mathbf{u})$, set

$$q(\mathbf{n}) = \sum_{kl\in E} n_k n_l + \sum_{k\in V} c_k \binom{n_k}{2},$$

that is, $q(\mathbf{n})$ is the number of edges of $G(\mathbf{c};\mathbf{n})$.

**Lemma 4.** *For any graph $G$,*

$$S(z,w;\mathbf{u}) = F(z;\mathbf{u})^w \text{ where } F(z;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} (1+z)^{q(\mathbf{n})}\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!}.$$

*Proof.* The key observation is that the connected components of a spanning subgraph of $G(\mathbf{c};\mathbf{n})$ are connected spanning subgraphs of graphs in the family $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$. (For instance, a spanning subgraph of a complete bipartite graph is the union of connected spanning subgraphs of complete bipartite graphs.) From this and general properties of generating functions it follows that $S(z,w;\mathbf{u}) = e^{C(z;\mathbf{u})w}$, where $C(z;\mathbf{u})$ is the exponential generating function for connected spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$, that is,

$$C(z;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} \frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!} \sum_{\substack{A\subseteq E(G(\mathbf{c};\mathbf{n})) \\ (V(G(\mathbf{c};\mathbf{n})),A) \text{ connected}}} z^{|A|}.$$

Now $F(z;\mathbf{u}) = e^{C(z;\mathbf{u})}$ is the exponential generating function of spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$ (counted by number of edges only), which is given by the expression in the statement of the theorem. □

Next we combine Lemmas 3 and 4 to get an alternative form of equation (2).

**Lemma 5.** *Let $G = (V,E)$ be a connected graph containing vertices $i$ and $j$ such that $ki \in E$ or $kj \in E$ for every $k \in V \setminus \{i,j\}$. Then equation (2) is equivalent to*

$$\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = 2, \tag{5}$$

*where $f(\mathbf{u}) = F(-2;\mathbf{u})$.*

*Proof.* By equations (3) and (4) and Lemma 4,

$$\begin{aligned}
\frac{\partial^2 T(x,-1;\mathbf{u})}{\partial u_i \partial u_j} &= \frac{1}{x-1}\frac{\partial^2 S(-2,-2(x-1);\frac{\mathbf{u}}{-2})}{\partial u_i \partial u_j} = \frac{1}{x-1}\frac{\partial^2 f(\frac{\mathbf{u}}{-2})^{-2(x-1)}}{\partial u_i \partial u_j} \\
&= \frac{-1}{2}\left((-2x+1)f(\frac{\mathbf{u}}{-2})^{-2x}\frac{\partial f(\frac{\mathbf{u}}{-2})}{\partial u_i}\frac{\partial f(\frac{\mathbf{u}}{-2})}{\partial u_j} + f(\frac{\mathbf{u}}{-2})^{-2x+1}\frac{\partial^2 f(\frac{\mathbf{u}}{-2})}{\partial u_i \partial u_j}\right), \\
T(2,-1;\mathbf{u}) = S(-2,-2;\frac{\mathbf{u}}{-2}) = f(\frac{u}{-2})^{-2}.
\end{aligned}$$

The desired equation is now obtained by setting $x = 1$ in both expressions. $\square$

Solving the differential equation (5) will put conditions on the quadratic form $q(\mathbf{n})$ that translate to structural conditions on the graph $G$ and the clique/coclique parameter $\mathbf{c}$ that specify the graph $G(\mathbf{c};\mathbf{n})$. This is Theorem 6 below.

We use $\mathbb{I}(P)$ to denote the indicator function, equal to 1 when the statement $P$ is true and 0 otherwise.

**Theorem 6.** *A pair $(G,\mathbf{c})$ satisfies equation (1) for all $\mathbf{n} \in N^V$ with $n_i, n_j \geq 1$ if and only if the following conditions hold:*

*(i) $ij \in E$;*

*(ii) for each $k \in V \setminus \{i,j\}$, $\mathbb{I}(ki \in E) + \mathbb{I}(kj \in E) = c_k + 1$;*

*(iii) for all $U \subseteq V \setminus \{j\}$, either $j$ has odd degree in $G[U \cup \{j\}]$ or there is a vertex $k \in U$ whose degree in the induced subgraph $G[U]$ has the same parity as $c_k$.*

*Proof.* Note that have already observed that each $k \in V \setminus \{i,j\}$ is adjacent to at least one of $i$ and $j$. We now wish to find those $f$ that solve equation (5).

Define the relation $k \sim l$ to hold if and only if $kl \in E$, or $k = l$ and $c_k = 1$. The graph $\widetilde{G}$ with edges $kl$ when $k \sim l$ is equal to the graph $G = (V,E)$ with loops added to each vertex $k$ such that $c_k = 1$. We have

$$2q(\mathbf{n}) = \sum_{\substack{(k,l)\in V\times V \\ k\sim l}} n_k n_l - \sum_{\substack{k\in V \\ k\sim k}} n_k. \tag{6}$$

We have also

$$f(\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})}\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}$$

from which we calculate

$$\frac{\partial f(\mathbf{u})}{\partial u_i} = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})+\Delta_i q(\mathbf{n})}\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!},$$

where $\Delta_i q(\mathbf{n}) = q(\ldots, n_i+1, \ldots) - q(\ldots, n_i, \ldots)$ is the forward difference of $q(\mathbf{n})$ in the $i$th component, and

$$\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})+\Delta_{i,j} q(\mathbf{n})}\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!},$$

where $\Delta_{i,j} q(\mathbf{n}) = q(\ldots, n_i+1, \ldots, n_j+1, \ldots) - q(\ldots, n_i, \ldots, n_j, \ldots)$.

Multiplying power series we find that

$$\begin{aligned}
&\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = \\
&\sum_{\mathbf{n}\in\mathbb{N}^V}\sum_{\mathbf{m}\leq\mathbf{n}} (-1)^{q(\mathbf{m})+q(\mathbf{n}-\mathbf{m})}\left((-1)^{\Delta_i q(\mathbf{m})+\Delta_j q(\mathbf{n}-\mathbf{m})} - (-1)^{\Delta_{i,j} q(\mathbf{m})}\right)\prod_k \binom{n_k}{m_k}\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}.
\end{aligned} \tag{7}$$

Here we write $\mathbf{m} \leq \mathbf{n}$ to mean $m_k \leq n_k$ for each $k \in V$.

Now we use (6) to rewrite the forward differences. For instance, $\Delta_i q(\mathbf{m}) = \sum_{k\sim i} m_k$. By doing the same for $\Delta_j q(\mathbf{n}-\mathbf{m})$ and $\Delta_{i,j} q(\mathbf{n})$, we get that the relative parity of $\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n}-\mathbf{m})$ and $\Delta_{i,j} q(\mathbf{m})$ is given by

$$\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n}-\mathbf{m}) + \Delta_{i,j} q(\mathbf{m}) \equiv \sum_{k\sim j} n_k + \mathbb{I}(i\sim j) \pmod 2, \tag{8}$$

which is independent of $\mathbf{m}$. If the right-hand side of equation (8) is zero then the coefficient of $\mathbf{u}^{\mathbf{n}}$ in equation (7) is equal to zero. Since the constant term ($\mathbf{n} = \mathbf{0}$) should be equal to 2 it is necessary that $i \sim j$. Given $i \sim j$, for any $\mathbf{n}$, if

$$\sum_{k\sim j} n_k \equiv 1 \pmod 2$$

then the coefficient of $\mathbf{u}^{\mathbf{n}}$ in equation (7) is zero.

Therefore, we need only concern ourselves with the coefficients of $\mathbf{u}^{\mathbf{n}}$ where $\sum_{k\sim j} n_k \equiv 0 \pmod 2$. The coefficient

$$\frac{1}{\mathbf{n}!}[\mathbf{u}^{\mathbf{n}}]\left(\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j}\right)$$

is given by

$$-2\sum_{\mathbf{m}\leq\mathbf{n}} (-1)^{q(\mathbf{m})+q(\mathbf{n}-\mathbf{m})+\Delta_{i,j}q(\mathbf{m})} \prod_k \binom{n_k}{m_k}.$$

So we wish to find necessary and sufficient conditions for this coefficient of $\frac{1}{\mathbf{n}!}\mathbf{u}^{\mathbf{n}}$ to equal zero for all $\mathbf{n} \neq \mathbf{0}$, subject to $\sum_{k\sim j} n_k \equiv 0 \pmod 2$ and $i \sim j$. Again using (6) to rewrite the exponent of $-1$ in the expression above, and after some manipulation, we find that the coefficient we are interested in vanishes if and only if:

$$\begin{aligned} 0 &= \sum_{\mathbf{m}\leq\mathbf{n}} (-1)^{\sum_k m_k \sum_{l\sim k}[n_l + \mathbb{I}(l=i)+\mathbb{I}(l=j)+\mathbb{I}(l=k)]} \prod_k \binom{n_k}{m_k} \\ &= \prod_k \sum_{m_k \leq n_k} (-1)^{\left[\sum_{l\sim k} n_l + \mathbb{I}(i\sim k)+\mathbb{I}(j\sim k)+\mathbb{I}(k\sim k)\right] m_k} \binom{n_k}{m_k} \\ &= \prod_k \left[1 + (-1)^{\sum_{l\sim k} n_l + \mathbb{I}(i\sim k)+\mathbb{I}(j\sim k)+\mathbb{I}(k\sim k)}\right]^{n_k}. \end{aligned} \tag{9}$$

By taking $\mathbf{n}$ to have all its entries equal to zero, except for one equal to 2, for the expression (9) to be zero it is necessary that, for each $k \in V$,

$$\mathbb{I}(i\sim k) + \mathbb{I}(j\sim k) + \mathbb{I}(k\sim k) \equiv 1 \pmod 2. \tag{10}$$

Thus if $c_k = 1$ in $G(\mathbf{c};\mathbf{n})$ (a clique) the vertex $k$ must be adjacent to both $i$ and $j$, whereas if $c_k = 0$ (a coclique) then the vertex $k$ must be adjacent to exactly one of $i, j$.

Since the nullity of the expression (9) depends only on the parity of each $n_k$, if the coefficients subject to $\sum_{k\sim j} n_k \equiv 0 \pmod 2$ and $n_k \in \{0,1\}$ are all zero apart from the constant term then the coefficients of $\mathbf{u}^{\mathbf{n}}$ are zero for all $\mathbf{n} \neq \mathbf{0}$. In terms of the graph $G$, this is to say we may assume each vertex $k$ is either deleted ($n_k = 0$) or is present as a single vertex ($n_k = 1$); if this induced subgraph satisfies the required conditions then so does $G(\mathbf{c};\mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}^V$.

Note also that the conditions $i \sim j$ and $\sum_{l\sim j} n_l \equiv 0 \pmod 2$ imply that we can assume $n_j = 0$, otherwise expression (9) is clearly zero. Now consider the set $U \subseteq V \setminus \{j\}$ given by $U = \{k \in V : n_k \neq 0\}$. Since we assume $\sum_{k\sim j} n_k \equiv 0 \pmod 2$ the induced subgraph $G[U]$ of $G$ has the property that the number of vertices $k \in U$ such that $kj \in E$ is even. A necessary and sufficient condition that the expression (9) is zero (under the assumption that $i \sim j$, $n_j = 0$ and $\sum_{k\sim j} n_k \equiv 0 \pmod 2$) is that for any such choice of $U$ there is a vertex $k$ whose degree in $G[U]$ has the same parity as $c_k$. □

We remark that the roles of $i$ and $j$ in the previous theorem are symmetric. Indeed, if in (7) we write $\Delta_j q(\mathbf{m}) + \Delta_i q(\mathbf{n} - \mathbf{m})$ instead of $\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n} - \mathbf{m})$, condition (iii) in the statement of the theorem would be stated in terms of $i$ instead of $j$. In other words, any consequence of Theorem 6 about $j$ holds as well for $i$.

A particular instance of this is the next lemma, that says that the values of $c_i$ and $c_j$ can be chosen freely (that is, whether they are replaced by cliques or cocliques does not affect the validity of equation (1)). This is clearly true for $j$, as the conditions of Theorem 6 do not depend on the value of $c_j$.

**Lemma 7.** *If $G = (V, E)$, $i, j \in V$ and $\mathbf{c} \in \{0, 1\}^V$ satisfy the conditions of Theorem 6, then so does $G$ and $\mathbf{c}'$ where $\mathbf{c}'$ is $\mathbf{c}$ with $c_i$ replaced by $1 - c_i$ or with $c_j$ replaced by $1 - c_j$ (or both).*

We would now like to deduce the induced subgraph characterization of Theorem 1. For the rest of this section, we shall use the following notation:

$$\begin{aligned}
A &= \{k \in V \setminus \{i, j\} : ki \in E, c_k = 0\},\\
B &= \{k \in V \setminus \{i, j\} : kj \in E, c_k = 0\},\\
C &= \{k \in V \setminus \{i, j\} : ki, kj \in E, c_k = 1\}.
\end{aligned}$$

Clearly, by condition (ii) these sets partition $V \setminus \{i, j\}$ (recall Figure 1).

**Lemma 8.** *The induced subgraphs $G[A]$ and $G[B]$ are cocliques and the induced subgraph $G[C]$ is a clique.*

*Proof.* Let $k, k'$ be two vertices in $A$. By taking $U = \{k, k'\}$, condition (iii) in Theorem 6 implies that at least one of $k$ and $k'$ must have even degree in the subgraph they induce, so they cannot be adjacent. An analogous argument shows that $G[B]$ is also a coclique and that $G[C]$ is a clique. □

Lemmas 7 and 8 imply that condition (iii) of Theorem 6 is satisfied if and only if:

($\star$) for all $U \subseteq V \setminus \{i, j\}$ such that $|U \cap (B \cup C)|$ is even, the induced subgraph $G[U]$ contains either a vertex in $A \cup B$ of even degree or a vertex in $C$ of odd degree.

*Proof of Theorem 1.* We show how to deduce the characterization of Theorem 1 from the condition ($\star$). Clearly if $G$ contains as an induced subgraph any of the subgraphs described in conditions (ii) and (iii) in Theorem 1, then this subgraph contradicts condition ($\star$). Next we prove the converse. (Recall the forbidden induced subgraphs depicted in Figure 1.)

Suppose that $G$ is as described in Theorem 1 and that condition ($\star$) fails for a subset $U \subseteq V \setminus \{i, j\}$; that is, $G[U]$ contains none of the five forbidden induced subgraphs, $|U \cap (B \cup C)|$ is even, all vertices in $U \cap (A \cup B)$ have odd degree, and all vertices in $U \cap C$ have even degree. We lose no generality by restricting the discussion to the graph $G[U \cup \{i, j\}]$, so we assume that $U \cap A = A, U \cap B = B, U \cap C = C$. We begin by deducing two facts about the structure of the neighbourhoods in $U$. For any vertex $x \in U$, let $A_x$ (respectively, $B_x, C_x$) be its set of neighbours in $A$ (respectively, in $B$, $C$).

**Claim 1.** Let $D$ be one of $A, B$, or $C$ and let $E$ be one of $\{A, B, C\} \setminus \{D\}$. If $x, y \in D$, then $E_x$ and $E_y$ are comparable sets.

*Proof of Claim 1.* We prove the case $D = A$ and $E = B$, the other five cases being dealt with analogously. Suppose there are $u \in B_x \setminus B_y$ and $v \in B_y \setminus B_x$; then the edges $xu$ and $yv$ are a copy of the forbidden induced subgraph $2P_2$. □

**Claim 2.** If $x$ is a vertex in $C$, then $A_x \cup B_x$ induces a complete bipartite graph.

*Proof of Claim 2.* If not, the fourth graph in Figure 1 would be an induced subgraph. □

Claim 1 with $D = C$ and $E = A$ implies that there is a vertex $a_0 \in A$ adjacent to all vertices of $C$ that have at least one neighbour in $A$.

Now let $B' \subseteq B$ be the set of those vertices that are not adjacent to any vertex of $C$. If $B'$ is non-empty, each of its vertices must be adjacent to at least one vertex in $A$, because vertices in $B$ have odd degree. Suppose $b \in B'$ is adjacent to $a \in A$. If $a$ is not adjacent to every vertex in $C$, then we find the fifth graph in Figure 1 as an induced subgraph, therefore $a$ must be adjacent to all vertices in $C$. Then $a$ must also be adjacent to every vertex in $B \setminus B'$, otherwise the fourth forbidden induced subgraph would appear. This makes the degree of $a$ equal to $|B \cup C|$, which is even, hence contradicting our assumption. Therefore, $B'$ must be empty.

Hence, every vertex in $B$ (if any) must be adjacent to at least one vertex in $C$. By Claim 1 again, there is a vertex in $C$ adjacent to all vertices in $B$. Any vertex with this property must be adjacent to some vertex in $A$, and hence to $a_0$ as well (otherwise it would have degree $|B|+|C|-1$, which is odd). Also, $a_0$ is adjacent to all of $B$ by Claim 2. Now, let $C'$ be the vertices in $C$ that are not adjacent to $a_0$. Since $a_0$ has odd degree and $|B \cup C|$ is even, $|C'|$ is odd. Any $c' \in C'$ cannot be adjacent to all of $B$, because we have just shown that in this case it would be adjacent to $a_0$ as well. But then, if $c'$ is not adjacent to, say, $b' \in B$, then the edge $a_0b'$ and vertex $c'$ form one of the forbidden induced subgraphs.

Therefore, we are forced to have $B = \emptyset$. Then either there is a vertex in $C$ not adjacent to $a_0$, and hence to no vertex in $A$, or $a_0$ is adjacent to every vertex in $C$. But in the former case there is a vertex in $C$ of odd degree and in the latter case $a_0$ has even degree. □

# 4 Bijective proofs

In this section we give a bijective proof of Merino's identity

$$t(K_{n+2}; 1, -1) = t(K_n; 2, -1). \tag{11}$$

To translate the identity into combinatorial terms, we recall the interpretation of $t(G; 1+x, y)$ given by Gessel and Sagan in [3]. Let $\mathcal{T}(G)$ and $\mathcal{F}(G)$ be the set of spanning trees and spanning forests of a graph $G$, respectively (assume $G$ is connected from now on). The following is an expression of the Tutte polynomial as a generating function of spanning forests according to the number of connected components and an "external activity" that will be described next (it is not the usual external activity for trees defined by Tutte):

$$t(G; 1+x, y) = \sum_{F \in \mathcal{F}(G)} x^{c(F)-1} y^{e(F)}. \tag{12}$$

The external activity $e(F)$ is computed in the following way. First, we fix an arbitrary linear order $<$ on the vertex set $V$ and root every connected component of the forest at its smallest vertex. We say that vertex $u$ *precedes* vertex $v$ in the forest $F$ if $u$ and $v$ are in the same connected component and $u$ lies in the path from the root to $v$. If $u$ precedes $v$ in $F$, we denote by $u_v$ the child of $u$ that lies in the path from $u$ to $v$. Define

$$\mathcal{E}(F) = \{uv \in E(G) \backslash F : u \text{ precedes } v, v < u_v\}.$$

The edges in $\mathcal{E}(F)$ are called *externally active with respect to* $F$. (See Figure 2 for examples.) Observe that in particular the edges joining different connected components are never externally active. The *external activity* of a forest $F$ is $e(F) = |\mathcal{E}(F)|$.

By taking $x \in \{0, 1\}$ and $y = -1$ in (12) we obtain expressions for the evaluations we are interested in:

$$t(G; 1, -1) = \sum_{T \in \mathcal{T}(G)} (-1)^{e(T)}, \tag{13}$$

$$t(G; 2, -1) = \sum_{F \in \mathcal{F}(G)} (-1)^{e(F)}. \tag{14}$$

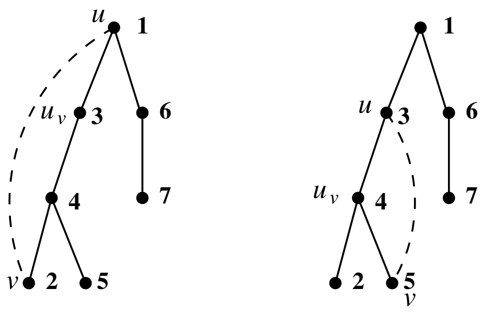


Figure 2: With respect to the given spanning tree of $K_7$, the edge $\{1, 2\}$ is externally active whereas the edge $\{3, 5\}$ is not.

Moreover, we wish to examine these expressions when $G$ is a complete graph. We take $[n]$ as the vertex set, with the usual order. In this case, it is easy to see that the external activity of a forest $F$ equals the number of inversions of $F$, where an *inversion* is a pair $(u, v)$ such that $u$ precedes $v$ in $F$ and $v$ is smaller than $u$. Therefore, $t(K_n; 1, y)$ is the generating function for inversions in trees with $n$ vertices (rooted at 1) and $t(K_n; 2, y)$ is the generating function for inversions in forests with $n$ vertices (each component rooted at its minimum). The polynomial $\sum_{T \in \mathcal{T}(K_n)} y^{\mathrm{inv}(T)}$, where $\mathrm{inv}(T)$ is the number of inversions of $T$ when rooted at 1, is called the *inversion polynomial.*

**Remark.** The identity between the Tutte polynomial of $K_n$ at $x = 1$ and the inversion polynomial was apparently first noticed by Björner. In [2, Exer. 7.7] he refers to Gessel and Wang [4] for a proof that the inversion polynomial is the generating function of connected subgraphs of $K_n$ counted by number of edges, and to Bessinger [1] for a bijection between trees counted by numbers of inversions and by (Tutte) external activity. Kuznetsov, Pak and Postnikov [5] prove the identity by showing that both polynomials satisfy the same recurrence relation.

Let $\mathcal{T}_n$ denote the set of labelled trees on $[n]$ rooted at 1 and $\mathcal{F}_n$ the set of labelled forests on $[n]$ where each component is rooted at its minimum. Identity (11) can be then rephrased as

$$\sum_{T \in \mathcal{T}_{n+2}} (-1)^{\mathrm{inv}(T)} = \sum_{F \in \mathcal{F}_n} (-1)^{\mathrm{inv}(F)}.$$

To prove this identity, we first cancel out some terms in the sums so that all remaining terms are positive. A forest $F \in \mathcal{F}_n$ is *increasing* if it has no inversions and it is *even* if all non-root vertices have an even number of children.

It is well known that increasing spanning trees of $K_{n+1}$ are in bijective correspondence with permutations of $[n]$; that is, the inversion polynomial evaluated at 0 is $(n-1)!$. Another relationship between inversions and permutations is that the inversion polynomial evaluated at $-1$ gives the number of up-down (or alternating) permutations. The first statement in Lemma 9 below says that the inversion polynomial evaluated at $-1$ is precisely the number of even increasing spanning trees of $K_n$. This connection between alternating permutations and even increasing spanning trees of $K_n$ was apparently first made by Viennot [10]. Pansiot [8] gives another proof by describing an involution on the trees that are not even increasing that reverses the parity of the number of inversions. Yet another proof is given in [5], together with other interpretations of the evaluation of the inversion polynomial at $-1$. There it is also briefly indicated (end of Section 3.3) how to construct an involution on trees that fixes even increasing trees and reverses the parity of the number of inversions in the remaining trees. Since we shall use similar ideas in the following section, we describe such an involution explicitly; let us remark that it is not exactly the same involution as the one given by Pansiot.

**Lemma 9.** *(i)* $\sum_{T\in\mathcal{T}_n}(-1)^{\mathrm{inv}(T)}$ *equals the number of even increasing trees in* $\mathcal{T}_n$.

*(ii)* $\sum_{F\in\mathcal{F}_n}(-1)^{\mathrm{inv}(F)}$ *equals the number of even increasing forests in* $\mathcal{F}_n$.

*Proof.* Let $\mathcal{U}_n$ be the collection of those trees in $\mathcal{T}_n$ that fail to be even increasing. We define an involution $\varphi$ of $\mathcal{U}_n$ such that $|\mathrm{inv}(\varphi(T)) - \mathrm{inv}(T)| = 1$. This suffices to prove both statements in the lemma, since the second follows by suitably applying the involution to the components of the forest.

A subtree of a rooted tree $T$ is a tree comprising a vertex of $T$ and all its descendants. We say that a subtree is *strictly even increasing* if it has no inversions and all its vertices, including the root, have an even number of children. Hence a tree is even increasing if and only if all its proper subtrees are strictly even increasing. Recall that the *depth* of a rooted tree is the length of the largest possible path starting from the root. Given a tree $T$ in $\mathcal{U}_n$, let $d$ be maximum with the property that all subtrees of $T$ of depth at most $d$ are strictly even increasing (clearly $d \geq 0$, since leaves are strictly even increasing). Among all subtrees of $T$ of depth $d+1$ that are not strictly even increasing, let $T'$ be the one that contains the smallest element; note that $T' \neq T$ as $T$ is not even increasing. Let the vertices of $T'$ be $\{a_1, \ldots, a_t\}$ with $a_1 > a_2 > a_3 > \cdots > a_t$. The fact that all proper subtrees of $T'$ are strictly even increasing implies that if the root of $T'$ has odd degree, then $t$ is even, whereas if the root of $T'$ has even degree then $t$ is odd. Also, if the root of $T'$ has even degree, it cannot be $a_t$, otherwise $T'$ would be increasing.

We now have all the ingredients to define $\varphi$. If the root of $T'$ is $a_{2i}$, where $1 \leq i \leq \lfloor t/2 \rfloor$, let $\varphi(T)$ be the tree obtained by interchanging vertices $a_{2i}$ and $a_{2i-1}$. In $T'$, an element is smaller than $a_{2i-1}$ if and only if it is smaller than $a_{2i}$, therefore the only effect of the swap of $a_{2i-1}$ and $a_{2i}$ is that $(a_{2i-1}, a_{2i})$ becomes an inversion. Similary, if the root of $T'$ is $a_{2i-1}$, we interchange $a_{2i-1}$ and $a_{2i}$ and the number of inversions goes down by one; it is straightforward to check that the tree $\varphi(T)$ obtained in this way is not even increasing.

For an example of the involution $\varphi$ see Figure 3. □

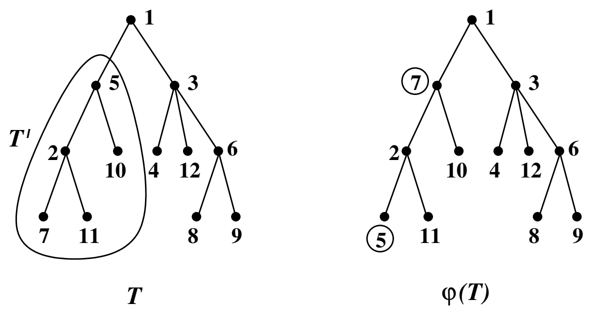


Figure 3: An example of the involution $\varphi$.

To complete the proof of identity (11), we give a bijection between even increasing trees with $n+2$ vertices and even increasing forests with $n$ vertices.

Let $T$ be an even increasing tree with $n+2$ vertices. The core of the bijection is the following easy lemma.

**Lemma 10.** *Let $T$ be an even increasing tree and let $u$ be any vertex of $T$. Then the forest $F$ obtained from $T$ by removing all edges in the unique $(1-u)$–path is even increasing.*

*Proof.* Clearly $F$ is increasing. Let $v$ be any vertex. If the $(1-u)$–path does not contain $v$, then the degree of $v$ in $F$ is the same as in $T$, hence even. If $v$ is in the $(1-u)$–path, then $v$ will become the root of one of the components of $F$, and it is not relevant whether its degree is even or odd. □

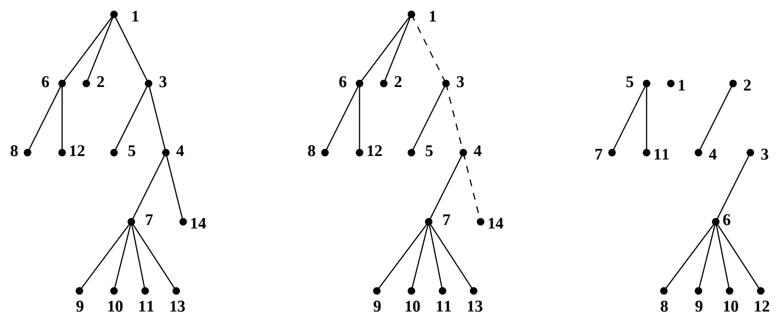


Figure 4: Obtaining an even increasing forest from an even increasing tree.

Hence, given an increasing even tree with $n+2$ vertices, by Lemma 10 if we remove the edges of the path that goes from 1 to $n+2$ then the result is an even increasing forest with $n+2$ vertices. Now remove vertices 1 and $n+2$ (the latter being an isolated vertex), obtaining an even increasing forest with $n$ vertices labelled from 2 to $n+1$. Relabel them from 1 to $n$ to obtain the desired forest. See Figure 4 for an illustration of this procedure.

Conversely, we show how to recover $T$ if $F$ is given. First, increase all the labels by 1, so that they run from 2 to $n+1$. Of the components of $F$, let $T_1, \dots, T_k$ be the ones where the root has even degree and let $U_1, \dots, U_l$ be the ones with odd root-degree; let the roots of these components be $r_1, \dots, r_k$ and $s_1, \dots, s_l$, respectively, and assume also that $s_1 < s_2 < \cdots < s_l$. Construct $T$ by adding vertices 1 and $n+2$ and edges $\{1, r_1\}, \dots, \{1, r_k\}, \{1, s_1\}, \{s_1, s_2\}, \dots, \{s_l, n+2\}$. It is clear that this procedure recovers $T$.

# 5 Evaluating $t(K_{n,m}; 2, -1)$ and $t(K_n + \overline{K}_m; 2, -1)$

We referred in Section 4 to the fact that $t(K_n; 2, -1)$ is the number of up-down permutations of $[n+1]$. The corresponding exponential generating function (EGF) is $\sec(t)(\tan(t) + \sec(t))$, recalling that the EGF for up-down permutations of $[n]$ is $\tan(t) + \sec(t)$.

In this section we shall find that the evaluations $t(K_{n,m}; 2, -1)$ and $t(K_n + \overline{K}_m; 2, -1)$ have similar if more complicated combinatorial interpretations in terms of up-down permutations.

Here are the first values of $t(K_{m,n}; 2, -1)$ for $1 \le m \le n$. (That the first column is given by $2^n$ and the second by $(3^{n+1} - 1)/2$ are facts easily proved from the definition and properties of the Tutte polynomial.)

| $n \backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | | | | | |
| 2 | 4 | 13 | | | | |
| 3 | 8 | 40 | 176 | | | |
| 4 | 16 | 121 | 736 | 4081 | | |
| 5 | 32 | 364 | 3008 | 21616 | 144512 | |
| 6 | 64 | 1093 | 12160 | 111721 | 927424 | 7256173 |

Set $t_{m,n} = t(K_{m,n}; 2, -1)$ and

$$B = B(u, v) = \sum_{m,n \ge 0} t(K_{m,n}; 2, -1) \frac{u^m}{m!} \frac{v^n}{n!}.$$

By Lemmas 3 and 4 and equation (4) for $G = K_2$,

$$\begin{aligned} B(u, v) &= S(-2, -2; -u/2, -v/2) = F(-2; -u/2, -v/2)^{-2} \\ &= \left( \sum_{m,n \ge 0} (-1)^{mn} \frac{u^m}{(-2)^m m!} \frac{v^n}{(-2)^n n!} \right)^{-2}. \end{aligned}$$

From $\sum_{m,n\geq 0}(-1)^{nm}\frac{u^m}{m!}\frac{v^n}{n!} = e^u\cosh(v) + e^{-u}\sinh(v)$ and after using some hyperbolic function identities, we obtain

$$B(u,v) = \frac{1}{\cosh(u)\cosh(v) - \sinh(u) - \sinh(v)}.$$

We would like to extract the coefficient of $u^m v^n$ from $B(u,v)$. Denote by $D^m$ the $m$-fold derivative with respect to $u$. Then

$$D^m(B(u,v))\big|_{u=0} = \sum_{n\geq 0} t(K_{m,n};2,-1)\frac{v^n}{n!}.$$

Let $g = \cosh(u)\cosh(v) - \sinh(u) - \sinh(v)$. Applying the rule for the derivative of a product to the equality $D^m(g\cdot g^{-1}) = 0$ we obtain the following recursion

$$gD^m(g^{-1}) = -\sum_{k=0}^{m-1}\binom{m}{k}D^{m-k}(g)D^k(g^{-1}).$$

It is easy to show by induction that, for $i \geq 1$, $D^{2i}(g) = \cosh(u)\cosh(v) - \sinh(u)$ and $D^{2i-1}(g) = \sinh(u)\cosh(v) - \cosh(u)$. By evaluating at $u = 0$ and using the above recurrence, we arrive at

$$D^m(g^{-1})\big|_{u=0} = -e^v\left(\sum_{k=0}^{m-1}\binom{m}{k}D^k(g^{-1})\big|_{u=0}\left(\delta^0_{k,m}\cosh(v) - \delta^1_{k,m}\right)\right),$$

where $\delta^0_{k,m}$ (respectively, $\delta^1_{k,m}$) is equal to 1 if $m$ and $k$ have the same parity (respectively, different parity), and zero otherwise.

Writing $b_m$ for $D^m(g^{-1})\big|_{u=0}$, we have

$$b_m = \sum_{k=0}^{m-1}\binom{m}{k}b_k\left(e^v\delta^1_{k,m} - \frac{1}{2}(1+e^{2v})\delta^0_{k,m}\right).$$

Since $b_0 = e^v$, it follows that $b_k$ is a linear combination of exponentials $e^{lv}$ with $l$ being at most $k+1$ and of parity opposite to $k$. The first $b_k$ are:

$$\begin{aligned} b_0 &= e^v, \\ b_1 &= e^{2v}, \\ b_2 &= \frac{1}{2}(3e^{3v} - e^v), \\ b_3 &= \frac{1}{2}(6e^{4v} - 4e^{2v}), \\ b_4 &= \frac{1}{2}(2e^v - 15e^{3v} + 15e^{5v}). \end{aligned}$$

Let $b_{m,j}$ be the coefficient of $e^{jv}$ in $b_m$, so that

$$t(K_{m,n};2,-1) = \sum_{j=1}^{m+1} b_{m,j}j^n.$$

The $b_{m,j}$ satisfy the recurrence

$$b_{m,j} = \sum_{k=0}^{m-1}\binom{m}{k}\left(b_{k,j-1}\delta^1_{k,m} - \frac{1}{2}(b_{k,j} + b_{k,j-2})\delta^0_{k,m}\right). \tag{15}$$

The first values of $b_{m,j}$ are given in the following table:

| $m\backslash j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 0 | 1 | | | | | | |
| 2 | -$\frac{1}{2}$ | 0 | $\frac{3}{2}$ | | | | | |
| 3 | 0 | -2 | 0 | 3 | | | | |
| 4 | 1 | 0 | -$\frac{15}{2}$ | 0 | $\frac{15}{2}$ | | | |
| 5 | 0 | $\frac{17}{2}$ | 0 | -30 | 0 | $\frac{45}{2}$ | | |
| 6 | -$\frac{17}{4}$ | 0 | $\frac{231}{4}$ | 0 | -$\frac{525}{4}$ | 0 | $\frac{315}{4}$ | |
| 7 | 0 | -62 | 0 | 378 | 0 | -630 | 0 | 315 |

Notice that the elements in the table satisfy the Pascal-like recurrence

$$b_{m,j} = \frac{j}{2}(b_{m-1,j-1} - b_{m-1,j+1}).$$

**Theorem 11.** *For $m \geq 0$, $t(K_{m,n}; 2, -1) = \sum_{j=1}^{m+1} b_{m,j} j^n$, where $b_{m,j}$ is given by*

$$\begin{aligned} &b_{0,1} = 1,\ b_{m,0} = 0,\ b_{m,m+1} = 0, \\ &b_{m,j} = \frac{j}{2}(b_{m-1,j-1} - b_{m-1,j+1}), \quad \textit{for } 1 \leq j \leq m. \end{aligned}$$

*In particular, $b_{m,j} = 0$ if $m$ and $j$ are of the same parity.*

*Proof.* We show by induction on $m$ that the sequence defined recursively by

$$\begin{aligned} &a_{0,1} = 1,\ a_{m,0} = 0,\ a_{m,m+1} = 0, \\ &a_{m,j} = \frac{j}{2}(a_{m-1,j-1} - a_{m-1,j+1}), \quad \text{for } 1 \leq j \leq m, \end{aligned}$$

satisfies recurrence (15).

$$\begin{aligned} a_{m,j} &= \frac{j}{2}(a_{m-1,j-1} - a_{m-1,j+1}) \\ &= \frac{j}{2}\sum_{k=1}^{m-2}\binom{m-1}{k}\left((a_{k,j-2} - a_{k,j})\delta^1_{k,m-1} - \frac{1}{2}(a_{k,j-1} + a_{k,j-3} - a_{k,j+1} - a_{k,j-1})\delta^0_{k,m-1}\right). \end{aligned} \tag{16}$$

From the recursion satisfied by the $a_{k,j}$, we have that

$$\frac{j}{2}(a_{k,j-2} - a_{k,j}) = a_{k+1,j-1} + \frac{1}{2}a_{k,j-2} - \frac{1}{2}a_{k,j-2}.$$

By applying the same trick to the other terms within the parentheses, the right-hand side of equation (16) becomes

$$\sum_{k=1}^{m-2}\binom{m-1}{k}\left((a_{k+1,j-1} + \frac{1}{2}a_{k,j-2} - \frac{1}{2}a_{k,j})\delta^1_{k,m-1} - \frac{1}{2}(a_{k+1,j} + a_{k+1,j-2} + a_{k,j-3} - a_{k,j-1})\delta^0_{k,m-1}\right).$$

We rewrite the previous line as

$$\binom{m-1}{m-2}a_{m-1,j-1} + \sum_{k=1}^{m-2}\binom{m-1}{k-1}a_{k,j-1}\delta^0_{k,m-1} + \tag{17}$$

$$\frac{1}{2}\sum_{k=1}^{m-2}\binom{m-1}{k}(a_{k,j-2} - a_{k,j})\delta^1_{k,m-1} + \tag{18}$$

$$-\frac{1}{2}\sum_{k=1}^{m-2}\binom{m-1}{k-1}(a_{k,j} + a_{k,j-2})\delta^1_{k,m-1} + \tag{19}$$

$$\frac{1}{2}\sum_{k=1}^{m-2}\binom{m-1}{k}(a_{k,j-1} - a_{k,j-3})\delta^0_{k,m-1}. \tag{20}$$

Adding up lines (18) and (19) we obtain

$$-\frac{1}{2}\sum_{k=1}^{m-2}\binom{m}{k}(a_{k,j}+a_{k,j-2})\delta^1_{k,m-1}+ \tag{21}$$

$$\sum_{k=1}^{m-2}\binom{m}{k}a_{k,j-2}\delta^1_{k,m-1}. \tag{22}$$

From lines (20) and (22) and the induction hypothesis we get

$$a_{m-1,j-1}+\sum_{k=1}^{m-1}\binom{m-1}{k}a_{k,j-1}\delta^0_{k,m-1}. \tag{23}$$

Finally, lines (17), (21) and (23) add up to

$$\binom{m}{m-1}a_{m-1,j-1}+\sum_{k=1}^{m-2}\binom{m}{k}\left(a_{k,j-1}\delta^1_{k,m}-\frac{1}{2}(a_{k,j}+a_{k,j-2})\delta^0_{k,m}\right),$$

which is what we needed to show. □

We use the preceding theorem to give a relationship between the numbers $b_{m,j}$ and up-down permutations. Observe that

$$\sum_{m,j} b_{m,j}x^j\frac{u^m}{m!}=\frac{1}{\frac{x+x^{-1}}{2}\cosh(u)-\frac{x-x^{-1}}{2}-\sinh(u)}.$$

From this it follows, by extracting the coefficient of $x$, that

$$\sum_m b_{m,1}\frac{u^m}{m!}=\frac{2}{1+\cosh(u)}.$$

Define $B_j(u)=\sum_{m\geq 0} b_{m,j}\frac{u^m}{m!}$. The recurrence for the $b_{m,j}$ in Theorem 11 becomes

$$B_{j+1}(u)=B_{j-1}(u)-\frac{2}{j}B_j'(u).$$

From the initial case $B_1=2/(1+\cosh(u))$ and induction it follows that

$$\begin{aligned}
B_{2i}(u) &= \frac{4i\sinh(u)(\cosh(u)-1)^{i-1}}{(1+\cosh(u)^{i+1})} = 2i\,\mathrm{sech}^2(\frac{u}{2})\tanh^{2i-1}(\frac{u}{2})\\
B_{2i+1}(u) &= \frac{2(2i+1)(\cosh(u)-1)^{i-1}}{(1+\cosh(u)^{i+1})} = (2i+1)\mathrm{sech}^2(\frac{u}{2})\tanh^{2i}(\frac{u}{2})
\end{aligned}$$

In conclusion,

$$B_j(u)=2(\tanh^j(\frac{u}{2}))'.$$

The coefficients of $\tanh^j(u)$ can be interpreted combinatorially in the following way. Recall that $\tan(x)$ is the EGF of up-down permutations of $[n]$, for odd $n$ (odd up-down permutations). Then $\tanh(x)$ is the EGF for *signed* odd up-down permutations, where the sign depends only on $n$ and is given by $(-1)^{(n-1)/2}$. Finally, $\tanh(x)^j$ is the EGF for sequences of $j$ signed odd up-down permutations.

For instance, consider $b_{3,2}$. There is one odd up-down permutation of [1] and two odd up-down permutations of [3]. There are thus 16 permutations of [4] that can be split as a sequence of two odd up-down permutations. Then the coefficient of $u^3$ in $2(\tanh^2(u/2))'$ is $2 \cdot 16/(3! \cdot 2^4) = 2/3!$, which agrees with $|b_{3,2}| = 2$.

Next we look at the evaluation of $t(K_m + \overline{K}_n; 2, -1)$, following the same strategy. Here is a table for the first values of $m$ and $n$. The first column is again $2^n$ and the sequence in the first row is the number of up-down permutations.

| $n\backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 16 | 61 | 272 | 1385 |
| 2 | 4 | 14 | 56 | 256 | 1324 | 7664 |
| 3 | 8 | 41 | 208 | 1141 | 6848 | 44981 |
| 4 | 16 | 122 | 800 | 5296 | 36976 | 275792 |
| 5 | 32 | 365 | 3136 | 25261 | 205952 | 1747745 |
| 6 | 64 | 1094 | 12416 | 122656 | 1173184 | 11357744 |

Define

$$C(u,v) = \left( \sum_{m,n\geq 0} (-1)^{mn+\binom{m}{2}} \frac{u^m}{(-2)^m m!} \frac{v^n}{(-2)^n n!} \right)^{-2}.$$

Since

$$\sum_{m,n\geq 0} (-1)^{nm+\binom{m}{2}} \frac{u^m}{m!} \frac{v^n}{n!} = e^v \cos(u) + e^{-v} \sin(u),$$

after some manipulation we obtain

$$C(u,v) = \frac{1}{\cosh(v) - \cos(u)\sinh(v) - \sin(u)}.$$

To extract the coefficient of $u^m$ in $C(u,v)$ we take derivatives as before. The calculations are routine and analogous to the previous ones, so we present the result directly. Setting $c_m = D^m(C(u,v))\big|_{u=0}$, we obtain the recurrence relation

$$\begin{aligned} c_m &= -e^v \sum_{k=0}^{m-1} \binom{m}{k} c_k \left( \sinh(v)(-1)^{\frac{m-k}{2}-1} \delta^0_{k,m} + (-1)^{\frac{m-k-1}{2}-1} \delta^1_{k,m} \right) \\ &= \sum_{k=0}^{m-1} \binom{m}{k} c_k \left( e^v (-1)^{\frac{m-k-1}{2}} \delta^1_{k,m} + \frac{e^{2v}-1}{2} (-1)^{\frac{m-k}{2}} \delta^0_{k,m} \right), \end{aligned}$$

with initial term $c_0 = e^v$. Then, again, each $c_m$ can be written as $\sum_{j=1}^{m+1} c_{m,j} e^{jv}$, where $c_{m,j}$ is zero when $m$ and $j$ have the same parity. Computing the first terms of the double sequence $\{c_{m,j}\}_{m,j}$ one is led to the conjecture that $c_{m,j} = |b_{m,j}|$.

**Theorem 12.** *For $m \geq 0$, $t(K_m + \overline{K}_n; 2, -1) = \sum_{j=1}^{m+1} c_{m,j} j^n$, where $c_{m,j} = b_{m,j}(-1)^{(m-j-1)/2}$.*

*Proof.* The proof follows easily by induction, using the recurrence for $c_m$ to write a recurrence for the $c_{m,j}$ very similar to the one for $b_{m,j}$. □

The EGF for the sequence $\{c_{m,j}\}_m$ follows immediately from the one for $\{b_{m,j}\}_m$:

$$\sum_{m\geq 0} c_{m,j} \frac{u^m}{m!} = (\tan(u)^j)'$$

## 6 An open question

We would like to find a bijective proof of the identity

$$t(K_{n+1,m+1}; 1, -1) = t(K_{n,m}; 2, -1). \tag{24}$$

The interpretation of Gessel and Sagan of $t(G; x, -1)$ allows us to translate this identity into more combinatorial terms, in a similar way to that followed in Section 4 for the corresponding identity for complete graphs. We first consider equations (13) and (14) when $G$ is a complete bipartite graph. Take $[n] \cup [m]' = [n] \cup \{1', 2', \ldots, m'\}$ as the vertex set of $K_{n,m}$ and call the vertices in $[n]$ *black* and the vertices in $[m]'$ *white*. Black vertices amongst themselves are ordered by the usual order; the same applies to white vertices. A black vertex is smaller than a white one. Trees are thus rooted at black vertices, unless they consist of a single white vertex.

If the edge $uv$ is externally active, the vertices $u_v$ and $v$ belong to the same class, therefore the external activity of a forest $F$ is the number of pairs of vertices $(x, y)$ such that $x$ precedes $y$, $x < y$ and $x$ and $y$ are both white or both black. We refer to these as *white inversions* and *black inversions*, respectively. Their union is the set of *monochromatic inversions of* $F$ and its cardinality is denoted by $\mathrm{binv}(F)$.

As before, let $\mathcal{T}_{n,m}$ be the set of spanning trees of $K_{n,m}$, rooted at 1, and let $\mathcal{F}_{n,m}$ be the set of spanning forests of $K_{n,m}$, each component rooted at its minimum vertex. Equality (24) is equivalent to

$$\sum_{T \in \mathcal{T}_{n,m}} (-1)^{\mathrm{binv}(T)} = \sum_{F \in \mathcal{F}_{n,m}} (-1)^{\mathrm{binv}(F)}.$$

Next, we would like to reduce this equality between alternating sums to an equality between cardinalities of sets. That is, to find an analogue to Lemma 9.

A forest in $\mathcal{F}_{n,m}$ is *$\chi$-increasing* if it has no monochromatic inversions, and it is *bi-even* if each non-root vertex has an even number of grandchildren (descendants at distance two).

**Lemma 13.** *(i)* $\sum_{T \in \mathcal{T}_{n,m}} (-1)^{\mathrm{binv}(T)}$ *equals the number of bi-even $\chi$-increasing trees in* $\mathcal{T}_{n,m}$.

*(ii)* $\sum_{F \in \mathcal{F}_{n,m}} (-1)^{\mathrm{binv}(F)}$ *equals the number of bi-even $\chi$-increasing forests of* $\mathcal{F}_{n,m}$.

*Proof.* The proof goes along the same lines as the case of the complete graph. □

So far we have not been able to find a bijection that proves identity (24).

**Problem 14.** *Find a bijection between bi-even $\chi$-increasing trees of $\mathcal{T}_{n+1,m+1}$ and bi-even $\chi$-increasing forests of $\mathcal{F}_{n,m}$.*

The numbers of bi-even $\chi$-increasing trees in $\mathcal{T}_{n,m}$ appear at the beginning of Section 5. As previously mentioned, even increasing trees are in bijection with (among other objects) up-down permutations. We do not know of any family of permutations equinumerous with bi-even $\chi$-increasing trees.

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