Monochromatic paths on edge colored digraphs and state splittings

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Abstract

We look at the behavior under state splitting of distinct kinds of properties regarding monochromatic paths on edge colored digraphs. These are absorbance and independence as well as the existence of kernels, semikernels, quasikernels and Grundy functions, all of them defined in terms of monochromatic paths.

Keywords: absorbance, independence, kernel, semikernel, quasikernel, Grundy function, monochromatic path, state splitting, duality

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1 Introduction

There has always been interest in studying properties of digraphs under different kinds of operations (see [1], [6], [9], [14]). Results concerning the line digraph operator have motivated other kind of operations to be studied. In [4] the authors look at simple state splitting (a fundamental operation in symbolic dynamics), and generalize results known for line digraphs. Here we look at state splittings on edge colored digraphs and compare absorbance and independence as well as the existence of kernels, semikernels, quasikernels and Grundy functions, all of them defined in terms of monochromatic paths, and also consider the dual definitions. In particular, we generalize previous results regarding kernels and line digraphs (see [5]). Some of these properties have been studied for other kinds of operations in edge colored digraphs (see [7]).

The importance of state splittings in symbolic dynamics comes from the fact that they constitute the basic blocks from which a conjugacy between shift spaces is formed. This results is known as the *Decomposition Theorem* (see e.g. [11] or [10]). *Shift spaces* are the central objects of study in symbolic dynamics and consist of sets of doubly infinite sequences of symbols, defined by specifying a set of *forbidden blocks*. For example, the set of all doubly infinite walks in an oriented graph is a shift space (here the symbols are the edges and a set of forbidden blocks consists of all ordered pair of edges not forming a path). Two shift spaces are *conjugated* if they are essentially the same, which means that there exists a bijective block code with inverse also a block code. A *block code* is a particular kind of map defined in terms of a *local rule* so that to determine the zero coordinate of the image it is only necessary to look at a fixed amount of coordinates to the past (the *memory*) and to the future (the *anticipation*). A useful standard technique that allows one to suppose that a block code can always be chosen so that its memory and anticipation are both equal to zero makes use of what is known as the *higher block presentation* of shift spaces. For every integer $n \geq 1$, then n-higher block presentation of a shift space is a shift space where the symbols are the blocks of length n of the original shift space. In the 1-higher block presentation of a shift space determined by an oriented graph, the blocks are the edges and the resulting shift space is the one determined by the corresponding line digraph. Now, the higher block presentation of a shift space is conjugated to the original shift space and hence, in virtue of the Decomposition Theorem, it can be obtained by sequences of state splittings. It is in this sense that state splitting is an operation that refines the line digraph operator.

The shift spaces that result from oriented graphs as described above are called *Shifts of Finite Type* for the set of forbidden blocks can be chosen to be finite. If instead we consider colored oriented graphs, the set of symbols is the set of colors and the resulting shift spaces are now called *Sofic Shifts* (a *finite* set of forbidden blocks may not exist). State splittings have been defined in the context of colored oriented graphs as well as many other concepts in graph theory. For example, independence by monochromatic paths was introduced by Sands, Sauer and Woodrow in [12]. Kernels by monochromatic paths were defined in [2] where sufficient conditions for their existence were obtained based on results on colored tournaments and monochromatic paths in [3] (see also [13]). Another instance is the absorbing sets by monochromatic paths that were introduced in [8]. Kernels by monochromatic paths and line digraphs were initially studied in [5] by Galeana-Sánchez and Pastrana-Ramírez and their results are generalized in this paper by combining the results on absorbance and independece, now in the context of state splittings.

2 Basic definitions

In this paper, a digraph D consists of a vertex set $V(D)$ and an arc set $A(D) \subset V(D) \times V(D)$ with no loops (so $(u,u) \notin A(D)$ for all $u \in V(D)$). For every $u \in V(D)$ let $\Gamma^-(u) = \{x \in V(D) \mid (x,u) \in A(D)\}$ and $\Gamma^+(u) = \{y \in V(D) \mid (u,y) \in A(D)\}$. Let $m \leq |A(D)|$ be a positive integer. An m-coloring (or m-*labeling*) is a function $L\colon A(D) \to [m]$ where $[m] = \{1, \ldots, m\}$ (we refer to its elements as *colors* or *labels*). We say that $\mathcal{D} = (D, L)$ is an m-colored digraph. A sequence of distinct vertices $T = (x_0, \ldots, x_k)$ such that $(x_{i-1}, x_i) \in A(D)$ for every $i = 1, \ldots, k$ is an x_0x_k-*path* in D that *starts* at x_0 and *ends* at x_k and determines a *colored path* by looking at the sequence of colors $(L(x_0, x_1), \ldots, L(x_{k-1}, x_k))$; it is *monochromatic* if $L(x_{i-1}, x_i) = L(x_{j-1}, x_j)$ for every $i, j = 1, \ldots, k$. We let $V(T) = \{x_0, \ldots, x_k\}$. For a vertex $u \in V(D)$ and a subset of vertices $A \subset V(D)$, a monochromatic uA-path is any monochromatic ux-path with $x \in A$, and define monochromatic Au-path similarly.

Recall *in-state splitting*:

Definition 2.1. Let $\mathcal{D} = (D, L)$ be an m-colored digraph. Let $u \in V(D)$ and $\mathcal{P} = \{F_1, \ldots, F_n\}$ be a partition of $\Gamma^-(u)$. Let $\mathcal{D}^- = (D^-, L^-)$ be the m-colored digraph with vertex set $V(D^-) = (V(D) \setminus \{u\}) \cup \{u_1, \ldots, u_n\}$ and arc set and m-coloring defined as follows:

- For every $x, y \in V(D) \setminus \{u\}$, $(x,y) \in A(D^-)$ if and only if $(x,y) \in A(D)$ and in this case we let $L^-(x,y) = L(x,y)$.
- For every $i \in \{1, \ldots, n\}$ and $z \in V(D) \setminus \{u\}$, $(z, u_i) \in A(D^-)$ if and only if $(z,u) \in F_i$ and in this case we let $L^-(z, u_i) = L(z, u)$.
- For every $i \in \{1, \ldots, n\}$ and $z \in V(D) \setminus \{u\}$, $(u_i, z) \in A(D^-)$ if and only if $(u, z) \in A(D)$ and in this case we let $L^-(u_i, z) = L(u, z)$.

Out-state splitting is defined similarly, substituting all the occurrences of '$-$' by '$+$' and "switching" the last two items so that we would have:

- For every $i \in \{1, \ldots, n\}$ and $z \in V(D) \setminus \{u\}$, $(z, u_i) \in A(D^+)$ if and only if $(z, u) \in A(D)$ and in this case we let $L^+(z, u_i) = L(z, u)$.
- For every $i \in \{1, \ldots, n\}$ and $z \in V(D) \setminus \{u\}$, $(u_i, z) \in A(D^+)$ if and only if $(u, z) \in F_i$ and in this case we let $L^+(u_i, z) = L(u, z)$.

Line digraphs can be obtained by sequences of state splittings. This is a well known fact in symbolic dynamics. When dealing with colored digraphs,

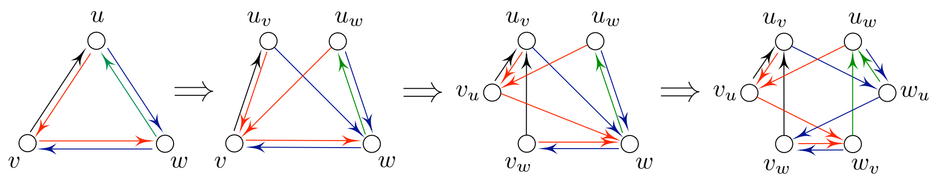

Figure 1: Elementary in-splittings that result in the in-colored line digraph.

the line digraph may be defined in two different ways. We shall state these two definitions and show how the resulting objets are decomposable into state splittings. The decomposition is exactly the same as the one shown in [4], here we just need to carry out the colorings.

Let $\mathcal{D} = (D, L)$ be an m-colored digraph. The *in-colored line digraph of* $\mathcal{D}$ is the m-colored digraph $\mathcal{L}^-(\mathcal{D}) = (\mathcal{L}(D), \mathcal{L}^-(L))$ where $\mathcal{L}(D)$ is the line digraph of D, that is, $V(L(D)) = A(D)$ and for every $(u, v), (x, y) \in A(D)$, $((u, v), (x, y)) \in A(L(D))$ if and only if $v = x$, and the coloring is defined by $\mathcal{L}^-(L)\left(((u, v), (x, y))\right) = L(u, v)$. The *out-colored line digraph of* $\mathcal{D}$ is the m-colored digraph $\mathcal{L}^+(L)(\mathcal{D}) = (\mathcal{L}(D), \mathcal{L}^+(L))$ where $L^+\left(((u, v), (x, y))\right) = L(x, y)$

Proposition 2.2. *Let $\mathcal{D} = (D, L)$ be an m-colored digraph. The in-colored line digraph $\mathcal{L}^-(\mathcal{D})$ can be obtained by a sequence of in-state splittings.*

Figure 1 describes the procedure of the proof of proposition 2.2.

Proof. Let $V(D) = \{u, v, \ldots, w\}$. Start with a complete in-splitting of vertex $u \in V(D)$, that is, consider the partition $\mathcal{P} = \{\{(x, u)\} \mid x \in \Gamma^-(u)\}$. Then u in-splits into $|\Gamma^-(u)|$ new vertices, one for each $x \in \Gamma^-(u)$. Denote the new vertices by u_x and let the coloring be defined as in the definition of in-state splitting so that $L^-(x, u_x) = L(x, u)$ and $L^-(u_x, y) = L(u, y)$ for every $y \in \Gamma^+(u)$. Next, choose a second vertex $v \notin \{u_x \mid x \in \Gamma^-(u)\}$. Its in-coming arcs are of the form (y, v) with $y \in \Gamma^-(v) \setminus \{u\}$ or of the form (u_x, v) if $u \in \Gamma^-(v)$ with $x \in \Gamma^-(u)$. In-split vertex v into $|\Gamma^-(v)|$ new vertices according to the partition formed by the singletons $\{y\}$ with $y \in \Gamma^-(v) \setminus \{u\}$ and $\{u_x \mid x \in \Gamma^-(u)\}$ if $u \in \Gamma^-(v)$. Denote by v_y the new vertices corresponding to the singletons, denote by v_u the last vertex corresponding to the partition element $\{u_x \mid x \in \Gamma^-(u)\}$ and let the coloring be carried out as indicated by the definition of in-splitting so that

- $L^-(y, v_y) = L(y, v)$
- $L^-(u_x, v_u) = L(u, v)$ if $u \in \Gamma^-(v)$,

- $L^-(v_y, z) = L(v, z)$ for all $z \in \Gamma^+(v) \setminus \{u\}$
- $L^-(v_y, u_x) = L(v, u)$ if $u \in \Gamma^+(v)$ and
- $L^-(v_u, u_v) = L(v, u)$ if $v \in \Gamma^-(u)$.

Continue in-splitting vertices in this way until a last vertex w in-splits according to the partition with $|\Gamma^-(w)|$ elements, one for each $x \in \Gamma^-(w)$ and defined by $\{x_y \mid y \in \Gamma^-(x)\}$. Each of the new vertices is determined by a unique $x \in \Gamma^-(w)$ and therefore we denote it by w_x. Let $\mathcal{L}^-(L)\colon A(L(D))$ be the resulting m-coloring.

For every arc $(x, y) \in A(D)$ there is a unique vertex y_x, and the map $(x, y) \mapsto y_x$ defines a bijection between $A(D)$ and the vertices of the resulting digraph. If $(x, y), (y, z) \in A(D)$, then (y_x, z_y) is an arc with $\mathcal{L}^-(L)(y_x, z_y) = L(x, y)$. Hence the map is an isomorphism of colored digraphs between $\mathcal{L}^-(D)$ and the resulting digraph. □

The proof of the following proposition is similar.

Proposition 2.3. *Let $\mathcal{D} = (D, L)$ be an m-colored digraph. The out-colored line digraph $\mathcal{L}^+(\mathcal{D})$ can be obtained by a sequence of out-state splittings.*

Lemma 2.4. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Let $T = (x_0, \ldots, x_k)$ be a monochromatic x_0x_k-path in $\mathcal{D}$.*

1. *If $x_0, x_k \neq u$, then there exists a monochromatic x_0x_k-path in $\mathcal{D}^-$.*
2. *If $x_0 = u$, then there exist monochromatic u_ix_k-paths in $\mathcal{D}^-$ for all $i = 1, \ldots, n$*
3. *If $x_k = u$, then there exists a monochromatic x_0u_q-path in $\mathcal{D}^-$ for some $q \in \{1, \ldots, n\}$.*

Proof. 1. If $u \notin V(T)$, then T is a monochromatic x_0x_k-path in $\mathcal{D}^-$. Otherwise, let $u = x_j$ for some $j = 1, \ldots, k-1$. Let $q \in \{1, \ldots, n\}$ be such that $(x_{j-1}, u) \in F_q$. Then $(x_0, \ldots, x_{j-1}, u_q, x_{j+1}, \ldots, x_k)$ is a monochromatic x_0x_k-path in $\mathcal{D}^-$.

2. Clearly, for every $i = 1, \ldots, n$, $(u_i, x_1, \ldots, x_k)$ is a monochromatic u_ix_k-path in $\mathcal{D}^-$.

3. Let $q \in \{1, \ldots, n\}$ be such that $(x_{k-1}, u) \in F_q$. Then $(x_0, \ldots, x_{k-1}, u_q)$ is a monochromatic x_0u_q-path in $\mathcal{D}^-$. □

Lemma 2.5. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Let $T = (x_0, \ldots, x_k)$ be a monochromatic x_0x_n-path in $\mathcal{D}^-$.*

1. *If $x_0, x_k \notin \{u_1, \dots, u_n\}$, then there exists a monochromatic x_0x_k-path in $\mathcal{D}$.*

2. *If $x_0 \in \{u_1, \dots, u_n\}$ and $x_k \notin \{u_1, \dots, u_n\}$, then there exists a monochromatic ux_k-path in $\mathcal{D}$.*

3. *If $x_0 \notin \{u_1, \dots, u_n\}$ and $x_k \in \{u_1, \dots, u_n\}$, then there exists a monochromatic x_0u-path in $\mathcal{D}$.*

4. *If $x_0, x_k \in \{u_1, \dots, u_n\}$, then there exists a monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$.*

Proof. 1. If $V(T) \cap \{u_1, \dots, u_n\} = \varnothing$, then T is a monochromatic x_0x_k-path in $\mathcal{D}$. Suppose that $V(T) \cap \{u_1, \dots, u_n\} \neq \varnothing$. Let

$$a = \min \{j \in \{0, 1, \dots, k\} \mid x_j \in \{u_1, \dots, u_n\}\} \tag{2.6}$$

$$b = \max \{j \in \{0, 1, \dots, k\} \mid x_j \in \{u_1, \dots, u_n\}\} \tag{2.7}$$

By assumption, $0 < a \leq b < k$. Then $(x_0, \dots, x_{a-1}, u, x_{b+1}, \dots, x_k)$ is a monochromatic x_0x_k-path in $\mathcal{D}$.

2. Let b be as in 2.7. By assumption $b < k$. Then $(u, x_{b+1}, \dots, x_k)$ is a monochromatic ux_k-path in $\mathcal{D}$.

3. Let a be as in 2.6. By assumption $a > 0$. Then $(x_0, \dots, x_{a-1}, u)$ is a monochromatic x_0, u-path in $\mathcal{D}$.

4. Let $c = \min\{j \in \{1, \dots, k\} \mid x_j \in \{u_1, \dots, u_n\}\}$. We have that $c > 1$ since D has no loops. Then $(u, x_1, \dots, x_{c-1}, u)$ is a monochromatic cycle C in $\mathcal{D}$ with $u \in C$. □

3 Absorbance

Let $\mathcal{D} = (D, L)$ be an m-colored digraph. A subset of vertices $S \subset V(D)$ is *absorbent by monochromatic paths* if for every $x \in V(D) \setminus S$ there exist $u \in S$ and a monochromatic xu-path.

Theorem 3.1. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in definition 2.1. If $S \subset V(D)$ is absorbent by monochromatic paths in $\mathcal{D}$, then*

$$S' = \begin{cases} (S \setminus \{u\}) \cup \{u_1, \dots, u_n\} \text{ if } u \in \mathcal{S} \\ \\ S \text{ otherwise} \end{cases} \tag{3.2}$$

is absorbent by monochromatic paths in $\mathcal{D}^-$.

Proof. Let $S \subset V(D)$ be absorbent by monochromatic paths in $\mathcal{D}$. We consider the two possible cases:

CASE I. Suppose that $u \in S$. In this case we have $S' = (S \setminus \{u\}) \cup \{u_1, \ldots, u_n\}$. Let $z \in V(D^-) \setminus S'$. Clearly $z \in V(D) \setminus S$ and since S is absorbent by monochromatic paths in $\mathcal{D}$, there exists a monochromatic zS-path in $\mathcal{D}$. By lemma 2.4, 1 or 3, there exists a monochromatic zS'-path in $\mathcal{D}^-$.

CASE II. Suppose that $u \notin S$. In this case we have that $S' = S$ and $S \subset V(D^-)$. Let $z \in V(D^-) \setminus S'$. We will prove that there exists a monochromatic zS'-path in $\mathcal{D}^-$.

- Suppose that $z \notin \{u_1, \ldots, u_n\}$. Then $z \in V(D) \setminus \{u\}$. Since S is absorbent by monochromatic paths in $\mathcal{D}$, there exists a monochromatic zS-path in $\mathcal{D}$. By 1 in lemma 2.4, there exists a monochromatic zS'-path in $\mathcal{D}^-$.

- Now suppose that $z = u_i$ for some $i \in \{1, \ldots, n\}$. Since $u \notin S$ and S is absorbent by monochromatic paths in $\mathcal{D}$, there exists a monochromatic uS-path in $\mathcal{D}$. By 2 in lemma 2.4, there exists a monochromatic $u_i S'$-path in $\mathcal{D}^-$.

□

Theorem 3.3. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in definition 2.1. If $S' \subset V(D^-)$ is absorbent by monochromatic paths in $\mathcal{D}^-$, then*

$$S = \begin{cases} S' \text{ if } S' \cap \{u_1, \ldots, u_n\} = \varnothing \\ \\ (S' \setminus \{u_1, \ldots, u_n\}) \cup \{u\} \text{ otherwise} \end{cases} \tag{3.4}$$

is absorbent by monochromatic paths in $\mathcal{D}$.

Proof. Let $S' \subset V(D^-)$ be absorbent by monochromatic paths in $\mathcal{D}^-$. We consider the two possible cases:

CASE I. Suppose that $S' \cap \{u_1, \ldots, u_n\} \neq \varnothing$. We will show that $S = (S' \setminus \{u_1, \ldots, u_n\}) \cup \{u\}$ is absorbent by monochromatic paths in $\mathcal{D}$. Let $z \in V(D) \setminus S$. By definition, $z \neq u$ and therefore $z \in V(D^-) \setminus \{u_1, \ldots, u_n\}$. Since S' is absorbent by monochromatic paths in $\mathcal{D}$ and $z \notin S'$, there exists a monochromatic zS'-path in $\mathcal{D}^-$. Let $w \in S'$ be the terminal vertex of such a path. If $w \notin \{u_1, \ldots, u_n\}$, then $w \in S$ and 1 in lemma 2.5 implies the existence of a monochromatic zw-path. If $w \in \{u_1, \ldots, u_n\}$, then 3 in lemma 2.5 implies the existence of a monochromatic zu-path (recall that $u \in S$).

Case II. Suppose that $S' \cap \{u_1, \ldots, u_n\} = \varnothing$. We will show that $S = S'$ is absorbent by monochromatic paths in $\mathcal{D}$. Let $z \in V(D) \setminus S$. We check the two possible cases:

- Suppose that $z \neq u$. It follows that $z \in V(D^-) \setminus \{u_1, \ldots, u_n\} = V(D) \setminus \{u\}$. Since $z \notin S$, $z \notin S'$. Then there exists a monochromatic zS'-path in $\mathcal{D}^-$. By 1 in lemma 2.5, there exists a monochromatic zS-path in $\mathcal{D}$.

- Suppose that $z = u$. For all $i = 1, \ldots, n$, $u_i \notin S'$ and since S' is absorbent by monochromatic paths in $\mathcal{D}^-$, there exists a monochromatic u_iS'-paths in $\mathcal{D}^-$ for each $i = 1, \ldots, n$. By 2 in lemma 2.5, there exists a monochromatic uS-path in $\mathcal{D}$.

□

4 Independence

Let $\mathcal{D} = (D, L)$ be an m-colored digraph. A subset of vertices $S \subset V(D)$ is *independent by monochromatic paths* if for every $u, v \in S$ there exists no monochromatic uv-path.

Theorem 4.1. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in definition 2.1. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. If $S \subset V(D)$ is independent by monochromatic paths in $\mathcal{D}$, then $S' \subset V(D^-)$ is independent by monochromatic paths in $\mathcal{D}^-$, where S' is as in 3.2.*

Proof. We consider the two possible cases:

Case I. Suppose that $u \in S$. We will show that $S' = (S \setminus \{u\}) \cup \{u_1, \ldots, u_n\}$ is independent by monochromatic paths in $\mathcal{D}^-$. Let $x, y \in S'$ with $x \neq y$. We need to show that there exists no monochromatic xy-path in $\mathcal{D}^-$. Suppose otherwise, that is, suppose that there exist monochromatic xy-paths in $\mathcal{D}^-$. We check the four possible cases:

- If $x, y \notin \{u_1, \ldots, u_n\}$, then 1 in lemma 2.5 implies the existence monochromatic xy-path in $\mathcal{D}$ contradicting that S is independent by monochromatic paths in $\mathcal{D}$.

- If $x \in \{u_1, \ldots, u_n\}$ and $y \notin \{u_1, \ldots, u_n\}$, then 2 in lemma 2.5 implies the existence a monochromatic uy-path in $\mathcal{D}$ contradicting that S is independent by monochromatic paths in $\mathcal{D}$.

- If $x \notin \{u_1, \dots, u_n\}$ and $y \in \{u_1, \dots, u_n\}$, then 3 in lemma 2.5 implies the existence a monochomatic xu-path in $\mathcal{D}$ contradicting that S is independent by monochromatic paths in $\mathcal{D}$.

- If $x, y \in \{u_1, \dots, u_n\}$, then 4 in lemma 2.5 implies the existence a monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$, against the hypothesis.

Case II. Suppose that $u \notin S$ and $S' = S$. Suppose that there exists a monochromatic xy-path in $\mathcal{D}^-$ for some $x, y \in S'$. Then 1 in lemma 2.5 implies that there exists a monochromatic xy-path in $\mathcal{D}$ contradicting that S is independent by monochromatic paths in $\mathcal{D}$. □

Theorem 4.2. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in definition 2.1. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Let $S' \subset V(D^-)$ be such that for every $z \in V(D^-) \setminus S'$ for which there exists a monochromatic $S'z$-path in $\mathcal{D}^-$, there exists a monochromatic zS'-path in $\mathcal{D}^-$. If S' is independent by monochromatic paths in $\mathcal{D}^-$, then $S \subset V(D)$ is independent by monochromatic paths in $\mathcal{D}$, where S is as in 3.4.*

Proof. We consider the two possible cases:

Case I. Suppose that $S' \cap \{u_1, \dots, u_n\} \neq \varnothing$. We will show that $S = (S' \setminus \{u_1, \dots, u_n\}) \cup \{u\}$ is independent by monochromatic paths in $\mathcal{D}$. Let $x, y \in S$ with $x \neq y$. Suppose that there exists a monochromatic xy-path in $\mathcal{D}$.

- If $x, y \neq u$, then 1 in lemma 2.4 implies the existence of a monochromatic xy-path in $\mathcal{D}^-$ contradicting that S' is independent by monochromatic paths.

- If $x = u$, then 2 in lemma 2.4 implies the existence of a monochromatic $u_i y$-paths in $\mathcal{D}^-$ for all $i = 1, \dots, n$ contradicting that S' is independent by monochromatic paths.

- If $y = u$, then 3 in lemma 2.4 implies the existence of a monochromatic xu_q-path in $\mathcal{D}^-$ for some $q \in \{1, \dots, n\}$. If $u_q \in S'$, then S' would not be independent by monochromatic paths in $\mathcal{D}^-$. Hence $u_q \notin S'$ and the additional hypothesis imply the existence of a $u_q S'$-path in $\mathcal{D}$. Let $w \in S'$ be the end of such a path. If $w \notin \{u_1, \dots, u_n\}$, then 2 in lemma 2.5 implies the existence of a monochromatic uw-path in $\mathcal{D}$ and hence 2 in lemma 2.4 implies the existence of monochromatic $u_i w$-paths in $\mathcal{D}^-$ for all $i = 1, \dots, n$, contradicting that S' is independent by monochromatic paths in $\mathcal{D}^-$. If $w \in \{u_1, \dots, u_n\}$, then 4 in lemma 2.5

implies the existence of a monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$, against the hypothesis.

CASE II. Suppose that $S' \cap \{u_1, \ldots, u_n\} = \varnothing$. We will show that $S = S'$ is independent by monochromatic paths in $\mathcal{D}$. Suppose that there exists a monochromatic xy-path in $\mathcal{D}$ for some $x, y \in S$. Then 1 in lemma 2.4 implies the existence of a monochromatic xy-path in $\mathcal{D}^-$ contradicting that S' is independent by monochromatic paths in $\mathcal{D}^-$. □

Let $\mathcal{D} = (D, L)$ be an m-colored digraph. A *kernel by monochromatic paths* is a set of vertices which is both independent and absorbent by monochromatic paths. We let $\mathcal{K}(\mathcal{D})$ denote the set of kernels by monochromatic paths in $\mathcal{D}$. Combining the results on absorbance and independence we get the following result.

The following corollary generalizes [5].

Corollary 4.3. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1 and suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Then $|\mathcal{K}(\mathcal{D})| = |\mathcal{K}(\mathcal{D}^-)|$.*

Proof. It suffices to show that if $S' \subset V(D^-)$ is a kernel by monochromatic paths in $\mathcal{D}^-$, then $S' \cap \{u_1, \ldots, u_n\} = \varnothing$ or $\{u_1, \ldots, u_n\} \subset S'$. Suppose that $u_i \in S'$ and $u_j \notin S'$ for some $i, j \in \{1, \ldots, n\}$. Since S' is a kernel in $\mathcal{D}^-$, there exists a monochromatic $u_j S'$ path in $\mathcal{D}^-$. Let $w \in S'$ be the end of such a path. If $w \neq u_i$, then there exists a monochromatic $u_i w$-path in $\mathcal{D}$, contradicting that S' is independent by monochromatic paths. But $w = u_i$ is impossible for there would exists a monochromatic cycle C with $u \in V(C)$ (observe that we do not require the additional hypothesis of theorem 4.2 for S to be independent). □

5 Semi-kernels

Definition 5.1. Let $\mathcal{D} = (D, L)$ be an m-colored digraph. A subset of vertices $S \subset V(D)$ is a *semikernel by monochromatic paths* if it is independent by monochromatic paths and for every $z \in V(D) \setminus S$ for which there exists a monochromatic Sz-path, there exists monochromatic zS-path.

Theorem 5.2. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Let S be a semikernel by monochromatic paths in $\mathcal{D}$. Then S' is a semikernel by monochromatic paths in $\mathcal{D}^-$, where S' is as in 3.2.*

Proof. Let $S \subset V(D)$ be a semikernel by monochromatic paths in $\mathcal{D}$ and let S' be as in 3.2. By theorem 4.1, S' is independent by monochromatic paths in $\mathcal{D}^-$. We consider the two possible cases:

Case I. Suppose that $u \in S$. We will show that $S' = (S \setminus \{u\}) \cup \{u_1, \ldots, u_n\}$ is a semikernel by monochromatic paths in $\mathcal{D}^-$. Let $z \in V(D^-) \setminus S'$ and suppose that there exists a monochromatic $S'z$-path in $\mathcal{D}^-$. By 1 or 2 in lemma 2.5, there exists a monochromatic Sz-path and hence there exists a monochromatic zS-path in $\mathcal{D}$. Then, by 1 or 3 in lemma 2.4, there exists a zS'-path in $\mathcal{D}^-$.

Case II. Suppose that $u \notin S$. We will show that $S' = S$ and $S \subset V(D^-)$ is a semikernel by monochromatic paths in $\mathcal{D}^-$. Let $z \in V(D^-) \setminus S'$ and suppose that there exists a monochromatic $S'z$-path in $\mathcal{D}^-$. Let $T = (x_0, x_1, \ldots, x_k = z)$ be such a path, with $x_0 \in S' = S$.

- If $x_k \notin \{u_1, \ldots, u_n\}$, then, by 1 in lemma 2.5, there exists a monochromatic x_0z-path in $\mathcal{D}$ implying the existence of a monochromatic zS-path in $\mathcal{D}$ and in virtue of 1 in lemma 2.4, there exists a monochromatic zS'-path in $\mathcal{D}^-$.

- If $x_k \in \{u_1, \ldots, u_n\}$, then, by 3 in lemma 2.5, there exists a monochromatic x_0u-path in $\mathcal{D}$ and since $u \notin S$, there exists a monochromatic uS-path in $\mathcal{D}$ and in virtue of 2 in lemma 2.4 there exists a monochromatic u_iS'-path in $\mathcal{D}^-$ for all $i = 1, \ldots, n$.

□

Theorem 5.3. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Let S' be a semikernel by monochromatic paths in $\mathcal{D}^-$. Then S is a semikernel by monochromatic paths in $\mathcal{D}$, where S is as in 3.4.*

Proof. Let $S' \subset V(D^-)$ be a semikernel by monochromatic paths in $\mathcal{D}^-$ and let S be defined as in 3.4. By theorem 4.2, S is independent by monochromatic paths in $\mathcal{D}$ (the additional hypothesis of theorem 4.2 is clearly satisfied by S' because it is a semikernel in $\mathcal{D}^-$). We consider the two possible cases:

Case I. Suppose that $S' \cap \{u_1, \ldots, u_n\} \neq \varnothing$. We will show that $S = (S' \setminus \{u_1, \ldots, u_n\}) \cup \{u\}$ is a semikernel by monochromatic paths in $\mathcal{D}$. Let $z \in V(D) \setminus S$ and suppose that there exists a monochromatic Sz-path in $\mathcal{D}$. Let $(x_0, \ldots, x_k = z)$ be such a path. If $x_0 = u$, then 2 in lemma 2.4 implies the existence of u_iz-paths in $\mathcal{D}^-$ for all $i = 1, \ldots, n$, otherwise 1 in lemma 2.4 implies the existence of a monochromatic x_0z-path in $\mathcal{D}^-$. It follows that there exists a monochromatic zS'-path in $\mathcal{D}^-$. Let $(y_0 = z, \ldots, y_r)$ be such a path with $y_r \in S'$. If $y_r \notin \{u_1, \ldots, u_n\}$, then $y_r \in S$ and 1 in lemma 2.5 implies the existence of a monochromatic zy_r-path in $\mathcal{D}$, otherwise 3 in lemma 2.5 implies the existence of a monochromatic zu-path in $\mathcal{D}$ (recall that $u \in S$).

Case II. Suppose that $S' \cap \{u_1, \ldots, u_n\} = \varnothing$. We will show that $S = S'$ is a semikernel by monochromatic paths in $\mathcal{D}$. Let $z \in V(D) \setminus S$ and suppose that there exists a monochromatic Sz-path in $\mathcal{D}$. If $z \neq u$, then 1 in lemma 2.4 implies the existence of a monochromatic $S'z$-path in $\mathcal{D}^-$, otherwise 3 in lemma 2.4 implies the existence of a monochromatic $S'u_q$-path in $\mathcal{D}^-$ for some $q \in \{1, \ldots, n\}$ (recall that $u_q \notin S'$). Hence there exists a monochromatic zS'-path in $\mathcal{D}^-$ or a monochromatic u_qS'-path in $\mathcal{D}^-$ respectively. Hence by 1 or 2 in lemma 2.5, there exists a monochromatic zS-path in $\mathcal{D}$. □

For an m-colored digraph D let $\mathcal{S}(D)$ denote the set of semikernels by moncochromatic paths in $\mathcal{D}$.

Corollary 5.4. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Then $|\mathcal{S}(D)| \leq |\mathcal{S}(D^-)|$.*

6 Quasi-kernels

Definition 6.1. Let $\mathcal{D} = (D, L)$ be an m-colored digraph. A subset of vertices $S \subset V(D)$ is a *quasi-kernel by monochromatic paths* if it is independent by monochromatic paths and for every $z \in V(D) \setminus S$, there exists a monochromatic zS-path or there exists $w \in V(D) \setminus S$, a monochromatic zw-path and a monochromatic wS-path.

Theorem 6.2. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Let S be a quasikernel by monochromatic paths in $\mathcal{D}$. Then S' is a quasikernel by monochromatic paths in $\mathcal{D}^-$, where S' is as in 3.2.*

Proof. Let $S \subset V(D)$ be a quasikernel by monochromatic paths in $\mathcal{D}$ and let S' be as in 3.2. By theorem 4.1, S' is independent by monochromatic paths in $\mathcal{D}^-$. We consider the two possible cases:

Case I. Suppose that $u \in S$. In this case we have $S' = (S \setminus \{u\}) \cup \{u_1, \ldots, u_n\}$. Let $z \in V(D^-) \setminus S'$. Clearly $z \in V(D) \setminus S$. If there exists a monochromatic zS-path in $\mathcal{D}$, then we argue exactly in the same way as in the proof of theorem 3.1, case I. Therefore we suppose that no such path exists and hence we assume that there exists $w \in V(D) \setminus S$, a monochromatic

uw-path in $\mathcal{D}$ and a monochromatic wS-path in $\mathcal{D}$. By 1 in lemma 2.4, there exists a monochromatic uw-path in $\mathcal{D}^-$, and by 1 or 3 in lemma 2.4, there exists a monochromatic wS'-path in $\mathcal{D}^-$.

Case II. Suppose that $u \notin S$. In this case we have that $S' = S$ and $S \subset V(D^-)$. Let $z \in V(D^-) \setminus S'$. We will prove that either there exists a monochromatic zS'-path in $\mathcal{D}^-$ or there exist $w \in V(D^-) \setminus S$, a monochromatic zw-path in $\mathcal{D}^-$ and a monochromatic wS'-path in $\mathcal{D}^-$.

- Suppose that $z \notin \{u_1, \ldots, u_n\}$. Then $z \in V(D)$. If there exists a monochromatic zS-path in $\mathcal{D}$, then we argue exactly in the same way as in the proof of theorem 3.1, case II, first item. Therefore we suppose that no such path exists and hence we assume that there exists $w \in V(D) \setminus S$, a monochromatic zw-path in $\mathcal{D}$ and a monochromatic wS-path in $\mathcal{D}$. If $w \neq u$, then by 1 in lemma 2.4 applied twice we have that there exists a monochromatic zw-path in $\mathcal{D}^-$ and a monochromatic wS'-path in $\mathcal{D}^-$. If $w = u$, then by 3 in lemma 2.4, there exists $q \in \{1, \ldots, n\}$ and a monochromatic zu_q-path in $\mathcal{D}^-$, and by 2 in lemma 2.4, there exists a monochromatic u_iS'-path in $\mathcal{D}^-$ for all $i = 1, \ldots, n$ (in particular for $i = q$).

- Now suppose that $z = u_i$ for some $i \in \{1, \ldots, n\}$. We have that $u \notin S$. If there exists a monochromatic uS-path in $\mathcal{D}$ then we argue exactly in the same way as in the proof of theorem 3.1, case II, second item. Therefore we suppose that no such path exists and hence we assume that there exists $w \in V(D) \setminus S$, a monochromatic uw-path in $\mathcal{D}$ and a monochromatic wS-path in $\mathcal{D}$ (necessarily $w \neq u$). By 2 in lemma 2.4, there exists monochromatic u_iw-paths in $\mathcal{D}^-$ for all $i = 1, \ldots, n$. By 1 in lemma 2.4, there exists a monochromatic wS'-path in $\mathcal{D}^-$.

□

Theorem 6.3. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Let S' be a quasikernel by monochromatic paths in $\mathcal{D}^-$. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Let $S' \subset V(D^-)$ be such that for every $z \in V(D^-) \setminus S'$ for which there exists a monochromatic $S'z$-path in $\mathcal{D}^-$, there exists a monochromatic zS'-path in $\mathcal{D}^-$. If S' is a quasikernel by monochromatic paths in $\mathcal{D}^-$, then S is a quasikernel by monochromatic paths in $\mathcal{D}$, where S is as in 3.4.*

Proof. Let $S' \subset V(D^-)$ be a quasikernel by monochromatic paths in $\mathcal{D}^-$ and let S be as in 3.4. By theorem 4.2, S is independent by monochromatic paths in $\mathcal{D}$. We consider the two possible cases:

Case I. Suppose that $S' \cap \{u_1, \ldots, u_n\} \neq \varnothing$. We will show that $S = (S' \setminus \{u_1, \ldots, u_n\}) \cup \{u\}$ is a quasikernel by monochromatic paths in $\mathcal{D}$. Let $z \in V(D) \setminus S$. Since $z \neq u$, $z \in V(D^-) \setminus S'$ and hence there exists a monochromatic zS'-path in $\mathcal{D}^-$ or there exists $w \in V(D^-) \setminus S'$ such that there exists a monochromatic zw-path in $\mathcal{D}^-$ and a monochromatic wS'-path in $\mathcal{D}^-$. By lemma 2.5, there exists a monochromatic zS-path in $\mathcal{D}$ or there exists a monochromatic uw-path in $\mathcal{D}$ and a monochromatic wS-path in $\mathcal{D}^-$.

Case II. Suppose that $S' \cap \{u_1, \ldots, u_n\} = \varnothing$. We will show that $S = S'$ is a quasikernel by monochromatic paths in $\mathcal{D}$. Let $z \in V(D) \setminus S$.

- Suppose that $z = u$. For each $i = 1, \ldots, n$, $u_i \notin S'$ and since S' is a quasikernel by monochromatic paths in $\mathcal{D}^-$, there exists a monochromatic u_iS'-path in $\mathcal{D}^-$ or there exists $w \in V(D^-) \setminus S'$ such that there exists a monochromatic u_iw-path in $\mathcal{D}^-$ and a monochromatic wS'-path in $\mathcal{D}^-$. By lemma 2.5, there exists a monochromatic uS-path in $\mathcal{D}$ or there exists a monochromatic uw-path in $\mathcal{D}$ and a monochromatic wS-path in $\mathcal{D}^-$.

- Suppose that $z \neq u$. Then $z \in V(D^-) \setminus S'$. Since S' is a quasikernel by monochromatic paths in $\mathcal{D}^-$, there exists a monochromatic zS'-path in $\mathcal{D}^-$ or there exists $w \in V(D^-) \setminus S'$ such that there exists a monochromatic zw-path in $\mathcal{D}^-$ and a monochromatic wS'-path in D^-. By lemma 2.5, there exists a monochromatic zS-path in $\mathcal{D}$ or there exists a monochromatic uw-path in $\mathcal{D}$ and a monochromatic wS-path in $\mathcal{D}^-$.

□

For an m-colored digraph D let $\mathcal{Q}(D)$ denote the set of quasikernels by moncochromatic paths in $\mathcal{D}$.

Corollary 6.4. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Suppose that there exists no monochromatic cycle C in $\mathcal{D}$ with $u \in V(C)$. Then $|\mathcal{Q}(D)| \leq |\mathcal{Q}(D^-)|$.*

7 Grundy functions

In this section, given $v \in V(D)$, we let

$$M_D^+(v) = \{x \in V(D) \mid \text{there exists a monochromatic } vx\text{-path}\}$$

$$M_D^-(v) = \{y \in V(D) \mid \text{there exists a monochromatic } yv\text{-path}\}$$

Definition 7.1. Let $\mathcal{D} = (D, L)$ be an m-colored digraph. A function $g \colon V(D) \to \mathbb{N}$ is a *Grundy function by monochromatic paths of* $\mathcal{D}$ if for every $x \in V(D)$,

$$g(x) = \min\left\{\mathbb{N} \setminus \{g(z) \in V(D) \mid z \in M_D^+(x)\}\right\}.$$

Let $\mathcal{G}(\mathcal{D})$ be the set of Grundy functions by monochromatic paths.

Theorem 7.2. *Let $\mathcal{D}$ and $\mathcal{D}^-$ be as in 2.1. Suppose that there exist no monochromatic cycles in $\mathcal{D}$. Then $|\mathcal{G}(\mathcal{D})| = |\mathcal{G}(\mathcal{D}^-)|$.*

Proof. Let $g \in \mathcal{G}_(\mathcal{D})$ and define $g_0 \colon V(D^-) \to \mathbb{N}$ by

$$g_0(x) = \begin{cases} g(x) & \text{if } x \notin \{u_1, \dots, u_n\} \\ \\ g(u) & \text{otherwise.} \end{cases}$$

Let $x \in V(D_0)$. If $x \in \{u_1, \dots, u_n\}$, then $M_{D^-}^+(x) = M_D^+(u)$ because there exist no monochromatic cycles in $\mathcal{D}$. Then $g_0(u_i) = \min\{\mathbb{N} \setminus \{g_0(z) \mid z \in M_D^+(u)\}\}$. Suppose that $x \notin \{u_1, \dots, u_n\}$. If $\{u_1, \dots, u_n\} \cap M_{D^-}^+(x) = \varnothing$, then $M_{D^-}^+(x) = M_D^+(x)$ and hence $g_0(x) = \min\{\mathbb{N} - \{g_0(z) \mid z \in M_{D^-}^+(x)\}$. Suppose that $\{u_1, \dots, u_n\} \cap M_{D^-}^+(x) \neq \varnothing$. Then $u \in M_D^+(x)$ and $M_D^+(x) \setminus \{u\}$ differs from $M_{D^-}^+(x)$ in at most $\{u_1, \dots, u_n\} \setminus M_{D^-}^+(x)$, hence $g_0(x) = \min\{\mathbb{N} \setminus \{g_0(z) \mid M_{D^-}^+(x)\}\}$. Thus $g_0 \in \mathcal{G}(\mathcal{D}^-)$. Clearly, the map $g \mapsto g_0$ is injective, so $|\mathcal{G}(\mathcal{D})| \leq |\mathcal{G}(\mathcal{D}^-)|$.

Now let $g_0 \in \mathcal{G}(D^-)$. Having no monochromatic cycles implies that $M_{D^-}^+(u_i) = M_{D^-}^+(u_j)$ for all $i, j \in \{1, \dots, n\}$, therefore $g_0(u_i) = g_0(u_j)$. Define $g \colon V(D) \to \mathbb{N}$ by

$$g(x) = \begin{cases} g_0(x) & \text{if } x \neq u \\ \\ g_0(u_i) & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $g \in \mathcal{G}(\mathcal{D}^-)$ and that the map $g_0 \mapsto g$ is actually the inverse of the map defined above. Therefore the result follows. □

8 Duality

Out-state duals in-state splittings for partitions of the out-going edges are considered. Obtaining lemmas similar to lemmas 2.4 and 2.5 is straightforward. With them we can rephrase all our results for out-splittings, we just need to state the corresponding duals of each concept. For example, let $\mathcal{D} = (D, L)$ be an m-colored digraph. A subset of vertices $S \subset V(D)$ is *dominant by monochromatic paths* if for every $z \in V(D) \setminus S$, there exists a monochromatic Sz-path in $\mathcal{D}$. So dominance by monochromatic paths *duals* absorbance by monochromatic paths. A *solution by monochromatic paths* is a subset of vertices which is both independent and dominant by monochromatic paths and this definition duals kernels by monochromatic paths. A *semisolution* and a *quasisolution by monochromatic paths* are defined similarly as well as *dual Grundy functions by monochromatic paths.* Stating the dual results and verifying their proofs is an exercise left to the reader.

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