(k, l)-kernels, (k, l)-semikernels, k-Grundy functions and duality for state splittings

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Abstract

Line digraphs can be obtained by sequences of *state splittings*, a particular kind of operation widely used in symbolic dynamics [12]. Properties of line digraphs inherited from the source have been studied, for instance in [7] Harminc showed that the cardinalities of the sets of kernels and solutions (kernel's dual definition) of a digraph and its line digraph coincide. We extend this for (k, l)-kernels in the context of state splittings and also look at (k, l)-semikernels, k-Grundy functions and their duals.

Keywords: state splitting; line digraph; kernel; Grundy function; duality

Mathematics Subject Classification: 05C20

1 Introduction

State splitting is a fundamental operation in symbolic dynamics (see [12] or [8]). A *shift of finite type* is a dynamical system (homeomorphic to a Cantor set) consisting of all possible doubly infinite paths in a digraph and correspond to doubly infinite sequences of symbols (the vertices). Performing state splittings induce conjugacies of shift spaces, and every conjugacy can be decomposed into a sequence of conjugacies induced by state splittings (a result called *decomposition theorem*). Particular kind of sequences of state splittings result in *higher block presentations* which consist of making the paths of certain length the symbols of the new shift space, and when this length equals one, the resulting digraph is isomorphic to the *line digraph*. Whence line digraphs can be obtained by sequences of state splittings (proposition 2.3). This is a well known fact in symbolic dynamics.

The line digraph is an object which has been widely studied. In [7] Harminc showed that the cardinalities of the sets of *kernels* in a digraph and in its line digraph coincide, and that the same holds for *solutions* (kernel's dual definition). In section §3 we show that this is also true for state splittings. Moreover, we actually present results for (k, l)-kernels, a generalization of the concept of kernels introduced by M. Kwaśnik in [9]. The existence of (k, l)-kernels in digraphs has been studied by several authors, for example, see [3], [10], in particular, in [4]

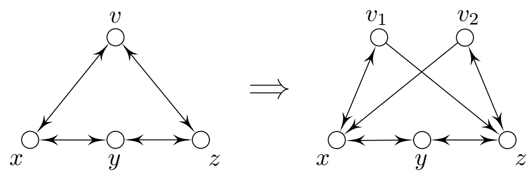

Figure 1: Elementary in-splitting.

the authors obtain results concerning the line digraph. In section §4 we look at (k,l)-semikernels in the context of state splittings. Kucharska and Kwaśnik introduced the concept of (k,l)-semikernel in [11], a generalization of the concept of semikernel introduced by Neumann-Lara in [13]. In [5] the authors showed that the number of smikernels of a digraph is less than or equal to the number of semikernels of its line digraph. They also look at *Grundy functions* of a digraph and of its line digraph, showing that their cardinalities must coincide. In section §5 we present results for k-Grundy functions in this context of state splittings. We consider all dual definitions and present results accordingly.

2 Sate splittings and line digraphs

In this section we define state splittings (use the terms "vertex" and "state" interchangeably). For a complete treatment of this operation see [12] or [8] (also, see [2] for generalizations of state splittings on digraphs presented by polynomial matrices, or more generally, see [6], where matrices over formal power series are considered). All digraphs are *simple*, which means that there are no loops nor multiple arcs. For general concepts see [1].

Let D be a digraph with vertex set $V(D)$ and arc set $A(D) \subset V(D) \times V(D)$ with no loops (so $(v,v) \notin A(D)$ for all $v \in V(D)$). For every $v \in V(D)$, let

$$\Gamma^-(v) = \Gamma^-_D(v) = \{x \in V(D) \mid \ (x,v) \in A(D)\}$$

and

$$\Gamma^+(v) = \Gamma^+_D(v) = \{y \in V(D) \mid \ (v,y) \in A(D)\}.$$

Definition 2.1. Let D be a digraph. For $v \in V(D)$, let $\Gamma^-(v) = X_1 \cup X_2$ be a partition of $\Gamma^-(v)$ inducing a partition of the incoming arcs to v. Let D_0 be the digraph that results from D by an *elementary in-splitting* of vertex v into two new vertices $v_1, v_2 \in V(D_0)$ in such a way that for $i = 1, 2$, we have $(x, v_i) \in A(D_0)$ whenever $x \in X_i$ and $(v_i, y) \in A(D_0)$ whenever $(v,y) \in A(D)$. Define *elementary out-splitting* similarly.

An example of an elementary in-splitting is depicted in figure 1. Sometimes we will have to consider the possibility $X_i = \varnothing$ for some $i = 1, 2$, (e.g. if $|\Gamma^-(v)| = 1$), in which case we let $D_0 = D$. If we say that an in-splitting of a vertex v is performed, it will be implicit, if not stated, the assumption $\Gamma^-(v) \neq \varnothing$, and similarly for out-splittings.

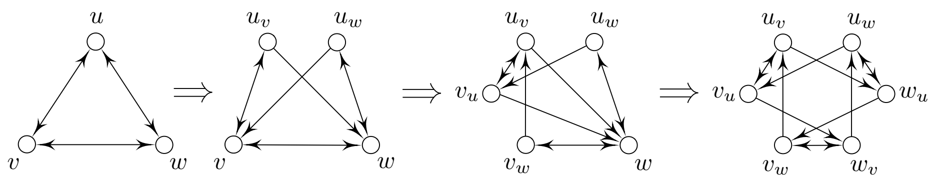

Figure 2: Elementary in-splittings that result in the line digraph.

Definitions of in-splittings and out-splittings (not necessarily elementary) are carried out similarly except for the partition which is allowed to have more than two elements (clearly, splittings can be obtained by sequences of elementary splittings). In particular, we let the *complete in-splitting* (resp. *complete out-splitting*) of a vertex be the digraph that results from in-splitting (resp. out-splitting) a vertex according to the partition with all its elements being singletons.

Definition 2.2. Let D be a digraph. The *line digraph* $L(D)$ has vertex set $V(L(D)) = A(D)$ and arc set $A(L(D))$ determined by the rule $((u,v),(v,w)) \in A(L(D))$ whenever $u,v,w \in V(D)$ are such that $(u,v),(v,w) \in A(D)$.

A well known result in symbolic dynamics is the following (see [12] or [8]).

Proposition 2.3. *If D is a digraph with $\Gamma^-(v) \neq \varnothing$ (resp. $\Gamma^+(v) \neq \varnothing$) for all $v \in V(D)$, then $L(D)$ can be obtained by sequences of in-splittings (resp. out-splittings).*

Figure 2 describes the procedure of the proof of proposition 2.3.

Proof. Let $V(D) = \{u, v, \ldots, w\}$. Start with a complete in-splitting of vertex $u \in V(D)$. Then u in-splits into $|\Gamma^-(u)|$ new vertices, one for each $x \in \Gamma^-(u)$, so we label the new vertices by u_x. Next, choose a second vertex $v \neq u_x$ for all $x \in \Gamma^-(u)$. Its in-coming arcs are of the form (y, v) with $y \in \Gamma^-(v) \setminus \{u\}$ or of the form (u_x, v) if $u \in \Gamma^-(v)$ with $x \in \Gamma^-(u)$. In-split vertex v into $|\Gamma^-(v)|$ new vertices according to the partition formed by the singletons $\{y\}$ with $y \in \Gamma^-(v) \setminus \{u\}$ and by $\{u_x \mid x \in \Gamma^-(u)\}$. Label the vertices corresponding to the singletons $\{y\}$ with $y \in \Gamma^-(v)\setminus\{u\}$ by v_y, and label the last vertex corresponding to the partition element $\{u_x \mid x \in \Gamma^-(u)\}$ by v_u. Continue in-splitting vertices in this way until a last vertex w in-splits according to the partition with $|\Gamma^-(w)|$ elements, one for each $x \in \Gamma^-(w)$ and defined by $\{x_y \mid y \in \Gamma^-(x)\}$. Each of the new vertices is determined by a unique $x \in \Gamma^-_D(w)$ and therefore we label them by w_x.

For every arc $(x, y) \in A(D)$ there is a unique vertex y_x, and the map $(x, y) \mapsto y_x$ defines a bijection between $A(D)$ and the vertices of the resulting digraph. If $(x, y), (y, z) \in A(D)$, then (y_x, z_y) is an arc in the resulting digraph because the labeling has the property that the incoming arcs to vertex z_y come from vertices y_x for all $x \in \Gamma^-(y)$, and since for (x_y, z_w) to be an arc in the resulting digraph it is actually necessary that $w = x$, the map is an isomorphism between $L(D)$ and the resulting digraph.

A sequence of out-splittings that results in a digraph isomorphic to $L(D)$ is found similarly.

$\square$

Definition 2.4. Let D be a digraph. A set of vertices $I \subset V(D)$ is *independent* if there exists no $x, y \in I$ with $(x, y) \in A(D)$.

Recall that a (*directed*) *path* in a digraph D is a sequence of distinct vertices $(x_0, \ldots, x_n)$ such that $(x_{i-1}, x_i) \in A(D)$ for every $i = 1, \ldots, n$, and its length is n. Given $x, y \in A(D)$, a *shortest path* from x to y is a path of minimal length.

Definition 2.5. Let D be a digraph. By the *directed distance* $d_D(x, y)$ from vertex $x \in V(D)$ to vertex $y \in V(D)$ we mean the length of a shortest directed path in D from x to y (so $d_D(x, y) = 0$ if and only if $x = y$).

The following definition generalizes independence.

Definition 2.6. Let D be a digraph and $k \geq 2$. A set of vertices $I \subset V(D)$ is k*-independent* if there exists no $x, y \in I$ with $d_D(x, y) < k$.

Lemma 2.7. *Let D be a digraph and $k \geq 2$. Suppose that there exist no cycles in D of length less than k. Let D_0 be the digraph that results from the elementary in-splitting (resp. out-splitting) of vertex $v \in V(D)$ according to a partition $\Gamma^-(v) = X_1 \cup X_2$ (resp. $\Gamma^+(v) = X_1 \cup X_2$).*

1. *If $A \subset V(D)$ is k-independent and $v \in A$, then $A_0 = (A \setminus \{v\}) \cup \{v_1, v_2\}$ is k-independent in D_0.*
2. *If $A \subset V(D)$ is k-independent and $v \notin A$, then $A_0 = A$ is k-independent in D_0.*
3. *If $A_0 \subset V(D_0)$ is k-independent and $v_1, v_2 \in A_0$, then $A = (A_0 \setminus \{v_1, v_2\}) \cup \{v\}$ is k-independent in D.*
4. *If $A_0 \subset V(D_0)$ is k-independent and $v_1, v_2 \notin A_0$, then $A = A_0$ is k-independent in D.*

Proof. First we prove 1. The fact that for every $x, y \in A_0$ we have $d_{D_0}(x, y) \geq k$ is clear except for the case when $x = v_i$ and $y = v_j$ with $i \neq j$, which follows from the hypothesis of having no cycles of length less than k.

Next we prove 2. If there exist $x, y \in A_0$ such that $d_{D_0}(x, y) < k$, then there exists a path in D_0 from x to y of length less than k, and such a path must come from a path in D of length less than k, contradicting that A is k-independent.

To show 3, observe that for every $x, y \in A$, there exists no path in D_0 of length less than k from x to y that intersects A except at the extreme vertices. Thus a shortest path in D_0 from x to y must come from a shortest path in D from x to y if $x, y \notin \{v_1, v_2\}$, from v to y if $x \in \{v_1, v_2\}$ and $y \notin \{v_1, v_2\}$, from x to v if $x \notin \{v_1, v_2\}$ and $y \in \{v_1, v_2\}$, and finally from v to v if $x, y \in \{v_1, v_2\}$, whence $d_D(x, y) = d_{D_0}(x, y) \geq k$.

Finally we show 4. Let $x, y \in A$ and suppose that $d_D(x, y) < k$. Then there exists a path in D from x to y of length less than k, and such a path becomes a path in D_0 from x to y of length less than k, contradicting that A_0 is k-independent.

$\square$

3 (k,l)-kernels

Definition 3.1. Let D be a digraph. An independent set of vertices $K \subset V(D)$ is a *kernel* if for every $v \in V(D) \setminus K$, there exists $y \in K$ such that $(v,y) \in A(D)$. Let $\mathcal{K}(D)$ be the set of kernels of D.

In [7] Harminc showed that $|\mathcal{K}(D)| = |\mathcal{K}(L(D))|$.

The following definition generalizes kernels.

Definition 3.2. Let D be a digraph, $k \geq 2$ and $l \geq 1$. A subset of vertices $K \subset V(D)$ is a (k,l)-*kernel* if the following are satisfied:

1. K is k-independent.
2. K is l-*absorbent*, which means that for every $u \in V(D) \setminus K$, there exists $x \in K$ such that $d_D(u,x) \leq l$.

Let $\mathcal{K}_{(k,l)}(D)$ be the set of (k,l)-kernels of D.

Observe that a $(2,1)$-kernel of a digraph D is a kernel in the sense of definition 3.1, that is, $\mathcal{K}(D) = \mathcal{K}_{(2,1)}(D)$.

Theorem 3.3. *Let D be a digraph, $k \geq 2$ and $l \geq 1$. Suppose that there exist no cycles in D of length less than k. Let D_0 be the digraph that results from the elementary in-splitting of vertex $v \in V(D)$ according to a partition $\Gamma^-(v) = X_1 \cup X_2$. Then $|\mathcal{K}_{(k,l)}(D)| \leq |\mathcal{K}_{(k,l)}(D_0)|$ with equality holding if $l < k$.*

Proof. Let $K \in \mathcal{K}_{(k,l)}(D)$. Suppose that $v \in K$ and let $K_0 = (K \setminus \{v\}) \cup \{v_1, v_2\}$. By 1 of lemma 2.7, K_0 is k-independent. For every $u \in V(D_0) \setminus K_0$ there exists $x \in K$ such that $d_D(u,x) \leq l$ because $u \in V(D) \setminus K$. If $x \neq v$, then $x \in K_0$ and therefore $d_{D_0}(u,x) \leq l$. If $x = v$, then a path in D of length $d_D(u,v)$ becomes a path in D_0 of equal length ending at v_i for some $i = 1,2$, thus $d_{D_0}(u,v_i) \leq l$. Therefore $K_0 \in \mathcal{K}_{(k,l)}(D_0)$.

Now suppose that $v \notin K$ and let $K_0 = K$. By 2 of lemma 2.7, K_0 is k-independent. Let $u \in V(D_0) \setminus K_0$. If $u \notin \{v_1, v_2\}$, then $u \in V(D) \setminus K$ and there exists $x \in K$ such that $d_D(u,x) \leq l$, that is, there exists a path in D from u to x of length at most l. If such a path does not intersect $\{v\}$, then it remains unchanged in D_0, otherwise it becomes a path of equal length that intersects $\{v_i\}$ for some $i = 1,2$. Hence, in both cases, we have $d_{D_0}(u,x) \leq l$ with $x \in K_0$. Now suppose that $u \in \{v_1, v_2\}$. There exists $y \in K$ such that $d_D(v,y) \leq l$, that is, there exists a path in D from v to y of length at most l. Such a path becomes two paths in D_0 of equal lengths, one starting at v_1 and the other starting at v_2, whence $d_{D_0}(u,y) \leq l$. Therefore $K_0 \in \mathcal{K}_{(k,l)}(D_0)$. Clearly, the map $K \mapsto K_0$ is injective, so $|\mathcal{K}_{(k,l)}(D)| \leq |\mathcal{K}_{(k,l)}(D_0)|$.

Next, suppose that $l < k$ and let $K_0 \in \mathcal{K}_{(k,l)}(D_0)$. Suppose that $v_1, v_2 \in K_0$ and let $K = (K_0 \setminus \{v_1, v_2\}) \cup \{v\}$. By 3 of lemma 2.7, K is k-independent. Let $u \in V(D) \setminus K$. Since $u \in K_0$, there exists $x \in K_0$ such that $d_{D_0}(u, x) \leq l$, that is, there exists a path in D_0 from u to x of length at most l, and such a path must come from a path in D of equal length from u to $x \in K$ if $x \notin \{v_1, v_2\}$ or from u to v if $x \in \{v_1, v_2\}$, whence $d_D(u, x) \leq l$. Therefore $K \in \mathcal{K}_{(k,l)}(D)$.

Now suppose that $v_1, v_2 \notin K_0$ and let $K = K_0$. By 4 of lemma 2.7, K is k-independent. Let $u \in V(D) \setminus K$. If $u \neq v$, then $u \in V(D_0) \setminus K_0$ and hence there exists $x \in K_0$ such that $d_{D_0}(u, x) \leq l$, that is, there exists a path in D_0 from u to x of length at most l, and such a path must come from a path in D from u to x of equal length, thus $d_D(u, x) \leq l$. If $u = v$, then there exists $x \in K_0$ such that $d_{D_0}(v_1, x) \leq l$ and so $d_D(v, x) \leq l$. Therefore $K \in \mathcal{K}_{(k,l)}(D)$.

Finally, suppose that $v_i \in K_0$ and $v_j \notin K_0$ for $i \neq j$. There exists $x \in K_0$ such that $d_{D_0}(v_j, x) \leq l$, so $d_{D_0}(v_i, x) \leq l < k$, contradicting that K_0 is k-independent. We conclude that the injective map $K_0 \mapsto K$ is the inverse of the map described above and hence $|\mathcal{K}_{(k,l)}(D)| = |\mathcal{K}_{(k,l)}(D_0)|$.

□

Corollary 3.4. *Let D be a digraph. Let D_0 be the digraph that results from the elementary in-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^-(v) = X_1 \cup X_2$. Then $|\mathcal{K}(D)| = |\mathcal{K}(D_0)|$.*

Proof. By assumption, D has no cycles of length less than $k = 2$. Since $l = 1 < 2$ and $\mathcal{K}(D) = \mathcal{K}_{(2,1)}(D)$, the result follows from theorem 3.3.

□

The following definition duals kernels.

Definition 3.5. Let D be a digraph. An independent set of vertices $K^* \subset V(D)$ is a *solution* if for every $u \in V(D) \setminus K^*$, there exists $x \in K^*$ such that $(x, u) \in A(D)$. Let $\mathcal{K}^*(D)$ be the set of solutions of D.

In [7] Harminc also showed that $|\mathcal{K}^*(D)| = |\mathcal{K}^*(L(D))|$.

The following definition generalizes solutions.

Definition 3.6. Let D be a digraph, $k \geq 2$ and $l \geq 1$. A subset of vertices $K \subset V(D)$ is a (k, l)-*solution* if the following are satisfied:

1. K is k-independent.
2. K is l-*dominant*, which means that for every $u \in V(D) \setminus K$, there exists $x \in K$ such that $d_D(x, u) \leq l$.

Let $\mathcal{K}^*_{(k,l)}(D)$ be the set of (k, l)-solutions of D.

Again, observe that a $(2, 1)$-solution of a digraph D is a solution in the sense of definition 3.5, that is, $\mathcal{K}^*(D) = \mathcal{K}^*_{(2,1)}(D)$.

Theorem 3.7. *Let D be a digraph, $k \geq 2$ and $l \geq 1$. Suppose that there exist no cycles in D of length less than k. Let D_0 be the digraph that results from the elementary out-splitting of vertex $v \in V(D)$ according to a partition $\Gamma^+(v) = X_1 \cup X_2$. Then $|\mathcal{K}^*_{(k,l)}(D)| \leq |\mathcal{K}^*_{(k,l)}(D_0)|$ with equality holding if $l < k$.*

Proof. The proof is similar to the proof of theorem 3.3. □

Corollary 3.8. *Let D be a digraph. Let D_0 be the digraph that results from the elementary out-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^+(v) = X_1 \cup X_2$. Then $|\mathcal{K}^*(D)| = |\mathcal{K}^*(D_0)|$.*

Proof. The proof is similar to the proof of corollary 3.4. □

Using proposition 2.3 and corollaries 3.4 and 3.8 we obtain as corollaries Harminc's results mentioned above.

4 (k,l)-semikernels

Definition 4.1. Let D be a digraph. An independent set of vertices $S \subset V(D)$ is a *semikernel* if for every $u \in V(D) \backslash S$, there exists $x \in S$ such that $(u,x) \in A(D)$ whenever there exists $y \in S$ such that $(y,u) \in A(D)$. We let $\mathcal{S}(D)$ be the set of semikernels of D.

In [5] Galeana Sánchez et al showed that $|\mathcal{S}(D)| \leq |\mathcal{S}(L(D))|$.

The following definition generalizes semikernels.

Definition 4.2. Let D be a digraph, $k \geq 2$ and $l > 1$. A subset of vertices $S \subset V(D)$ is a (k,l)-*semikernel* if the following are satisfied:

1. S is k-independent.
2. For every $u \in V(D) \setminus S$ there exists $x \in S$ such that $d_D(u,x) \leq l$ whenever there exists $y \in S$ such that $d_D(y,u) \leq l$.

Let $\mathcal{S}_{(k,l)}(D)$ be the set of (k,l)-semikernels of D.

Theorem 4.3. *Let D be a digraph, $k \geq 2$ and $l > 1$. Suppose that there exist no cycles in D of length less than k. Let D_0 be the digraph that results from the elementary in-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^-(v) = X_1 \cup X_2$. Then $|\mathcal{S}_{(k,l)}(D)| \leq |\mathcal{S}_{(k,l)}(D_0)|$.*

Proof. Let $S \in \mathcal{S}_{(k,l)}(D)$. Suppose that $v \in S$ and let $S_0 = (S \setminus \{v\}) \cup \{v_1, v_2\}$. By 1 of lemma 2.7, S_0 is k-independent. For every $u \in V(D_0) \setminus S_0 = V(D) \setminus S$, there exists $x \in S$ such that $d_D(u,x) \leq l$ whenever there exists $y \in S$ such that $d_D(y,u) \leq l$. First suppose that $y \neq v$. Then $y \in S_0$ and therefore $d_{D_0}(y,u) \leq l$ since a path in D from y to u of minimal length remains unchanged under the in-split because it does not intersect $\{v\}$. If $x \neq v$, then $x \in S_0$ and again we have $d_{D_0}(u,x) \leq l$, otherwise $x = v$ and since a path in D from u to v of minimal length becomes a path in D_0 of equal length from u to v_i for some $i = 1,2$, $d_{D_0}(u,v_i) \leq l$. Now suppose that $y = v$. Thus a path in D from v to u of minimal length becomes two paths in D_0 of equal length, one starting at v_1 and the other at v_2, whence $d_{D_0}(v_i,u) \leq l$ for every $i = 1,2$. Again, if

$x \neq v$, then $x \in S_0$ and $d_{D_0}(u,x) \leq l$, otherwise $d_{D_0}(u,v_i) \leq l$ for some $i = 1, 2$. Therefore, in this case, $S_0 \in \mathcal{S}_{(k,l)}(D_0)$.

Now suppose that $v \notin S$ and let $S_0 = S$. By 2 of lemma 2.7, S_0 is k-independent. Let $u \in V(D_0) \setminus S_0$ and suppose that there exists $y \in S_0$ such that $d_{D_0}(y,u) \leq l$. First suppose that $u \notin \{v_1, v_2\}$. Then $u \in V(D) \setminus S$ and therefore $d_D(y,u) \leq l$ since a path in D_0 from y to u implies the existence of a path in D from y to u of equal length. Hence there exists $x \in S$ such that $d_D(u,x) \leq l$, and so $d_{D_0}(u,x) \leq l$ since a path in D from u to x implies the existence of a path in D_0 from u to x of equal length. Now suppose that $u = v_i$ for some $i = 1, 2$. It follows that $d_D(y,v) \leq l$, hence there exists $x \in S$ such that $d_D(v,x) \leq l$ and so $d_{D_0}(v_i,x) \leq l$ since $x \in S_0$. Therefore, also in this case, $S_0 \in \mathcal{S}_{(k,l)}(D_0)$. Clearly, the map $S \mapsto S_0$ is injective, so $|\mathcal{S}_{(k,l)}(D)| \leq |\mathcal{S}_{(k,l)}(D_0)|$.

□

Corollary 4.4. *Let D be a digraph. Let D_0 be the digraph that results from the elementary in-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^-(v) = X_1 \cup X_2$. Then $|\mathcal{S}(D)| \leq |\mathcal{S}(D_0)|$.*

Proof. By assumption, D has no cycles of length less than $k = 2$. Since $l = 1 < 2$ and $\mathcal{S}(D) = \mathcal{S}_{(2,1)}(D)$, the result follows from theorem 4.3.

□

The following definition duals (k,l)-semikernels.

Definition 4.5. Let D be a digraph, $k \geq 2$ and $l > 1$. A subset of vertices $S^* \subset V(D)$ is a (k,l)-*semisolution* if the following are satisfied:

1. S^* is k-independent.
2. For every $u \in V(D) \setminus S$ there exists $y \in S$ such that $d_D(y,u) \leq l$ whenever there exists $x \in S$ such that $d_D(u,x) \leq l$.

Let $\mathcal{S}^*_{(k,l)}(D)$ be the set of (k,l)-semisolutions of D.

Theorem 4.6. *Let D be a digraph, $k \geq 2$ and $l > 1$. Suppose that there exist no cycles in D of length less than k. Let D_0 be the digraph that results from the elementary out-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^+(v) = X_1 \cup X_2$. Then $|\mathcal{S}^*_{(k,l)}(D)| \leq |\mathcal{S}^*_{(k,l)}(D_0)|$.*

Proof. The proof is similar to the proof of theorem 4.3.

□

Corollary 4.7. *Let D be a digraph. Let D_0 be the digraph that results from the elementary out-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^+(v) = X_1 \cup X_2$. Then $|\mathcal{S}^*(D)| \leq |\mathcal{S}^*(D_0)|$.*

Proof. The proof is similar to the proof of theorem 4.4.

□

Readers can verify that 4.3 and 4.4 are valid for out-splittings and that 4.6 and 4.7 are valid for in-splittings. Using proposition 2.3 and corollary 4.4 we obtain as a corollary the result mentioned above by Galeana-Sánchez et al.

5 k-Grundy functions

Definition 5.1. Let D be a digraph. A function $g\colon V(D) \to \mathbb{N}$ is a *Grundy function* if for every $x \in V(D)$, $g(x) = \min\{\mathbb{N} - \{g(z) \mid z \in \Gamma^+(x)\}\}$. We let $\mathcal{G}(D)$ be the set of Grundy functions of D.

In [5] Galeana-Sánchez et al showed that $|\mathcal{G}(D)| \leq |\mathcal{G}(L(D))|$.

The following definition generalizes Grundy functions.

Definition 5.2. Let D be a digraph and $k \geq 1$. A function $g\colon V(D) \to \mathbb{N}$ is a *k-Grundy function* if for every $x \in V(D)$,

$$g(x) = \min\{\mathbb{N} - \{g(z) \mid 1 \leq d_D(x,z) \leq k\}\}.$$

Let $\mathcal{G}_k(D)$ be the set of k-Grundy functions of D.

Theorem 5.3. *Let D be a digraph and $k \geq 1$. Suppose that there exists no cycles in D of length less than $k+1$. Let D_0 be the digraph that results from the elementary in-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^-(v) = X_1 \cup X_2$. Then $|\mathcal{G}_k(D)| = |\mathcal{G}_k(D_0)|$.*

Proof. Let $g \in \mathcal{G}_k(D)$ and define $g_0\colon V(D_0) \to \mathbb{N}$ by

$$g_0(x) = \begin{cases} g(x) & \text{if } x \notin \{v_1, v_2\} \\ g(v) & \text{otherwise.} \end{cases}$$

Let $u \in V(D_0)$. If $u = v_i$ for some $i = 1,2$, then $\{z \in V(D_0) \mid 1 \leq d_{D_0}(v_i,z) \leq k\} = \{z \in V(D) \mid 1 \leq d_D(v,z) \leq k\}$ because there exist no cycles of length less than $k+1$, hence $g_0(v_i) = \min\{\mathbb{N} - \{g_0(z) \mid 1 \leq d_{D_0}(u,z) \leq k\}\}$. Suppose that $u \notin \{v_1, v_2\}$. If $v_i \notin \{z \in V(D_0) \mid 1 \leq d_{D_0}(u,z) \leq k\}$ for all $i = 1,2$, then $\{z \in V(D_0) \mid 1 \leq d_{D_0}(u,z) \leq k\} = \{z \in V(D) \mid 1 \leq d_D(u,z) \leq k\}$ and hence $g_0(u) = \min\{\mathbb{N} - \{g_0(z) \mid 1 \leq d_{D_0}(u,z) \leq k\}\}$. If $v_i \in \{z \in V(D_0) \mid 1 \leq d_{D_0}(u,z) \leq k\}$ for some $i = 1,2$, then $v \in \{z \in V(D) \mid 1 \leq d_D(u,z) \leq k\}$ and $\{z \in V(D) \mid 1 \leq d_D(u,z) \leq k\} - \{v\}$ differs from $\{z \in V(D_0) \mid 1 \leq d_{D_0}(u,z) \leq k\}$ in at most $\{v_j\}$ with $j \neq i$, hence $g_0(u) = \min\{\mathbb{N} - \{g_0(z) \mid 1 \leq d_{D_0}(u,z) \leq k\}\}$. Thus $g_0 \in \mathcal{G}_k(D_0)$. Clearly, the map $g \mapsto g_0$ is injective, so $|\mathcal{G}_k(D)| \leq |\mathcal{G}_k(D_0)|$.

Now let $g_0 \in G_k(D_0)$. Having no cycles of length less than $k+1$ implies that $\{z \in V(D_0) \mid 1 \leq d_{D_0}(v_1,z) \leq k\} = \{z \in V(D_0) \mid 1 \leq d_{D_0}(v_2,z) \leq k\}$, therefore $g_0(v_1) = g_0(v_2)$. Define $g\colon V(D) \to \mathbb{N}$ by

$$g(x) = \begin{cases} g_0(x) & \text{if } x \neq v \\ g_0(v_i) & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $g \in \mathcal{G}_k(D_0)$ and that the map $g_0 \mapsto g$ is actually the inverse of the map defined above. Therefore the result follows. ∎

The following definition duals k-Grundy functions

Definition 5.4. Let D be a digraph and $k \geq 1$. A function $g\colon V(D) \to \mathbb{N}$ is a *dual k-Grundy function* if for every $x \in V(D)$,

$$g(x) = \min \{\mathbb{N} - \{g(y) \mid \ 1 \leq d_D(y,x) \leq k\}\}.$$

We let $\mathcal{G}_k^*(D)$ be the set of dual k-Grundy functions of D.

Theorem 5.5. *Let D be a digraph $k \geq 1$. Suppose that there eixsts no cycles in D of length less than $k+1$. Let D_0 be the digraph that results from the elementary out-splitting of a vertex $v \in V(D)$ according to a partition $\Gamma^+(v) = X_1 \cup X_2$. Then $|\mathcal{G}_k^*(D)| = |\mathcal{G}_k^*(D_0)|$.*

Proof. The proof is similar to the proof of theorem 5.3.

□

Again, using proposition 2.3 and corollary 4.4 we obtain as a corollary the result mentioned above by Galeana-Sánchez et al.

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