Geometric Spanning Cycles in Bichromatic Point Sets

B. Joeris *   I. Urrutia †   J. Urrutia ‡

Abstract

Given a set of points in the plane each colored either red or blue, we find non-self-intersecting geometric spanning cycles of the red points and of the blue points such that each edge of the red spanning cycle is crossed at most three times by the blue spanning cycle and vice-versa.

1 Introduction

A *geometric graph* is a graph embedded in the plane with edges that are straight-line segments. A set of points is in general position if no three points of the set are collinear. In this paper, a bichromatic point set is a finite set of points S in general position, partitioned into two disjoint color classes S_R and S_B (red and blue.)

Several problems have been studied that involve finding geometric graphs on sets of red and blue points. Alternating paths in bichromatic point sets in convex position were studied in [2]. Alternating paths in general position were studied in [1]. Alternating paths in points with more than two colors were studied in [6]. [7] examined non-self-intersecting geometric spanning trees of the red points and the blue points and found a tight bound on the minimum number of intersection points between the red and blue spanning trees. [3] considered the case of more than two colors, and studied the number of intersections for monochromatic spanning trees and for monochromatic spanning cycles. [5] obtained a tight bound on the number of intersections in monochromatic perfect matchings. [4] looked at points and lines in the

*University of Waterloo, Waterloo, Canada. Email: bjoeris@uwateloo.ca

†University of Waterloo, Waterloo, Canada. Email: ihurruti@uwaterloo.ca

‡Instituto de Matemáticas, UNAM, Mexico. Email: urrutia@matem.unam.mx. Research supported by project number 178379 Conacyt, México.

plane lattice, and studied the number of crossings for alternating matchings and monochromatic spanning trees.

In [7], Tokunaga also showed that there exist non-intersecting geometric spanning paths P_R and P_B of the red and blue points respectively, such that each edge of P_R is crossed at most once by P_B, and vice-versa.

One may then wonder if a similar result is possible for cycles; i.e., is it possible to construct spanning cycles with "few" intersections on bichromatic point sets. In particular, we wonder for what values of k does the following statement hold: there exist non-intersecting geometric spanning cycles C_R and C_B of the red and blue points respectively, such that each edge of C_R is crossed at most k times by C_B, and vice-versa. In [7], Tokunaga conjectured that this statement is true when $k = 2$. It is easy to see that the statement is false with $k = 1$. We show here that the statement is true when $k = 3$.

Theorem 1. *Given any bichromatic point set in general position, there exists a non-self-intersecting geometric spanning cycle of the red points and a non-self-intersecting geometric spanning cycle of the blue points such that each edge of the red spanning cycle is crossed by at most three edges of the blue spanning cycle, and vice-versa.*

1.1 Definitions and Notation

If X is a set of points and y is a point outside the convex hull of X, we say that y *sees* a point $x \in X$ (with respect to X), if the line segment (x, y) intersects the convex hull of X only at x. In other words, if the convex hull of X were opaque, then y could see x. In particular, x must be a vertex of the convex hull in order for y to see it. If X and Y are two sets of points with disjoint convex hulls, and $x \in X$ and $y \in Y$, we say that x and *y see each other* (with respect to X and Y) if y sees x with respect to X and x sees y with respect to Y.

Let X be a set of points in the plane and $p \notin X$. The *radial order* of X about p is the cyclic list $(x_1, x_2, \ldots, x_n)$ taking all the elements $x \in X$ ordered clockwise around p. The *interval between x_i and x_j in the radial order* is the set consisting of $\{x_i, x_{i+1}, \ldots, x_j\}$, if $i \leq j$, or $\{x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_j\}$ if $j < i$. If y is in the interval between x_i and x_j in the radial order, then we say that y lies *between x_i and x_j in the radial order*.

If R and B are disjoint sets of red and blue points, respectively, then a *blob* in the radial order of $R \cup B$ about p is a maximal monochromatic set of consecutive elements in the radial order. Because the blobs are monochromatic, it is natural to speak of *red blobs* and *blue blobs*, containing red and blue points respectively.

Given a radial order $(x_1, \ldots, x_n)$ of $R \cup B$ with respect to a point $p \notin R \cup B$, for any $i \in \{1, \ldots, n\}$, x_{i-1} is called the point *before* x_i and x_{i+1} is called the point *after* x_i, where the indices are taken modulo n. For any blob X in this radial ordering, the *first point of* X is the point $x_i \in X$ such that $x_{i-1} \notin X$. Similarly, the *last point of* X is the point $x_i \in X$ such that $x_{i+1} \notin X$. We say that Y is the *previous red (resp. blue) blob before* X if Y is a red (resp. blue) blob and there is no red (resp. blue) point between the last point of Y and the first point of X in the radial order. Similarly, we say that Z is the *next red (resp. blue) blob after* X if there is no red (resp. blue) point between the last point of X and the first point of Z in the radial order.

1.2 Jump Configurations

If X is a blob, and Y is the next blob after X of the same color as X, then a *jump edge* from X to Y is a line segment (x, y) where $x \in X$, $y \in Y$ such that x and y see each other with respect to X and Y and the angle between x and y with respect to the point p is at most π. This last condition ensures that each point z on the line segment (x, y) would, if added to the radial order, lie between x and y. See Figure 1.

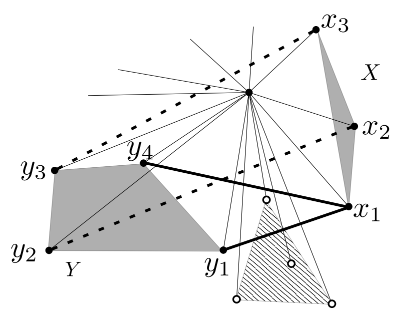


Figure 1: Jump edges from a blob X to the next blob of the same color, Y. Both x_1 to y_1 and x_1 to y_4 are jump edges. x_2 to y_2 is not a jump edge; it intersects the interior of the convex hulls of the blobs containing x_2 and y_2. x_3 to y_3 is not a jump edge because the angle from x_3 to y_3 is greater than π.

A *jump configuration* J is a collection of jump edges, one blue edge from each blue blob to the next blue blob and one red edge from each red blob to the next red blob, such that the blue edges do not cross each other, the red edges do not cross each other, and, for each blob X, the two edges with endpoints in X do not share a common endpoint unless X contains only one point.

In this section we show that a jump configuration can be completed to a pair of non-intersecting spanning cycles of the two color classes by adding a spanning path within each blob, such that each edge of these cycles will be crossed at most three times, except where a specific structure, called a 4-forcing, appears. 4-forcings will be defined later in this section.

Lemma 1. *If there is a jump configuration J in the radial order with respect to p, then for all blobs X, the angle from the first point in X to the last point in X is strictly less than π. Hence, for any z in the convex hull of X, if z were added to the radial order, it would lie in the interval from the first point in X to the last point in X, which implies that the convex hulls of the blobs are disjoint.*

Proof. Let (b, a) be the jump edge in J between the blob before X and the blob after X. By the definition of a jump edge, the angle from b to a is strictly less than π. Also, X is contained in the interval between b and a in the radial order, so the angle from the first point in X to the last point in X is strictly less than the angle from b to a, which is less than π. $\square$

The following two lemmas describe how jump edges in a jump configuration can intersect, and how jump edges can intersect the convex hulls of blobs.

Lemma 2. *If X_0 is a blob, and, for $i \in \{1, 2, 3, 4\}$, X_i is the next blob after X_{i-1}, then for any jump configuration J, the only jump edges in J that can cross the jump edge from X_1 to X_3 are the jump edges from X_0 to X_2 and from X_2 to X_4. In particular, no jump edges of the same color cross, and each jump edge is crossed by at most two jump edges of the opposite color.*

Proof. Let (x, y) be the jump edge from X_1 to X_3 in the jump configuration J, and let (x', y') be any jump edge in J such that (x, y) and (x', y') meet at a point z. By construction, if added to the radial order, z would lie between x and y, and between x' and y'. Therefore the interval I between x and y and the interval I' between x' and y' intersect. If $I' \supseteq I$, then (x', y') must be a jump edge from X_1 to X_3, so $(x', y') = (x, y)$. Otherwise, either $x' \in I$ or $y' \in I$, so, without loss of generality, x' lies between x and y in the radial order. Therefore $x' \in X_1 \cup X_2 \cup X_3$. If $x' \in X_1$ and y' is in the previous blob of the same color, then (x, y) and (x', y') do not cross by construction, and similarly if $x' \in X_3$ and y' is in the next blob of the same color. If $x' \in X_2$, then (x', y') is either the jump edge from X_0 to X_2 or the jump edge from X_2 to X_4. □

Lemma 3. *If X_0 is a blob, and, for $i \in \{1, 2, 3, 4\}$, X_i is the next blob after X_{i-1}, then for any jump configuration J, the only jump edges that can intersect the convex hull of X_2 are the jump edges from X_0 to X_2, from X_1 to X_3, and from X_2 to X_4.*

Proof. Let (y_1, y_2) be the jump edge between blobs Y_1 and Y_2 in the jump configuration J such that (y_1, y_2) intersects the convex hull of X_1 at a point z. Then z, if added to the radial order, would lie between the first point of X_1 and the last point of X_1. But z lies between a point in Y_1 and a point in Y_2 in the radial order, so X_1 must be Y_1, Y_2, or the blob between Y_1 and Y_2 in the radial order. □

When a jump edge passes through the convex hull of a blob X (which can only happen when it is the jump edge from the previous blob before X to the next blob after X), we want to find a spanning path within X that crosses the jump edge as few times as possible. The following lemma gives a construction for such a path.

Lemma 4. *If X is a finite set of points in general position, $x, y \in X$ are distinct points, and ℓ is a line, then there exists a non-self-intersecting spanning path P of X such that*

(i) if X lies entirely on one side of ℓ, then P does not cross ℓ;

(ii) if x and y lie on opposite sides of ℓ, then P crosses ℓ exactly once;

(iii) if x and y lie on the same side of ℓ, and X contains points on both sides of ℓ, then P crosses ℓ exactly twice.

Proof. By induction on $n = |X|$. If $n = 2$, then $X = \{x, y\}$, and so the one-edge path from x to y crosses ℓ zero times, if x and y are on the same side of ℓ; or once if x and y are on opposite sides of ℓ.

If $n > 2$, then $X \setminus \{x\}$ contains at least two points, and so x sees at least two points of $X \setminus \{x\}$. One of these points, x', is not y. If there are multiple choices for x', choose x' to lie on the same side of ℓ as x. By the induction hypothesis, there exists a spanning path P' of $X \setminus \{v\}$ from x' to y satisfying (i) (ii) and (iii). P' lies entirely within the convex hull of $X \setminus \{x\}$, and the edge from x to x' intersects the convex hull only at x', so $P = P' \cup (x, x')$ is a non-self-intersecting spanning path of X from x to y.

If X lies entirely on one side of ℓ, then P lies entirely on one side of ℓ because it is contained in the convex hull of X, so (i) holds.

Now suppose x and y lie on opposite sides of ℓ. If x' lies on the same side of ℓ as x then P crosses ℓ as many times as P', which is once. If x' lies on the opposite side of ℓ from x, then $X \setminus \{x\}$ cannot contain any point on the same side of ℓ as x, or else x would see some point x'' on the same side of ℓ, and $x'' \neq y$ because y is on the opposite side of ℓ, so x'' would have been chosen over x'. Therefore when x' lies on the opposite side of ℓ from x, P' does not cross ℓ, so P crosses ℓ exactly once.

Finally, suppose x and y lie on the same side of ℓ, and that X contains some point z on the opposite side of ℓ from x and y. If x' lies on the same side of ℓ as x and y, then P crosses ℓ as many times as P', which is twice. If x' lies on the opposite side of ℓ from x and y, then P' crosses ℓ once, so P crosses ℓ twice. $\square$

We now show how to construct a pair of monochromatic spanning cycles from a jump configuration by adding spanning paths within each blob.

The next lemma shows that there is exactly one case in which adding spanning paths can force our monochromatic spanning cycles to have edges that are crossed 4 times. We call this case a *4-forcing* and define it as follows: Suppose we are given a jump configuration J. Let X_0, Y_1, X_1, Y_2, X_2 be consecutive blobs such that the jump edge (y_1, y_2) between Y_1 and Y_2 in J intersects the convex hull of X_1 and crosses both the jump edge from X_0 to X_1 and the jump edge from X_1 to X_2, and such that the endpoints in X_1 of the jump edges from X_0 and to X_2 both lie on the same side of (y_1, y_2). Then this is called a *4-forcing* in J, and (y_1, y_2) is called the *center edge* of the 4-forcing. See Figure 2.

Lemma 5. *Given any jump configuration J, spanning paths of each blob can be added to J to construct a pair of non-self-intersecting spanning cycles of the red and blue points respectively such that, if an edge e is crossed more than three times by the opposite color cycle, then e is in J (and not in one of the spanning paths), and e is the center edge of a 4-forcing.*

Proof. For each blob X, let $a_X \in X$ be the point incident with the jump edge to X from the previous blob of the same color, and let $b_X \in X$ be the point incident with the jump edge from X to the next blob of the same color. If W is the previous blob before X and Y is the next blob after X, let ℓ_X be the line through b_W and a_Y, and note that (b_W, a_Y) is the only jump

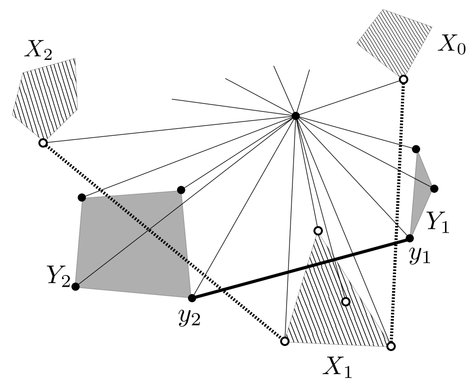


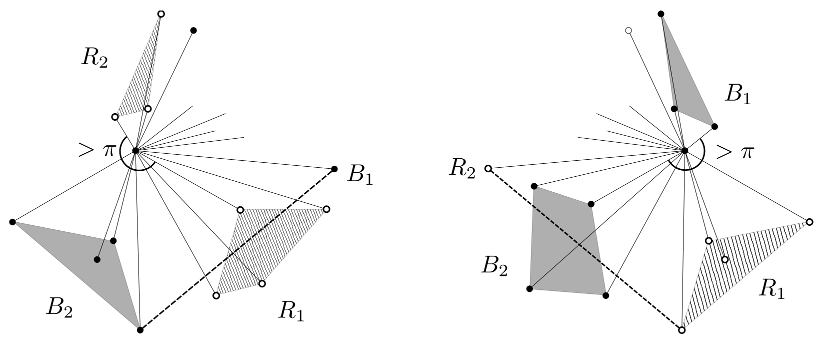
Figure 2: 4-forcing

edge that can cross through the convex hull of X. Let P_X be a spanning path of X from a_X to b_X minimizing the number of crossings of the ℓ_X.

By Lemma 3, no jump edge of the same color as X can cross the convex hull of X, and hence no jump edge can cross P_X. Also, for any blob $X' \neq X$, the convex hulls of X' and X are disjoint, so P_X and $P_{X'}$ cannot cross. By Lemma 2, two jump edges of the same color cannot cross, so the edges of $J \cup \bigcup_{\textit{all blobs } X} P_X$ form a pair of non-self-intersecting spanning-cycles.

Let e be an edge of one of these spanning cycles which is crossed at least 4 times by the other cycle. By Lemma 3, any edge of P_X can only be crossed by the jump edge (b_W, a_Y), and hence cannot be crossed 4 times, so e is a jump edge. Without loss of generality, suppose e is the jump edge between W and Y. By Lemma 2, e is crossed by at most two other jump edges (the two jump edges to and from X). The only blob for which e could cross the convex hull is X, so the only blob whos spanning-path e could cross is P_X. By Lemma 4, P_X crosses e at most two times, with equality only if a_X and b_X lie on the same side of ℓ_X. Therefore e must be crossed by both of the jump edges incident with X, so e is the center edge of a 4-forcing. $\square$

In the remainder of the paper, we focus on finding a good jump configuration such that when we add spanning cycles of each blob, as in the previous lemma, we have no 4-crossings —this will result in a pair of monochromatic spanning cycles in which each edge is crossed at most 3 times. In the following sections we show that it is possible to avoid 4-crossings by choosing p carefully.



(a) Red monster-jump between R_1 and R_2.

(b) Blue monster-jump between B_1 and B_2.

Figure 3: Red and blue monster-jumps

2 Monster-Jumps

If B_1, R_1, B_2, R_2 are consecutive blobs such that B_1 and B_2 are blue and R_1 and R_2 are red, then we say that R_1 to R_2 is a *red monster-jump* if $|R_1| > 1$ and the angle from the *second* point in R_1 to the *first* point in R_2 is at least π, and the line segment between the last point in B_1 and the first point in B_2 intersects the convex hull of R_1. See Figure 3a.

If B_1, R_1, B_2, R_2 are consecutive blobs such that B_1 and B_2 are blue and R_1 and R_2 are red, then we say that B_1 to B_2 is a *blue monster-jump* if $|B_2| > 1$ and the angle from the *last* point in B_1 to the *second to last* point in B_2 (i.e., the point before the last point in B_2) is at least π, and the line segment between the last point in R_1 and the first point in R_2 intersects the convex hull of B_2. See Figure 3b.

Note the slight asymmetry between the definition of red and blue monster-jump. In particular, if the colors are reversed and the plane reflected, then red monster-jumps become blue monster-jumps, and vice-versa.

In this section we show that if point p is chosen such that the radial ordering about p has no monster-jumps, then we can construct a jump configuration which can be completed to a pair of monochromatic spanning cycles with no 4-crossings.

2.1 Monster-Jumps and 4-forcings

Lemma 6. *Suppose that, for each blob X, the angle from the last point in X to the first point in the next blob of the same color is less than π. Then*

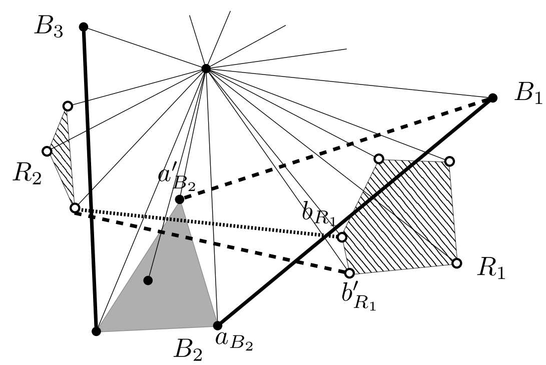


Figure 4: In the proof of Lemma 7, if there is a 4-forcing on the red jump edge between R_1 and R_2, then the blue-red crossing between the jump edge from B_1 to B_2 and the jump edge from R_1 to R_2 can be removed by replacing b_{R_1} by b'_{R_1} and replacing a_{B_2} by a'_{B_2}.

the collection of jump-edges consisting of, for each blob X, the edge from the last point of X to the first point of the next blob of the same color, is a valid jump configuration.

Proof. For each blob X, let a_X be the first point in X, and b_X be the last point in X. Then for each blob X, if X' is the next blob of the same color, then the angle from b_X to $a_{X'}$ is less than π. Therefore every point on the line segment $(b_X, a_{X'})$ would, if added to the radial order, lie in the interval between b_X and $a_{X'}$. If Y is the blob before X and Y' is the blob after X, then X is contained in the interval between the last point of Y and the first point of Y', which has angle less than π, so, for any point z in the convex hull of X, if z was added to the radial order, z would lie between the first point of X and the last point of X. Hence, $(b_X, a_{X'})$ can only intersect the convex hull of X at b_X. Similarly, $(b_X, a_{X'})$ can only intersect the convex hull of X' at $a_{X'}$. Therefore $(b_X, a_{X'})$ is a valid jump edge.

If X'' is the next blob of the same color after X', then the interval between b_X and $a_{X'}$ and the interval between $b_{X'}$ and $a_{X''}$ are either disjoint, if $|X'| > 1$, or meet at the point $a_{X'} = b_{X'}$, if $|X'| = 1$. Therefore the line segments $(b_X, a_{X'})$ and $(b_{X'}, a_{X''})$ are either disjoint, if $|X'| > 1$; or meet at $a_{X'} = b_{X'}$, if $|X'| = 1$. Thus, the collection J of jump edges consisting of, for each blob X with next blob of the same color X', $(b_X, a_{X'})$, is a valid jump configuration. $\square$

Lemma 7. *If a radial order contains no red monster-jump and no blue monster-jump and, for each blob X, the angle from the last point in X to the first point in the next blob of the same color is less than π, then there exists a jump configuration which contains no 4-forcing.*

Proof. A *blue-red crossing* in a jump configuration occurs when there are consecutive blobs B_1, R_1, B_2, R_2 such that B_1 and B_2 are blue and R_1 and R_2 are red, and the jump edge from B_1 to B_2 crosses the jump edge from R_1 to R_2. (Notice that a blue-red crossing is an assymetrical concept. This type of crossing is named "blue-red" instead of "red-blue" to emphasize that we are considering crossings between a blue edge and a red edge, where the blue edge is 'left' of the red edge in the radial ordering. Thus, a blue-red crossing is not the same as a *red-blue crossing*, which would consist of blobs R_1, B_1, R_2, B_2 such that R_1 and R_2 are red and B_1 and B_2 are blue and the jump edge from R_1 to R_2 crosses the jump edge from B_1 to B_2.)

For each red blob R, let a_R be the first point in R. For each blue blob B, let b_B be the last point in B.

For each red blob R choose $b_R \in R$, and for each blue blob B choose $a_B \in B$ such that the collection J of edges consisting of, for each blob X, $(b_X, a_{X'})$, where X' is the next blob of the same color, is a valid jump configuration, and that the number of blue-red crossings is minimized, with ties broken by minimizing the number of 4-crossings. Note that such a jump configuration exists by Lemma 6.

Suppose that the jump configuration J contains a 4-forcing. By swapping the colors and reflecting the plane, if necessary (so that we preserve the fact that there are no red or blue monster-jumps), we may assume that there are consecutive blobs B_1, R_1, B_2, R_2, B_3 such that B_1, B_2, B_3 are blue and R_1, R_2 are red, and that the red jump edge (b_{R_1}, a_{R_2}) from R_1 to R_2 is the center edge of a 4-forcing. Then (b_{B_1}, a_{B_2}) and (b_{B_2}, a_{B_3}) both cross (b_{R_1}, a_{R_2}), a_{B_2} and b_{B_2} are on the same side of the line ℓ through b_{R_1} and a_{R_2}, and B_2 contains points on both sides of ℓ.

If b'_{R_1} is the last point in R_1, then replacing (b_{R_1}, a_{R_2}) with (b'_{R_1}, a_{R_2}) in J will give another valid jump configuration J', and cannot increase the number of blue-red crossings, because the only blue-red crossing that a jump edge from R_1 to R_2 could be involved in is with the jump edge from B_1 to B_2, which (b_{R_1}, a_{R_2}) already crosses. Therefore, by minimality of J, (b'_{R_1}, a_{R_2}) must be the center edge of a 4-forcing in J'.

This means that a_{B_2} and b_{B_2} are both on the same side of the line ℓ' through b'_{R_1} and a_{R_2}, and B_2 contains points on both sides of ℓ', and that b_{B_1} and a_{B_3} are on the opposite side of ℓ' from a_{B_2} and b_{B_2}. There is some

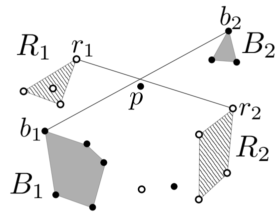


Figure 5: The choice of p when the red and blue convex hulls properly overlap (Lemma 8).

point $a'_{B_2} \in B_2$ on the same side of ℓ' as b_{B_1}, which b_{B_1} sees with respect to B_2. The angle from b_{B_1} to a'_{B_2} is at most the angle from b_{B_1} to the second to last point in B_2, which is less than π because B_1 to B_2 is not a blue monster-jump. Thus a'_{B_2} sees b_{B_1} with respect to B_1, so (b_{B_1}, a'_{B_2}) is a valid jump edge, and replacing (b_{B_1}, a_{B_2}) by (b_{B_1}, a'_{B_2}) in J' gives a valid jump configuration J''. The only blue-red crossing which (b_{B_1}, a'_{B_2}) can be involved in is with a jump edge from R_1 to R_2, and (b_{B_1}, a'_{B_2}) does not cross (b'_{R_1}, a_{R_2}), so J'' has fewer blue-red crossings than the chosen jump configuration J, a contradiction. See Figure 4. □

2.2 Avoiding Monster-Jumps

We have shown that if we choose p such that the radial order about p gives us a jump configuration with no monster-jumps, then we can complete this jump configuration to a pair of monochromatic spanning cycles with no 4-crossings. In this section we show that it is possible to choose a point p that avoids monster-jumps. This is broken down into two cases: when the blue and red convex hulls properly overlap (Lemma 8) and when the red convex hull contains the blue convex hull (Lemma 9). The only other alternative is that the red and blue convex hulls are disjoint, in which case the desired spanning cycles exist trivially.

Lemma 8. *Suppose the bichromatic point set S contains at least three red points and at least three blue points and that the convex hulls of the red points and of the blue points properly overlap; i.e., the intersection of the convex hulls is non-empty, and the blue convex hull is not contained in the red convex hull, nor vice-versa. Then after possibly swapping the color classes, there exists a point p in the intersection of the interior of the convex hulls such that the radial order about p contains neither a red nor a blue monster-jump.*

Proof. Note that because the convex hulls properly overlap, the boundaries of the convex hulls must intersect at some point q. Let (r_1, r_2) be the red segment containing q, such that r_2 follows r_1 in the clockwise ordering of the vertices of the red convex hull. Similarly, let (b_1, b_2) be the blue segment containing q, such that b_2 follows b_1 in the clockwise ordering of the vertices of the blue convex hull. By swapping colors if necessary, we may assume that b_1, r_1, b_2, r_2 appear in that order in the (clockwise) radial order about q. Therefore r_1 is outside the blue convex hull and b_2 is outside the red convex hull.

For all $\epsilon > 0$, there is a point p in the intersection of the interiors of the red and blue convex hulls such that $|p - q| < \epsilon$.

We will show that for ϵ sufficiently small, there is no red or blue monster-jump in the radial order about p. If ϵ is sufficiently small, then the radial orders of the bichromatic point set with respect to p and with respect to q coincide.

Any point between b_1 and b_2 in the radial order with respect to p is outside the blue convex hull, and hence must be red. Therefore there is exactly one red blob R_1 between b_1 and b_2, and $r_1 \in R_1$. Similarly there is exactly one blue blob, B_2, between r_1 and r_2, and $b_2 \in B_2$.

Let B_1 be the blue blob containing b_1 and R_2 be the red blob containing r_2. Note that b_1 is the last point in B_1 and b_2 is the first point in B_2, and the blob R_1 lies entirely on one side of the line through b_1 and b_2, so R_1 to R_2 is not a red monster-jump. Similarly, r_1 is the last point in R_1, r_2 is the first point in R_2, and B_2 lies entirely on one side of the line between r_1 and r_2, so B_1 to B_2 is not a blue monster-jump.

Let B be any blue blob which is not B_1, and let B' be the next blue blob after B. If $|B'| = 1$, then B to B' is not a blue monster-jump, so suppose $|B'| > 1$. Then the last point, b, in B and the second to last point, b', in B' lie in the interval between b_2 and b_1 in the radial order with respect to p. Note that $b \neq b_1$ because $B \neq B_1$ and $b' \neq b_1$, because b_1 is the not the second to last point in B_1. Therefore if b'_1 is the point after b_1 in the radial order with respect to p, then the angle from b to b' is at most the angle from b'_1 to b_2, which, for ϵ sufficiently small, is less than π. So, B to B' is not a blue monster-jump. By a symmetric argument, if R is a red blob which is not R_1 and R' is the next red blob after R and ϵ is sufficiently small, then R to R' is not a red monster-jump. Therefore, if ϵ is sufficiently small, then the radial order of the bichromatic point set with respect to p contains no red or blue monster-jump. □

Lemma 9. *Suppose the bichromatic point set S contains at least three red*

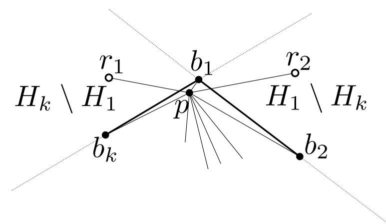


Figure 6: Case 1: The blue convex hull is contained in the red convex hull, and there are red points r_1 in $H_k \setminus H_1$ and r_2 in $H_1 \setminus H_k$.

points and at least three blue points and that the red convex hull contains the blue convex hull. Then there exists a point p in the interior of the blue convex hull such that the radial order of the bichromatic point set with respect to p contains no red or blue monster-jump.

Proof. Let $(b_1, \ldots, b_k)$ be the vertices of the blue convex hull in clockwise order. For $i \in \{1, \ldots, k\}$, let H_i be the open half-plane bounded by the line through b_i and $b_{i+1 \pmod k}$ which contains no blue points. Note that $\bigcup_{i=1}^{k} H_i \setminus H_{i+1 \pmod k}$ is the complement of the blue convex hull, and so, for some $i \in \{1, \ldots, k\}$, $H_i \setminus H_{i+1 \pmod k}$ contains a red point. Without loss of generality, there is some red point $r_1 \in H_k \setminus H_1$. Because b_1 is in the interior of the red convex hull, there is some red point in H_1; choose a red point r_2 in H_1 minimizing the angle from r_2 to b_2 with respect to b_1; that is the angle $\angle r_2 b_1 b_2$.

Case 1: $r_2 \in H_1 \setminus H_k$ (see Figure 6). In this case, you can choose p inside the triangle with vertices b_1, b_2, b_k and such that $|p - b_1 < \epsilon$, and consider the radial order of the bichromatic points with respect to p.

For $i \in \{1, 2, k\}$, let B_i be the blue blob containing b_i. For $i \in \{1, 2\}$, let R_i be the red blob containing r_i.

If ϵ and δ are sufficiently small, R_1 lies entirely on one side of the line through b_1 and b_k, which are the first point of B_1 and last point of B_k respectively, so R_1 to R_2 is not a red monster-jump. Again, if ϵ and δ are sufficiently small, R_2 lies entirely on one side of the line through b_1 and b_2, so R_2 to the next red blob is not a red monster-jump. If R is a red blob that is not R_1 or R_2 and R' is the next red blob and ϵ and δ are sufficiently small, then R lies entirely on the same side of the line through b_1 and b_2 as r_1, so the angle between the second point of R and the first point of R' is at most the angle between the second point of R and r_1, which is less than π. Therefore the radial order with respect to p contains no red monster-jump.

For ϵ and δ sufficiently small, the points before and after b_1 are both red,

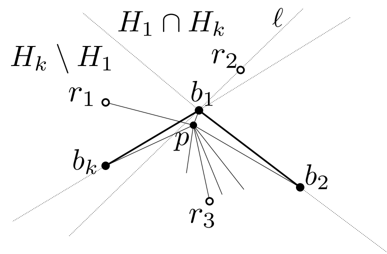


Figure 7: Case 2: The blue convex hull is contained in the red convex hull and there are red points r_1 in $H_k \setminus H_1$ and r_2 in $H_1 \cap H_k$.

so $B_1 = \{b_1\}$. Therefore B_k to B_1 is not a blue monster-jump. Let b'_k be the previous blue point before b_k in the radial order. For any blue blob B that is not B_k, if B' is the next blue blob, then the angle between the last point B and the second last point of B' is at most the angle between b_1 and b'_k, which is less than π if ϵ and δ are sufficiently small. Therefore the radial order with respect to p contains no blue monster-jump.

Case 2: $r_2 \in H_1 \cap H_k$ (see Figure 7). Let ℓ be the line through r_2 and b_1. Because b_1 is in the interior of the red convex hull, there is a red point r_3 on the opposite side of ℓ from r_1. However, r_3 cannot lie between r_2 and b_2 in the radial order with respect to b_1, so r_3 lies between b_2 and b_k in the radial order with respect to b_1, and the angle from r_2 to r_3 with respect to b_1 ($\angle r_2 b_1 r_3$) is less than π.

Choose p within the triangle $b_1 b_2 b_k$, close enough to b_1 such that the angles $\angle r_2 b_1 p$ and $\angle r_2 p r_3$ are less than π, and consider the radial order of the bichromatic points with respect to p. For $i \in \{1, 2, k\}$, let B_i be the blue blob containing b_i, and, for $i \in \{1, 2\}$, let R_i be the red blob containing r_i.

If ϵ and δ are sufficiently small, then R_1 lies entirely on one side of the line between b_k and b_1, which are the last point of B_k and first point of B_1 respectively, so R_1 to the next red blob is not a red monster-jump. For ϵ and δ sufficiently small, b_1 is the point before r_2 and b_2 is the point after r_2 so $R_2 = \{r_2\}$. Therefore, R_2 to the next red blob is not a red monster-jump. If R is a red blob that is neither R_1 nor R_2, and R' is the next red blob, then R lies entirely on the same side of the line through b_1 and b_2 as r_1, so the angle from the second point in R to the first point in R' is at most the angle from b_2 to r_1, which is less than π. Therefore the radial order with respect to p contains no blue monster-jump.

If ϵ and δ are sufficiently small, then the points before and after b_1 are both red, so $B_1 = \{b_1\}$. Therefore B_k to B_1 is not a blue monster-jump. If

ϵ and δ are sufficiently small, then r_3 lies between b_2 and b_k, and the angle from b_1 to r_3 is less than π. The blob B_2 ends before r_3, so the angle from b_1 to any point in B_2 is less than π, and B_1 to B_2 is not a blue monster-jump. If B is a blue blob that is not B_k or B_1, and B' is the next blue blob, then B and B' lie between b_2 and b_k in the radial order, so the angle from any point in B to any point in B' is less than π, and hence B to B' is not a blue monster-jump. Therefore the radial order with respect to p contains no blue monster-jump.

In both cases, p can be chosen such that the radial order with respect to p contains no red monster-jump and no blue monster-jump. □

Proof of Theorem 1. If the red convex hull and the blue convex hull are disjoint, then any pair of red and blue spanning cycle will be disjoint, so assume the red and blue convex hulls intersect. By Lemma 5, it suffices to show that for some point p, the radial order about p contains a jump-configuration with no 4-forcing. By Lemma 7, it suffices to show that there exists a point p such that the radial order about p contains no red or blue monster-jump.

Either the red and blue convex hulls properly overlap, or the blue convex hull is contained in the red convex hull, or the red convex hull is contained in the blue convex hull. If the red and blue convex hulls properly overlap, then by Lemma 8, there is a point p such that the radial order with respect to p contains no red or blue monster-jump. If the red convex hull is contained in the blue convex hull, we may swap the colors, so that the blue convex hull is contained in the red convex hull, and in that case, by Lemma 9, there is a point p such that the radial order with respect to p contains no red or blue monster-jump. □

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