Convexifying monotone polygons while maintaining internal visibility

O. Aichholzer[1], M. Cetina[2], R. Fabila-Monroy[3], J. Leaños[4], G. Salazar[2], J. Urrutia[5]

[1] Institute for Software Technology, University of Technology, Graz, Austria
oaich@ist.tugraz.at

[2] Instituto de Física, Universidad Autónoma de San Luis Potosí, México
mcetina@ifisica.uaslp.mx, gsalazar@ifisica.uaslp.mx

[3] Departamento de Matemáticas, Cinvestav, México
ruyfabila@math.cinvestav.edu.mx

[4] Unidad Académica de Matemáticas, Universidad Autónoma de Zacatecas, México
jleanos@mate.reduaz.mx

[5] Instituto de Matemáticas, Universidad Nacional Autónoma de México
urrutia@matem.unam.mx

Abstract. Let P be a simple polygon on the plane. Two vertices of P are visible if the open line segment joining them is contained in the interior of P. In this paper we study the following questions posed in [**6, 7**]: (1) Is it true that every non-convex simple polygon has a vertex that can be continuously moved such that during the process no vertex-vertex visibility is lost and some vertex-vertex visibility is gained? (2) Can every simple polygon be convexified by continuously moving only one vertex at a time without losing any internal vertex-vertex visibility during the process?

We provide a counterexample to (1). We note that our counterexample uses a monotone polygon. We also show that question (2) has a positive answer for monotone polygons.

Introduction

Let P be a simple polygon with vertices $\{p_1, \ldots, p_n\}$. We say that two vertices of P are P-*visible* if the relative interior of the line segment joining them is contained in the interior of P. The *visibility graph* $VG(P)$ of P is the graph with vertex set $\{p_1, \ldots, p_n\}$ in which two vertices of P are adjacent if they are P-visible. A classical problem in computational geometry is that of convexifying simple polygons; that is, using a given fixed set of transformations that can be applied to the vertices and edges of P, try to transform P into a convex polygon in such a way that some properties of P are preserved. The first formulation of a problem of this kind was proposed by Erdős [**4**], who proposed a strategy to convexify a non-convex polygon by using *flipturns*. Perhaps the most celebrated result in this area concerns the solution of the Carpenter's Rule conjecture [**3, 10**]; see also [**1, 2, 8, 9**].

[1]Partially supported by the FWF under grant S9205-N12.
[2]Supported by CONACYT Grant 106432.
[5]Supported by CONACYT Grant 80268 and project MTM2006-03909 (Spain).

Our starting point is the following question posed by Satyan L. Devadoss in the Open Problem Session at CCCG 2008 [**6, 7**]:

Question. Given a simple polygon P and its visibility graph $VG(P)$, can the vertices of P (one at a time or simultaneously) be moved continuously along paths so that:

- the simplicity of the polygon P is maintained all the time, and
- the visibility graph of P never loses edges, only gains them.

In discussions after the workshop, the following specific questions were raised [**5**]:

(1) Has every non-convex simple polygon a vertex p that can be continuously moved so that $VG(P)$ gains at least one extra edge, and never loses any?
(2) Can every simple polygon be convexified by continuously moving several vertices in sequence, but only one at a time, such that $VG(P)$ never loses any edge?

We will prove that Question (2) has a positive answer for monotone polygons. On the other hand, we give an example that shows that the answer to Question (1) is negative, even for monotone polygons. For recent results on this topic, see also [**7**].

1 Polygons and visibility

Let P be a simple polygon as defined above. The interior of P is the area bounded by P and we consider this area as an open set, i.e., vertices and edges of P do not belong to the interior of P. Let u and v be the leftmost and rightmost vertices of P. There are two edge-disjoint paths contained in P joining u to v, which are called the *upper chain* of P, and the *lower chain* of P, respectively. If any vertical line intersects the interior of P in at most one connected component then P is x-monotone, where, for simplicity, we will simply use the term monotone. Finally, we suppose without loss of generality that no vertical line passes through two vertices of P.

A basic operation that we use in this paper is to move the vertices of P around the plane. Strictly speaking, the polygon P defined by its vertices changes. Nevertheless, abusing our terminology a bit, we will always refer to it as P. Moreover, we will restrict our point moves to those that do not destroy the simplicity of P.

We say that the two vertices u and v of P are P-*visible* if the relative interior of the line segment $\overline{uv}$ joining them is contained in the interior of P. We call $\{u, v\}$ a *visibility pair*. Note that, according to our definition, consecutive vertices of P are not visible. Let $\mathcal{N}(P)$ be the set of pairs of vertices of P that are not P-visible. As consecutive vertices of P are not P-visible, $|\mathcal{N}(P)| \geq n$. Note that if the vertices of P move, the set of visible pairs of P may change, and in turn the visibility graph $VG(P)$ may also change.

We say that a vertex move is *visibility-preserving* if the following holds: If p_j and p_k were P-visible, they remain P-visible while p_i moves. If in addition the number of edges of $VG(P)$ increases, then we call it a *visibility-increasing* vertex move.

Our main results are the following:

Theorem 1.1. *There are polygons that have no visibility-increasing vertex moves.*

Theorem 1.2. *Every monotone polygon can be convexified with a sequence of visibility-preserving moves.*

2 A counterexample to Question (1)

Consider the monotone polygon P shown in Figure 1. The coordinates of the vertices of P are $a = (-100, 0)$, $b = (-63, 40)$, $c = (-61, 40)$, $d = (-33, 2)$, and $e = (0, 45)$. The points $\{f, g, h, i\}$ are obtained from the points $\{a, b, c, d\}$ by reflecting them along the y-axis. Points b' to h' are obtained from the points b to h by a reflection along the x-axis.

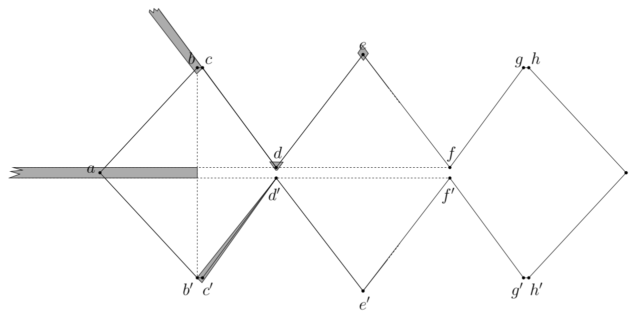

FIGURE 1. A monotone polygon without visibility-increasing vertex moves. Shaded areas indicate visibility-preserving regions. For point a, dashed lines indicate the boundary of its visibility-preserving region.

To show that P does not admit any visibility-increasing vertex move, it is sufficient to consider the vertices of P in the set $\{a, b, c', d, e\}$. The remaining cases follow by symmetry. For each of these vertices, we show in Figure 1 the open shaded region into which any of these points can be translated without losing any visibility pairs in P. It is now easy to see that there is no single vertex move that is visibility-increasing.

3 Visibility-preserving vertex moves

For a point $q \in \mathbb{R}^2$ and some $\delta > 0$, we denote by $B_\delta(q)$ the closed disk with radius δ with center at point q. Let $P = \{p_0, \ldots, p_{n-1}\}$ be a set of points in the plane in general position. We say that $\delta > 0$ is a *safe threshold of P* if there are no three elements p_i, p_j, and p_k of P such that $B_\delta(p_i)$, $B_\delta(p_j)$, and $B_\delta(p_k)$ are all intersected by a line ℓ. Equivalently, we can say that δ is a safe threshold of P if there are no three points $p_i, p_j, p_k \in P$ such that when we translate each of them to a point within δ distance of them, they become aligned.

It is not hard to see that every point set in the plane in general position has a safe threshold δ and that if a vertex move is not visibility-preserving, then at some point while moving the vertex it becomes collinear with two other vertices of P. However, the following lemma shows that collinearity is no problem for our approach. With $V^\circ(P)$ we denote the set of vertices interior to the convex hull of P.

Lemma 3.1. *Let P be a monotone polygon. Then there is a sequence of visibility-preserving vertex moves of some vertices of P such that at the end of the sequence, the vertices of P are in general position, P remains monotone, and $|V^\circ(P)| + |\mathcal{N}(P)|$ does not increase during the vertex movements.*

We are now ready to give a brief sketch of the proof of Theorem 1.2. By Lemma 3.1, we can assume that $V(P)$ is in general position. We proceed by induction on the sum of the number of interior vertices plus the number of non-visible pairs. If the vertices of P are in convex position, there is nothing to prove. Observe that P is convex if $|V^\circ(P)| + |\mathcal{N}(P)| = n$. Suppose then that $|V^\circ(P)| + |\mathcal{N}(P)| > n$ and assume that the theorem holds for all polygons Q with $|V^\circ(Q)| + |\mathcal{N}(Q)| < |V^\circ(P)| + |\mathcal{N}(P)|$.

Since P is not convex, suppose without loss of generality that there are $k \geq 1$ interior vertices of P on its upper chain. Relabel them as $v_1, v_2, \ldots, v_k$, in increasing order with respect to their x-coordinate. Let $\delta > 0$ be a safe threshold for the *initial position* of $V(P)$. Our algorithm starts by executing the following basic procedure **BP**:

BP: *One at a time from left to right, move $v_1, v_2, \ldots, v_k$ upwards, by a distance δ.*

Once $v_1, v_2, \ldots, v_k$ have all been moved, we execute **BP** repeatedly (using always the same δ!) until one of the following occurs: (1) a vertex in $\{v_1, v_2, \ldots, v_k\}$ reaches the convex hull of P, (2) a new visible pair occurs, or (3) the visibility-preserving property is lost. If we stop because (1) or (2) occurs, then we are done, by our induction hypothesis. Using monotonicity of P we can show that (3) does not happen before a visibility-increasing event, which proves the theorem. Details are omitted in this extended abstract.

4 Conclusion

Several open questions arise from our work: How many vertex moves do we need to convexify a monotone polygon? Can this number be bounded by a polynomial? If we allow only vertical moves we can construct a polygon where the number of vertex moves is unbounded, but how about general moves? What happens if we allow more than one vertex to move at a time? We conclude with the following conjecture.

Conjecture 4.1. *Every simple polygon can be convexified by a sequence of visibility-preserving* 1*-vertex moves.*

Acknowledgments

We would like to thank Erik Demaine and Stefan Langerman for valuable discussions on the topic, as well as the anonymous referees for their comments.

References

[1] O. Aichholzer, C. Cortés, E. Demaine, V. Dujmović, J. Erickson, H. Meijer, M. Overmars, B. Palop, S. Ramaswami, and G. Toussaint. *Flipturning polygons.* Discrete and Computational Geometry, Vol. 28, 2002, pp. 231–253.

[2] T. C. Biedl, E. Demaine, S. Lazard, S. M. Robbins, and M. A. Soss. *Convexifying monotone polygons.* Lecture Notes in Computer Science 1741, pp. 415–424, 1999.

[3] R. Connelly, E. D. Demaine and G. Rote, *Straightening polygonal arcs and convexifying polygonal cycles.* Discrete Comput. Geom. **30** (2003), 205–239.

[4] P. Erdős, *Problem* 3763. Amer. Math. Monthly **42** (1935), p. 627.

[5] E. D. Demaine, S. Langerman. Personal communication, Montreal, 2008.

[6] E. D. Demaine, J. O'Rourke. *Open Problems from CCCG 2008.* Proceedings of the 21st Canadian Conference on Computational Geometry (CCCG 2009), pp. 75–78, 2009.

[7] S. L. Devadoss, R. Shah, X. Shao, and E. Winston. *Visibility graphs and deformations of associahedra.* arXiv:0903.2848, March 2009.

[8] B. Grünbaum. *How to convexify a polygon.* Geocombinatorics **5** (1995), pp. 24–30.

[9] B. de Sz.-Nagy. *Solution of problem* 3763. Amer. Math. Monthly **49** (1939), pp. 176–177.

[10] I. Streinu, *Pseudo-triangulations, rigidity and motion planning.* Discrete Comput. Geom. **34** (2005), 587–635.