Covering the Convex Quadrilaterals of Point Sets

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Abstract. For a point set P on the plane, a four element subset $S \subset P$ is called a 4-*hole* of P if the convex hull of S is a quadrilateral and contains no point of P in its interior. Let R be a point set on the plane. We say that a point set B *covers* all the 4-holes of R if any 4-hole of R contains an element of B in its interior. We show that if $|R| \geq 2|B| + 5$ then B cannot cover all the 4-holes of R. A similar result is shown for a point set R in convex position. We also show a point set R for which any point set B that covers all the 4-holes of R has approximately $2|R|$ points.

Key words. k-Hole, Bicolored point set, Covering

1. Introduction

Throughout this paper, P denotes a point set on the plane. P is said to be in *general position* if no three of its elements lie on a line. We denote by $\text{Conv}(P)$ the convex hull of P, and we say that P is in *convex position* if P is in general position and all of its elements lie on the boundary of $\text{Conv}(P)$.

For an integer $k \geq 3$ and any point set P on the plane, a k-subset S of P is called a k-*hole* of P if S is in convex position and no element of P lies in the interior of $\text{Conv}(S)$. A k-*hole* is often identified with its convex hull. In 1931, Esther Klein proved that any point set P in general position with at least 5 elements contains a 4-subset in convex position [4,6]. It is easy to see that it also contains a 4-hole.

We say that P is a *bicolored* point set if P is the union of disjoint point sets R and B. Call the elements of R and B the red and blue points of P respectively. A *monochromatic* 4-*hole* of P is a 4-hole of P such that all its elements are either in R or in B. In [2], O. Devillers et al. showed many results on k-holes of m-*colored* point sets, $k \geq 3$, $m \geq 2$. Concerning monochromatic 4-holes of a bicolored point set, they conjecture:

Conjecture A. *Let P be a bicolored point set in general position consisting of a sufficiently large number of points. Then P contains a monochromatic 4-hole.*

For a 4-hole S of a red point set R and a blue point $b \in B$, we say that b *covers* S if $\text{Conv}(S)$ contains b in its *interior*. Furthermore, we say that B covers all the 4-holes

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of R if each 4-hole of R is covered with a point of B. In this paper, we will study the following question: Given a red point set R, how many points must a blue point set B have so that it covers all the 4-holes of R? Our objective here is to continue the study of a similar problem concerning coverings of 3-holes of point sets. The problem of coverings of 3-holes has been studied by M. Katchalski et al.[5] and independently by J. Czyzowicz et al.[1]. They proved:

Theorem A. *For any red point set R in general position, $2|R| - K - 2$ blue points are necessary and sufficient to cover all 3-holes of R, where K denotes the number of points of R on the boundary of* $\mathrm{Conv}(R)$.

Observe that any triangulation of R (in the language of this paper, a set of 3-holes of R with disjoint interiors such that their union is $\mathrm{Conv}(R)$) contains exactly $2|R| - K - 2$ triangles. Thus it follows that if a point set B of $2|R| - K - 2$ blue points covers all the triangles of R, then each 3-hole of R contains exactly one element of B.

In Section 2 we prove the following results:

Theorem 1. *Let $P = R \cup B$ be a bicolored point set such that R is in general position. Then if $|R| \geq 2|B| + 5$, P contains a red 4-hole.*

(The inequality of Theorem 1 is sharp when $|B| = 0$ and $|B| = 1$.) If R is in convex position, we obtain a better bound.

Theorem 2. *Let $P = R \cup B$ be a bicolored point set such that R is in convex position. Then if $|R| \geq \frac{3}{2}|B| + 4$, P contains a red 4-hole.*

For a red point set R, let $\beta(R)$ denote the minimum number of blue points that cover all 4-holes of R. From Theorems 1 and 2, it follows that $\beta(R) > \frac{|R|-5}{2}$ for any red point set R in general position, and $\beta(R) > \frac{2|R|-8}{3}$ for any red point set R in convex position. In Section 4 we prove:

Theorem 3. *Let n be a positive integer. Then*

$$\max_{\substack{|R| = n \\ R \text{ is in convex position}}} \beta(R) = n + o(n).$$

For a positive integer n, let

$$\beta_n = \max_{|R|=n} \beta(R).$$

It follows from Theorem A that $\beta_n \leq 2n - 5$ (it is not difficult to show $\beta_n \leq 2n - 6$ for the problem of covering red *4-holes*). In Section 4, we also show that there exist red point sets R for which approximately $2|R|$ blue points are needed to cover all 4-holes of R:

Theorem 4. $\beta_n = 2n + o(n)$.

2. Proof of Theorems 1 and 2

For a point set $\{p_1, p_2, \ldots, p_m\}$, we denote by $p_1p_2 \ldots p_m$ the polygon whose vertices in the clockwise are $p_1, p_2, \ldots, p_m$.

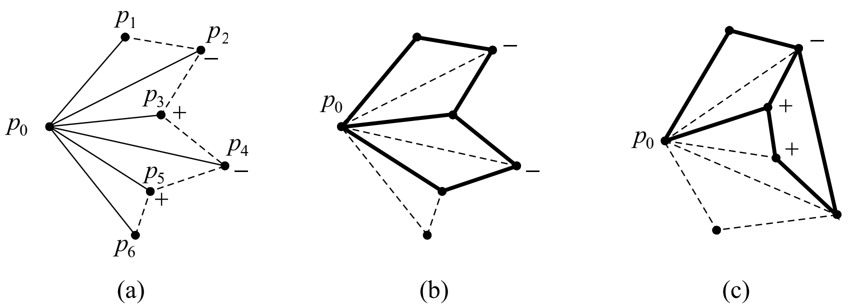

Fig. 1. Labeling of the points and two convex quadrilaterals with disjoint interiors.

2.1. Proof of Theorem 1

First note that B covers every red 4-hole if and only if so does a blue point set obtained by any slight perturbation of points in B. Thus it suffices to prove Theorem 1 for the case where $P = R \cup B$ is in general position.

We proceed by induction on $|B|$. The result is true for the case where $|B| = 0$ (the Esther Klein's Theorem mentioned above). Next we show the result for the case where $|B| = 1$. For this purpose, it suffices to prove the following proposition:

Proposition 1. *Any point set R with exactly seven elements in general position contains the vertices of two convex quadrilaterals with disjoint interiors.*

Proof. Choose the leftmost vertex on the convex hull of R, assuming without loss of generality that this point is unique, and let it be labeled p_0. Label the elements of $R - \{p_0\}$ by $p_1, \ldots, p_6$ in descending order according to the slope of the segments joining p_i to p_0, $i = 1, \ldots 6$; see Fig. 1(a). For each $i = 2, \ldots, 5$, assign the signature $+$ or $-$ to p_i according to whether the inner angle at p_i of the quadrilateral $p_0p_{i-1}p_ip_{i+1}$ is greater than or less than 180°.

If the sequence of signatures assigned to $p_2, \ldots, p_5$ contains two non-consecutive minus signs, then our result follows (Fig. 1(b)). Our result also follows if the sequence contains a minus sign and consecutive plus signs, see Fig. 1(c). The remaining cases to be analyzed, are for the sequences $+--+$ or $++++$. For the latter case, $p_1p_2p_3p_4$ and $p_1p_4p_5p_6$ are convex quadrilaterals with disjoint interiors.

Assume then that our sequence is $+--+$. Let l denote the straight line connecting p_2 and p_5. If at least one of p_1 or p_6 is in the same side of l as p_0, then $p_2p_3p_4p_5$ and at least one of $p_0p_1p_2p_5$ or $p_0p_2p_5p_6$ are convex quadrilaterals with disjoint interiors. Thus assume that both of p_1 and p_2 are in the opposite side of l to that containing p_0. Let m denote the straight line connecting p_3 and p_4, and D the half-plane bounded by m and containing p_2 and p_5. If $p_1 \in D$ (resp. $p_6 \in D$), then $p_1p_2p_4p_3$ and $p_0p_2p_4p_5$ (resp. $p_3p_5p_6p_4$ and $p_0p_2p_3p_5$) are convex quadrilaterals with disjoint interiors. Assume next that $p_1 \notin D$ and $p_6 \notin D$. In this case, $p_1p_3p_4p_6$ and $p_2p_3p_4p_5$ are convex quadrilaterals with disjoint interiors. □

Returning to the proof of Theorem 1, we consider the case where there are at least two blue points. Let p and p' be consecutive vertices of the boundary of $\mathrm{Conv}(B)$, and let D denote a half-plane bounded by the straight line pp', and containing no element of $B - \{p, p'\}$. If D contains at least five red points, then the result follows from the Esther Klein's Theorem. Thus we may assume that D contains at most four red points. Then if

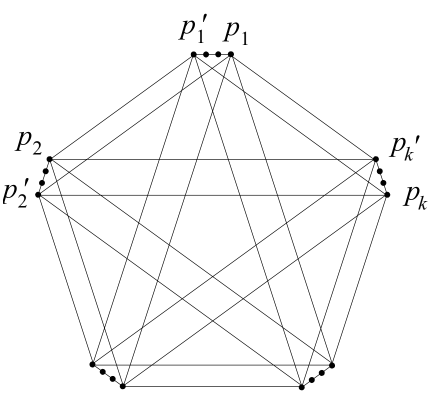

Fig. 2. p_i and p_i'.

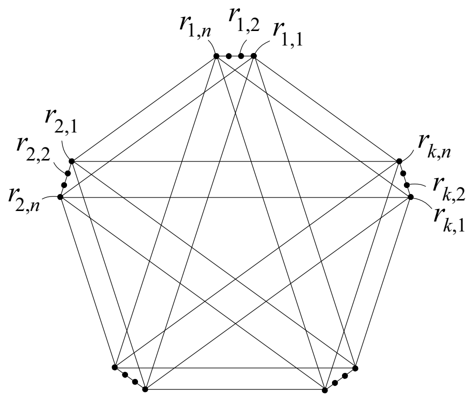

Fig. 3. Red point set $R = R(k, \varepsilon, n)$.

we write $D' = I\!R^2 - D$, $|R \cap D'| \geq |R| - 4 \geq 2(|B| - 2) + 5 = 2|B \cap D'| + 5$, and the desired conclusion follows from the induction hypothesis. $\square$

2.2. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1. In place of Proposition 1, we use the fact that any point set R with $|R| = 6$ in convex position contains two convex quadrilaterals with disjoint interiors. $\square$

3. Point set $R(k, \varepsilon, n)$ and Theorem 5

3.1. Point set $R(k, \varepsilon, n)$

Let $k \geq 3$ be an odd integer and n an integer, and consider a regular k-gon $P_0 = p_1 p_2 \dots p_k$ inscribed in a unit circle. Rotate each point p_i by a sufficiently small angle ε around the center of P_0 to obtain a point p_i' (Fig. 2). We may assume that

$$\frac{p_1 p_k'}{p_1 p_1'} > kn \tag{1}$$

in particular (we use this inequality in the proof of Lemma 1). Furthermore, since no three diagonals of P_0 meet at a point (see [3]), we may assume that no three quadrilaterals $p_{i_1} p_{i_1}' p_{j_1} p_{j_1}'$, $p_{i_2} p_{i_2}' p_{j_2} p_{j_2}'$ and $p_{i_3} p_{i_3}' p_{j_3} p_{j_3}'$ ($i_1, i_2, i_3, j_1, j_2, j_3$ are all different) have a common point.

Take n red points $r_{i,1} = p_i$, $r_{i,2}$, $\dots$, $r_{i,n-1}$, $r_{i,n} = p_i'$ at regular intervals on segment $p_i p_i'$ (Fig. 3). Set $R_i = \{r_{i,1}, r_{i,2}, \dots, r_{i,n}\}$ and define the set $R = R(k, \varepsilon, n)$ consisting of kn red points by $R = R(k, \varepsilon, n) = \cup_{i=1}^{k} R_i$.

3.2. Theorem 5

To prove Theorems 3 and 4, we show the following result:

Theorem 5. *Let k and n be positive integers. Then*

$$\frac{\beta(R(k, \varepsilon, n))}{|R(k, \varepsilon, n)|} \to 1 \text{ as } n \to \infty \text{ and } \frac{k}{n} \to \infty.$$

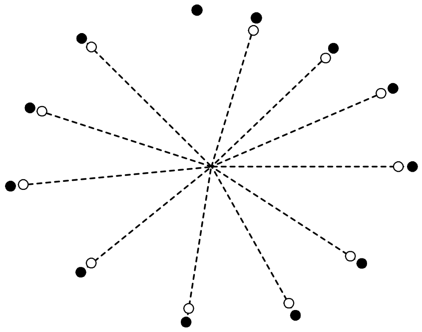

Fig. 4. •: red point, ∘: blue point.

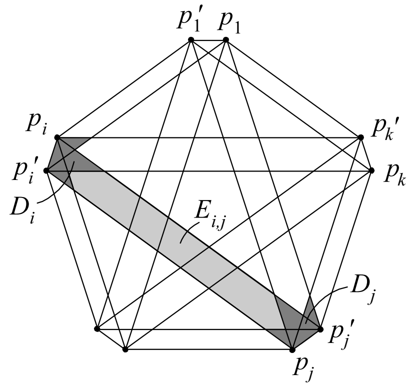

Fig. 5. D_i and $E_{i,j}$.

Observe that for a red point set R in convex position, if we place $|R|-1$ blue points inside the convex hull of R (shown in Fig. 4 as small empty circles) all the 4-holes of R are covered. From a similar observation, we see that

$$\beta(R(k,\varepsilon,n)) \le kn-1. \tag{2}$$

3.3. Proof of Theorem 5

We will prove that

$$\frac{\beta(R(k,\varepsilon,n))}{kn} \to 1 \text{ as } n\to\infty \text{ and } \frac{k}{n}\to\infty.$$

Define D_i and $E_{i,j}(=E_{j,i})$ by

$$D_i = \bigcup_{\substack{1\le l<m\le k\\ l\ne i,\ m\ne i}} (p_ip_i'p_lp_l' \cap p_ip_i'p_mp_m')$$

$$E_{i,j} = p_ip_i'p_jp_j' - (D_i\cup D_j), \quad \text{where } i\ne j\text{; see Fig. 5.}$$

Lemma 1. *The intersection of quadrilaterals $r_{i,m}r_{i,m+1}r_{j,l}r_{j,l+1}$ and $r_{i,m'}r_{i,m'+1}r_{j,l'}r_{j,l'+1}$ is contained in $E_{i,j}$ for any i, j, m, m', l, l', $1\le i<j\le k$, $2\le m+1<m'\le n-1$ and $2\le l+1<l'\le n-1$.*

Proof. It suffices to consider the case where $p_ip_j' < p_i'p_j$ (Fig. 6). Take any point x in the intersection of quadrilaterals $r_{i,m}r_{i,m+1}r_{j,l}r_{j,l+1}$ and $r_{i,m'}r_{i,m'+1}r_{j,l'}r_{j,l'+1}$. For this x, we can take points p, q, p' and q' such that they are on the segments $r_{i,m}r_{i,m+1}$, $r_{j,l}r_{j,l+1}$, $r_{i,m'}r_{i,m'+1}$ and $r_{j,l'}r_{j,l'+1}$, respectively, and the intersection point of the segments pq and $p'q'$ is x. Take the point r such that $\overrightarrow{q'r}=\overrightarrow{p_ip_i'}$, and let y be the intersection point of pr and $p'q'$. Then we have $p'x>p'y$ and $\frac{p'y}{q'y}=\frac{pp'}{rq'}\ge\frac{1}{n-1}$, and hence

$$p'x > \frac{p'q'}{n} > \frac{p_ip_j'}{n} \ge \frac{p_1p_k'}{n}.$$

On the other hand, since the convex hull of D_i is a regular k-gon which has p_ip_i' as one of its sides (Fig. 7), the diameter of D_i is less than kp_ip_i', which is the perimeter of the regular k-gon. Since $kp_ip_i' < \frac{p_1p_k'}{n}$ by (1), $x\notin D_i$, as desired. □

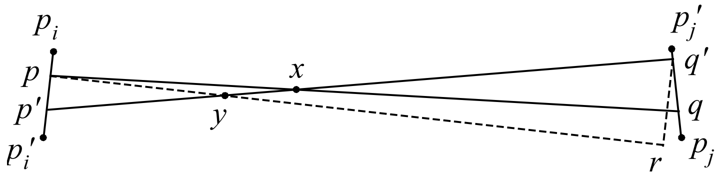

Fig. 6. A point x in the intersection and the point y.

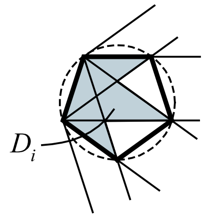

Fig. 7. D_i and its convex hull.

Let B be a set of blue points of minimum cardinality which cover all the 4-holes of R. For each $1 \leq i \leq k$, let $E_i = \cup_{j \neq i} E_{i,j}$. Then two cases arise:

Case 1. For each $1 \leq i \leq k$, either $|D_i \cap B| > n - \sqrt{n}$ or $|E_i \cap B| > 4(n - \sqrt{n})$.
Case 2. There exists i with $1 \leq i \leq k$ such that $|D_i \cap B| \leq n - \sqrt{n}$ and $|E_i \cap B| \leq 4(n - \sqrt{n})$.

Case 1. Let $I = \{1, 2, \ldots, k\}$,

$$I_1 = \{i \,|\, |D_i \cap B| > n - \sqrt{n}, 1 \leq i \leq k\} \quad \text{and}$$
$$I_2 = I - I_1.$$

Then

$$|E_i \cap B| > 4(n - \sqrt{n}) \quad \text{for each } i \in I_2.$$

Since any point $b \in \cup_{1 \leq i \leq k}(E_i \cap B)$ is contained in at most four E_i's, we have $|\cup_{i \in I_2}(E_i \cap B)| \geq \frac{1}{4}\sum_{i \in I_2}|E_i \cap B|$. Thus

$$\begin{aligned} |B| &\geq \left|\bigcup_{i \in I_1}(D_i \cap B)\right| + \left|\bigcup_{i \in I_2}(E_i \cap B)\right| \\ &\geq \sum_{i \in I_1}|D_i \cap B| + \frac{1}{4}\sum_{i \in I_2}|E_i \cap B| \\ &> |I_1|(n - \sqrt{n}) + \frac{1}{4}(k - |I_1|) \cdot 4(n - \sqrt{n}) \\ &= k(n - \sqrt{n}) \;=\; k(n + o(n)). \end{aligned}$$

From this, together with (2), we obtain the desired conclusion.

Case 2. Take i with $1 \leq i \leq k$ such that $|D_i \cap B| \leq n - \sqrt{n}$ and $|E_i \cap B| \leq 4(n - \sqrt{n})$. From this point on, we fix i. Let

$$J = \{j \mid E_{i,j} \cap B = \emptyset, j \neq i\}.$$

Since $|\{j' \mid E_{i,j'} \cap B \neq \emptyset, j \neq i\}| \leq |E_i \cap B| \leq 4(n - \sqrt{n})$,

$$|J| \geq (k-1) - 4(n - \sqrt{n}) = k + o(k) \tag{3}$$

for k, n with $\frac{k}{n} \to \infty$. Take any $j \in J$. Since $n-1$ blue points arranged suitably close to the midpoints of segments $r_{j,m}r_{j,m+1}$, $1 \leq m \leq n-1$, are sufficient to cover all 4-holes of R containing these segments, it follows from the minimality of $|B|$ that

$$|D_j \cap B| \leq n - 1. \tag{4}$$

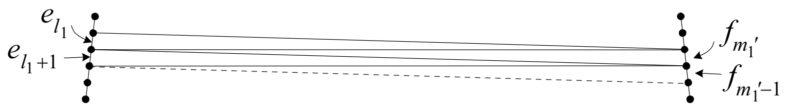

Fig. 8. m_2 must be greater than or equal to m_1'.

Lemma 2. $|D_j \cap B| = n + o(n)$.

To prove this lemma, we introduce some notation. Let e_l be the segment connecting $r_{i,l}$ and $r_{i,l+1}$, and f_m be the segment connecting $r_{j,m}$ and $r_{j,m+1}$, where $1 \le l \le n-1$ and $1 \le m \le n-1$. We denote by $[e_l, f_m]$ the quadrilateral containing e_l and f_m as its sides.

Let $b \in D_i \cap B$. Then it can be easily observed that the set $\mathcal{Q}_b$ of quadrilaterals $[e_l, f_m]$ which are covered with b is expressed in the following form:

$$\mathcal{Q}_b = \{[e_{l_1}, f_m] \,|\, m_1 \le m \le m_1'\} \cup \{[e_{l_1+1}, f_m] \,|\, m_2 \le m \le m_2'\} \cup \\ \ldots \cup \{[e_{l_1+h}, f_m] \,|\, m_{h+1} \le m \le m_{h+1}'\},$$

where l_1 and h are integers with $1 \le l_1 \le l_1 + h \le n-1$, and the m_t and the m_t', $1 \le t \le h+1$, are integers with $1 \le m_t \le m_t' \le n-1$.

Lemma 3. $h \in \{0, 1\}$*; and in the case where* $h = 1$, $m_2 \ge m_1'$.

Proof. We have $h \in \{0,1\}$ from Lemma 1. Next assume $h = 1$. Then since $b \in \mathrm{Int}([e_{l_1}, f_{m_1'}]) \cap \mathrm{Int}([e_{l_1+1}, f_{m_2}])$, where $\mathrm{Int}(X)$ denotes the interior of X, we must have $\mathrm{Int}([e_{l_1}, f_{m_1'}]) \cap \mathrm{Int}([e_{l_1+1}, f_{m_2}]) \ne \emptyset$, and hence $m_2 \ge m_1'$ (Fig. 8).

□

Consider an $(n-1) \times (n-1)$ table whose (l, m)-component corresponds to the quadrilateral $[e_l, f_m]$. By Lemma 3, each $\mathcal{Q}_b$, $b \in D_i \cap B$, is expressed as a set of components as shown in Fig. 9 (the case where $m_2 = m_1'$). We identify such a set of components with the set $\mathcal{Q}_b$. Set

$$\mathcal{Q} = \bigcup_{b \in D_i \cap B} \mathcal{Q}_b \qquad \text{(Fig. 10)}.$$

For each $b \in D_i \cap B$ such that $\mathcal{Q}_b$ is expressed in the form of $\mathcal{Q}_b = \{[e_{l_1}, f_m] \,|\, m_1 \le m \le m_1'\} \cup \{[e_{l_1+1}, f_m] \,|\, m_1' \le m \le m_2'\}$, let $U_b = [e_{l_1+1}, f_{m_1'}]$ (Fig. 9), and let $\mathcal{U}$ denote the set of all such U_b's. We have

$$|\mathcal{U}| \le |D_i \cap B| \le n - \sqrt{n}. \tag{5}$$

For each m with $1 \le m \le n-1$, call the column consisting of $n-1$ components $[e_1, f_m]$, $[e_2, f_m]$, $\ldots$, $[e_{n-1}, f_m]$ the *column* f_m. Let $\mathcal{F}$ be the set of columns f_1, f_2, $\ldots$, f_{n-1}, and $\mathcal{F}^*(\subseteq \mathcal{F})$ the set of columns f_m each of which contains at most $\sqrt[4]{n}$ components corresponding to elements of $\mathcal{U}$. Since we consider the case where n and k are sufficiently large, we may assume $n \ge 7$ in particular.

Lemma 4. *Let* $f_m \in \mathcal{F}^*$*. Then there is a component* $[e_l, f_m]$ *which does not belong to any* $\mathcal{Q}_b$*, i.e., there is a quadrilateral* $[e_l, f_m]$ *which is covered with some* $b' \in D_j \cap B$.

Proof. We identify the column f_m with the set of components contained in it, i.e., we let $f_m = \{[e_l, f_m] | 1 \le l \le n-1\}$. Since

$$|\mathcal{Q} \cap f_m| \le |D_i \cap B| + |\mathcal{U} \cap f_m| \le (n - \sqrt{n}) + \sqrt[4]{n},$$

there are at least $\sqrt{n} - \sqrt[4]{n} - 1(> 0 \text{ for } n \ge 7)$ components $[e_l, f_m] \in f_m$ which do not belong to any $\mathcal{Q}_b$. □

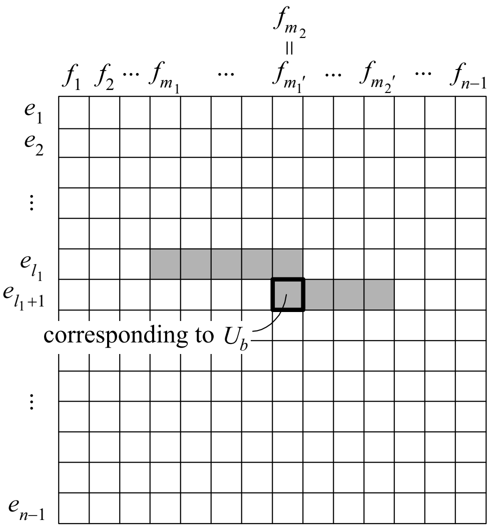

Fig. 9. Components corresponding to $\mathcal{Q}_b$.

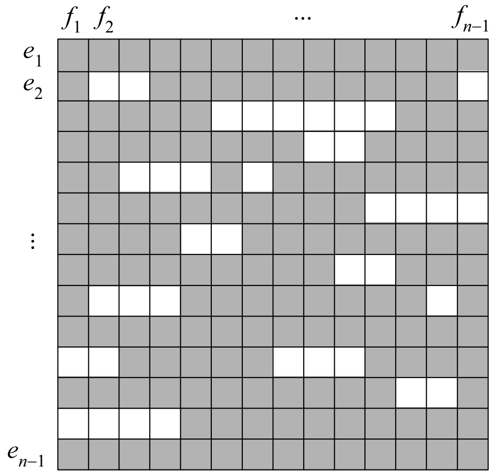

Fig. 10. Components corresponding to $\mathcal{Q}$.

Lemma 5. $|\mathcal{F}^*| \geq n - \sqrt[4]{n^3} + \sqrt[4]{n} - 1$.

Proof. By way of contradiction, suppose that $|\mathcal{F}^*| < n - \sqrt[4]{n^3} + \sqrt[4]{n} - 1$. Then we must have

$$\begin{aligned} |\mathcal{U}| &> \sqrt[4]{n} \times |\mathcal{F} - \mathcal{F}^*| \\ &> \sqrt[4]{n} \times (\sqrt[4]{n^3} - \sqrt[4]{n}) \\ &= n - \sqrt{n}. \end{aligned}$$

This contradicts (5). □

Now consider points of $D_j \cap B$. For $b' \in D_j \cap B$, we use the same notation used above: denote by $\mathcal{Q}_{b'}$ the set of quadrilaterals $[e_l, f_m]$ which are covered with b'. In the same way as in the proof of Lemma 3, the following lemma follows:

Lemma 6. *$\mathcal{Q}_{b'}$ is in the form of $\{[e_l, f_m] \,|\, l_1 \leq l \leq l_1'\}$ or $\{[e_l, f_m] \,|\, l_1 \leq l \leq l_1'\} \cup \{[e_l, f_{m+1}] \,|\, l_2 \leq l \leq l_2'\}$, where $l_2 \geq l_1'$ (Fig. 11).*

Let $\mathcal{S} (\subseteq \mathcal{F}^*)$ be the set of columns $f_m \in \mathcal{F}^*$ which satisfy at least one of the following conditions (i), (ii) or (iii):

(i) $m = 1$;
(ii) f_{m-1} is not contained in $\mathcal{F}^*$;
(iii) no two quadrilaterals $[e_l, f_m]$ and $[e_{l'}, f_{m-1}]$ are covered with a single point $b' \in D_j \cap B$.

Let $s = |\mathcal{S}|$ and write $\mathcal{S} = \{f_{m_1}, f_{m_2}, \ldots, f_{m_s}\}$. Then $\mathcal{F}^*$ is expressed in the following form:

$$\mathcal{F}^* = \{f_{m_1}, f_{m_1+1}, \ldots, f_{m_1'}\} \cup \{f_{m_2}, f_{m_2+1}, \ldots, f_{m_2'}\} \cup \ldots \cup \{f_{m_s}, f_{m_s+1}, \ldots, f_{m_s'}\}. \quad (6)$$

Let

$$\mathcal{F}_t^* = \{f_{m_t}, f_{m_t+1}, \ldots, f_{m_t'}\}, \quad 1 \leq t \leq s$$

(so $\mathcal{F}^* = \mathcal{F}_1^* \cup \mathcal{F}_2^* \cup \ldots \cup \mathcal{F}_s^*$), and let B_t denote the set of points $b' \in D_j \cap B$ each of which covers a quadrilateral of $\{[e_l, f_m] \,|\, 1 \leq l \leq n-1,\ f_m \in \mathcal{F}_t^*\}$. From Lemmas 4, 6 and the definition of $\mathcal{F}_t^*$, the following lemma follows:

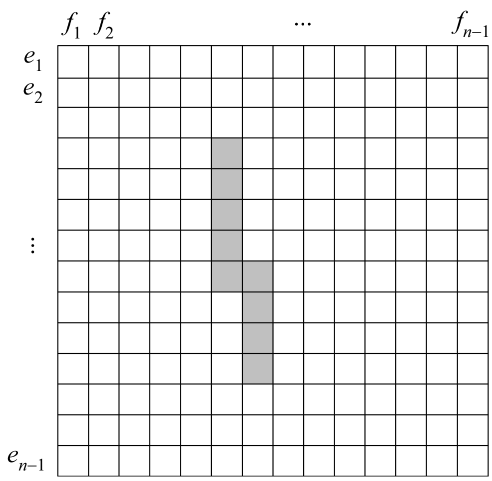

Fig. 11. Components corresponding to $\mathcal{Q}_{b'}$.

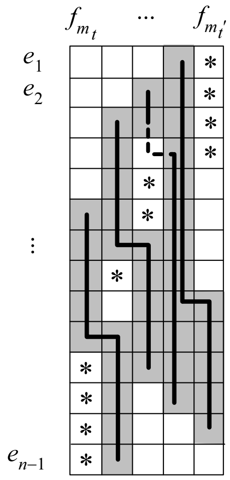

Fig. 12. Components corresponding to $\mathcal{Q}_{b'}$'s.

Lemma 7. $|B_t| \geq |\mathcal{F}_t^*| - 1$ *for* $1 \leq t \leq s$.

Furthermore, the following holds.

Lemma 8. *Let* $1 \leq t \leq s$*, and suppose* $|\mathcal{F}_t^*| \leq \sqrt[4]{n} - 1$*. Then* $|B_t| \geq |\mathcal{F}_t^*|$.

Proof. By way of contradiction, suppose $|B_t| \leq |\mathcal{F}_t^*| - 1$ (so $|B_t| = |\mathcal{F}_t^*| - 1$ by Lemma 7). Then it follows from Lemmas 4, 6 and the definition of $\mathcal{F}_t^*$ again that for each m with $m_t \leq m \leq m_t' - 1$, there exists exactly one point $b' \in B_t$ such that

$$\mathcal{Q}_{b'} = \{[e_l, f_m] \mid l_1 \leq l \leq l_1'\} \cup \{[e_l, f_{m+1}] \mid l_2 \leq l \leq l_2'\}, \tag{7}$$

where l_1, l_1', l_2, l_2' are some positive integers with $l_1 \leq l_1'$, $l_2 \leq l_2'$ and $l_1' \leq l_2$; and furthermore, there exists no point $b'' \in D_j \cap B$ such that $\mathcal{Q}_{b''}$ is expressed in the form $\{[e_l, f_m] \mid l_1 \leq l \leq l_1'\}$.

For $b' \in B_t$ such that $l_1' = l_2$ holds in the expression (7), let $V_{b'} = [e_{l_2}, f_{m+1}]$, and let $\mathcal{V}$ denote the set of all such $V_{b'}$'s. Furthermore, for each l with $1 \leq l \leq n-1$, call the (sub)row consisting of $[e_l, f_{m_t}]$, $[e_l, f_{m_t+1}]$, $\ldots$, $[e_l, f_{m_t'}]$, the *row* e_l (a row of the table shown in Fig. 12). Let $\mathcal{R}$ denote the set of rows e_l containing no element of $\mathcal{V}$. Since $|\mathcal{V}| \leq |B_t| \leq |\mathcal{F}_t^*| - 1$ by assumption, $|\mathcal{R}| \geq (n-1) - |\mathcal{V}| \geq n - |\mathcal{F}_t^*|$. On the other hand, among $|\mathcal{F}_t^*|$ components of each row $e_l \in \mathcal{R}$, at most $|B_t|(\leq |\mathcal{F}_t^*| - 1)$ components correspond to quadrilaterals which are covered with points of B_t. Hence, from each $e_l \in \mathcal{R}$, we can take one component $[e_l, f_m]$ which is not covered with any point of B_t (e.g. each component marked with $*$ in Fig. 12). Let $\mathcal{W}$ be the set of these components. Then

$$|\mathcal{W}| = |\mathcal{R}| \geq n - |\mathcal{F}_t^*|. \tag{8}$$

Since distinct components of $\mathcal{W} - \mathcal{U}$ must be covered with distinct points of $D_i \cap B$, we must have

$$|\mathcal{W} - \mathcal{U}| \leq |D_i \cap B| \leq n - \sqrt{n} \tag{9}$$

from the assumption of Case 2. On the other hand, since the columns f_{m_t}, $\ldots$, $f_{m_t'}$ belong to $\mathcal{F}^*$, it follows from the definition of $\mathcal{F}^*$ and (8) that

$$|\mathcal{W} - \mathcal{U}| \geq (n - |\mathcal{F}_t^*|) - \sqrt[4]{n}\,|\mathcal{F}_t^*|$$

$$
\begin{aligned}
&= n - (\sqrt[4]{n}+1)|\mathcal{F}_t^*| \\
&\geq n - (\sqrt[4]{n}+1)(\sqrt[4]{n}-1) \quad \text{(by the assumption of Lemma 8)} \\
&= n - \sqrt{n} + 1,
\end{aligned}
$$

which contradicts (9). □

Now let $T = \{1, 2, \ldots, s\}$, $T_1 = \{t \in T \mid |\mathcal{F}_t^*| > \sqrt[4]{n} - 1\}$ and $T_2 = T - T_1$. Since

$$
|T_1| < \frac{|\mathcal{F}^*|}{\sqrt[4]{n}-1} \leq \frac{|\mathcal{F}|}{\sqrt[4]{n}-1} \leq \frac{n-1}{\sqrt[4]{n}-1} = O(n^{\frac{3}{4}}), \tag{10}
$$

and since $B_t \cap B_{t'} = \emptyset$ for $1 \leq t < t' \leq s$, it follows from Lemmas 7 and 8 that

$$
\begin{aligned}
|D_j \cap B| &\geq \sum_{t\in T_1} |B_t| + \sum_{t\in T_2} |B_t| \\
&\geq \sum_{t\in T_1} (|\mathcal{F}_t^*| - 1) + \sum_{t\in T_2} |\mathcal{F}_t^*| \\
&= \sum_{t\in T} |\mathcal{F}_t^*| - |T_1| \\
&= |\mathcal{F}^*| - |T_1| \\
&= n + o(n) \qquad \text{(by Lemma 5 and (10))}.
\end{aligned}
$$

From this together with (4), Lemma 2 follows. This completes the proof of Lemma 2.

Now the conclusion of Theorem 5 follows in Case 2 as well from (2), (3) and Lemma 2.

4. Proofs of Theorems 3 and 4

Concerning Theorem 5, note that the same conclusion holds even if we construct the red point set by taking n red points at regular intervals, on each *arc* p_ip_i', $1 \leq i \leq k$, of the unit circle in which the regular k-gon P_0 is inscribed (recall the construction of $R(k, \varepsilon, n)$ stated in Section 3.1). The point set we obtain in this way is in convex position, and hence Theorem 3 holds.

We can construct another red point set with the desired property by placing n red points at regular intervals, around consecutive k vertices of a regular k'-gon, $k' > k$, as shown in Fig. 13 (all the points lie on a circle). Let $R_{k'}(k, \varepsilon, n)$ denote a point set obtained in this way. We now prove Theorem 4. First choose n, and k sufficiently large with respect to n, and K sufficiently large with respect to kn, and construct $R(K, \varepsilon, kn)$ for ε sufficiently small. To obtain the final point set, we replace each point set $\{r_{i,1}, r_{i,2}, \ldots, r_{i,kn}\}$ by a copy of $R_{k'}(k, \varepsilon', n)$ as shown in Fig. 14, where k' is a sufficiently large number with respect to K, so that Lemma 1 can be applied. Denote by R^* the red point set we obtain in this way. Then we have $|R^*| = Kkn$ and if we let $m = kn$ and $M = Kkn (= Km)$,

$$
\beta(R^*) = K(m + o(m)) + (M + o(M)) = 2M + o(M)
$$

as $n \to \infty$, $\frac{k}{n} \to \infty$ and $\frac{K}{kn} \to \infty$ (though not all pairs of adjacent points of $R_{k'}(k, \varepsilon', n)$ are at regular intervals), and hence Theorem 4 holds.

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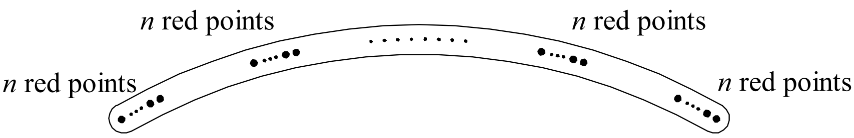

Fig. 13. Red point set R' with $|R'| = kn$.

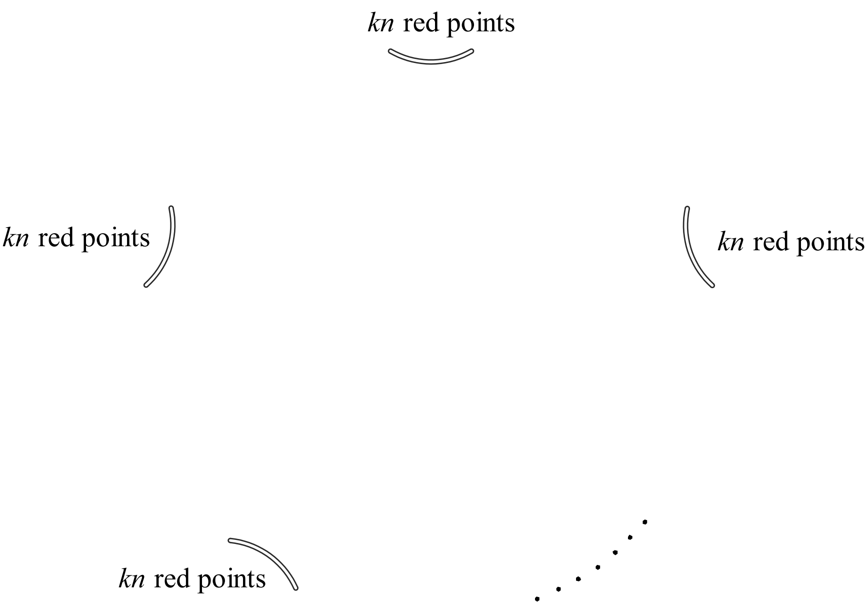

Fig. 14. Red point set R^* with $|R^*| = Kkn$.

References

1. Czyzowicz, J., Kranakis, E., Urrutia, J.: Dissections, cuts and triangulations. in proceedings of 11th Canadian Conference on Computational Geometry, CCCG-99, 154-157, Vancouver Aug. 15-18 (1999)

2. Devillers, O., Hurtado, F., Károlyi, G., Seara C.: Chromatic variants of the Erdős-Szekeres theorem on points in convex position. Computational Geometry: Theory and Applications **26**, 193-208 (2003)

3. Heineken, H.: Regelmassige Vielecke und ihre Diagonalen. Enseignement Math. **2**, ser. 8 (1962) 275-278

4. Kalbfleisch J. D., Kalbfleisch, J. G., Stanton, R. G.: A combinatorial problem on convex n-gons. Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing, 180-188 (1970)

5. Katchalski, M., Meir, A.: On empty triangles determined by points in the plane. Acta Math. Hungar. **51**, 323-328 (1988)

6. Moser, W.: *Research Problems in Discrete Geometry* (McGill Univ., 1981)