CROOKED DIAGRAMS WITH FEW SLOPES

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Abstract. A natural and practical criterion in the preparation of diagrams of ordered sets is to minimize the number of different slopes used for the edges. Any diagram requires at least the maximum number of upper covers (or of lower covers) of any element. While this maximum degree is not always enough we show that it is as long as any edge joining a covering pair may be bent, to produce a crooked diagram.

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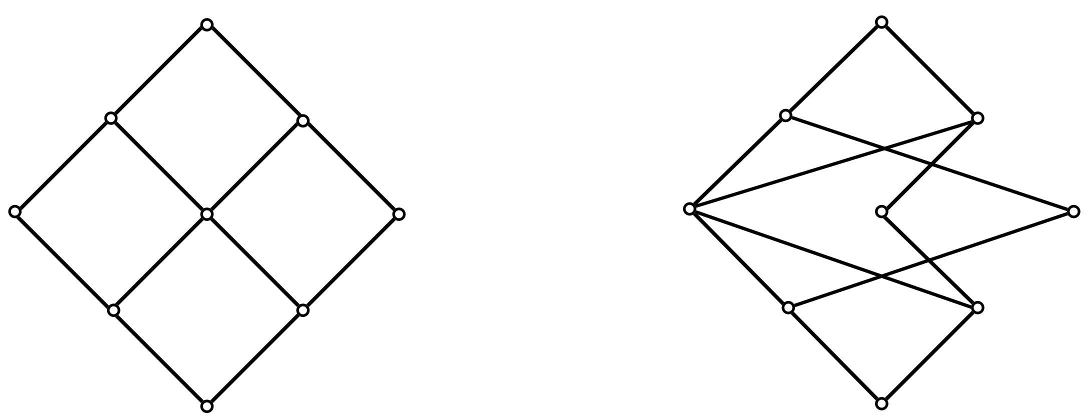
Ordered sets have become widespread in computational problems in scheduling, sorting and searching. One consequence is the increasing interest in efficient data structures to code and store ordered sets. Graphical data structures are particularly useful, too, in human decision-making problems in areas as disparate as social choice or even geography. For ordered sets the most important graphical data structure is the *diagram*, according to which the elements of an ordered set P are represented on the plane by small circles, arranged in such a way that, for a and b in P, the circle corresponding to a is higher than the circle corresponding to b whenever $a > b$ and a monotonic arc (usually a line segment) is drawn to connect them just if a covers b, that is, for each x in P, $a > x \geq b$ implies $x = b$. If a covers b we also write $a >\!\!- b$. Although the diagram determines the order, there is much variation possible in the actual pictorial rendering of the diagram.

What is a "good" diagram? For use as a structure for the presentation of ordered data, the order among the elements must, of course, be readily apparent. Thus, for elements a and b represented by vertices on the plane with different y-coordinates we must decide whether there is a monotonic polygonal path consisting of line segments, from the vertex a to the vertex b. A vertical path may, for instance, be the easiest to discern.

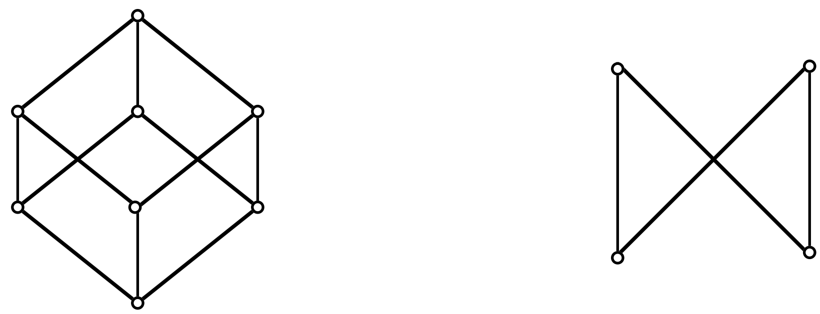


A planar diagram

A non–planar diagram of the same ordered set.

Figure 1

There are certain criteria that we may use to compare the "readability" of diagrams. An obvious one is planarity. Presumably a diagram inwhich line segments never cross, and meet only at vertices, is easier to read (cf. Figure 1). Some ordered sets have no such diagram at all. Indeed, the entire subject of planariuty for ordered sets is well-studied especially for lattices (cf. [3], [6]). Moreover, its application to a dimension theory of ordered sets is deep and surprising (cf [4], [7]).

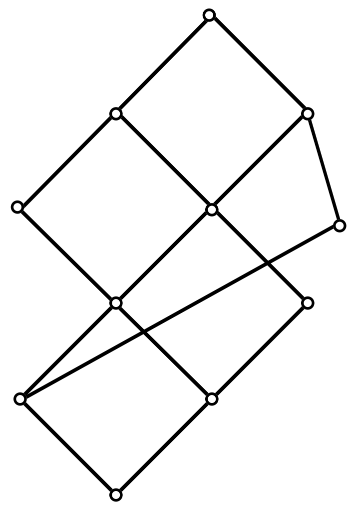


A diagram with three slopes and maximum degree three.
(a)

A diagram with three slopes and maximum degree two.
(b)

Figure 2

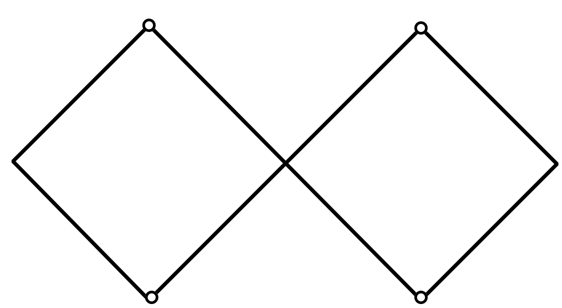
Another quite natural criterion is to use few slopes in drawing the covering edges for the diagram. This may be quite important in comparing diagrams according to their "drawability". Indeed, the steepness of the line segments has for some time remained a preoccupation of diagram drawing schemes. For an element a in P let *down degree* of a mean the number of lower covers of a, that is, the number of x in P such that a covers x. Dually, let *up degree* of a mean the number of upper covers of a. For simplicity let *maximum degree* of P mean the largest value from among the down degrees and up degrees of the elements of P. It is obvious that for any ordered set, the number of different slopes required in a diagram of P is at least maximum degree of P (see Figure 2) and, although this cannot hold generally (cf. Figure 2(b)), B. Sands (1984) had conjectured that the minimum number of slopes required to draw lattices, at any rate, is the maximum degree. Recently, Czyzowicz, Pelc and Rival (1987) showed that this is not true even in the case that the maximum degree is two (see Figure 3). Moreover, Czyzowicz (1987) has constructed lattices of maximum degree n, for all $n > 2$, which cannot be drawn using n slopes, thus confirming a conjecture put forth in [2].



A lattice with maximum degree two
which requires at least three slopes.

Figure 3

These examples notwithstanding, the slope criterion seems to be in wide favour. Our aim in this paper is to show how the simple artifice of introducing "bends" on the line segments joining vertices in the covering relation of the diagram enables us to draw the diagram with only maximum degree slopes. Actually there is already precedent for the idea of "bends", for example, in VLSI circuit design in which a planar graph is presented on a given rectilinear grid (cf. [9], [10]). Thus, for instance, the ordered set illustrated in Figure 2(b) has a diagram using "crooked" edges each with at most one bend in which only two different slopes are ever used for all of the line segments (see Figure 4). Such an artifice, of course, involves relaxing the usual edge constraint, for the covering relation need no longer be represented by a line segment. Comparable vertices a and b will still be located at the ends of a monotonic polygonal path.



A crooked two–slope diagram

Figure 4

Our first main result is that *any finite ordered set has a "one-bend" diagram using maximum degree many slopes.* Thus, every covering edge is constructed using at most two line segments and all line segments are parallel to one of maximum degree many lines. Thus, without using any bends, the ordered set illustrated in Figure 3 requires three slopes, although its maximum degree is two. Using bends, two slopes suffice (see Figure 5).

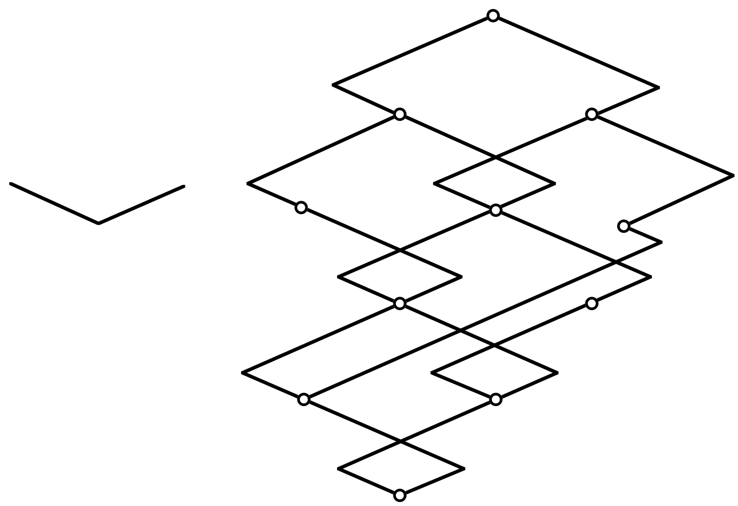


Figure 5

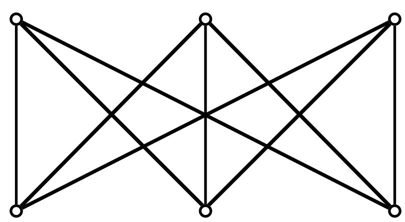
We shall see, too, from the proof of this result, that the complexity of a one-bend, maximum degree–slope drawing is intimately linked to the complexity of edge colourings of bipartite graphs. It will follow that such drawings can be implemented in O(e) where e is the number of edges. Drawing a diagram itself may take as much (cf. [5]).

In implementing such a "crooked" diagram we may well ask whether it can be done starting from any given diagram, keeping its vertices in place, and joining its vertices by appropriate crooked edges. We shall prove that the answer is yes, provided that the maximum degree is even but, that there are diagrams with odd maximum degree, whose vertices must be moved on the plane if they are to be joined by one-bend edges using only maximum degree slopes.

In any case we shall prove that, *for any diagram of a finite ordered set there is a two–bend, maximum degree–slope diagram on the same set of vertices* and the praparation of these diagrams may even be simpler to implement automatically.

Are crooked diagrams useful? It is too early to tell. A drawback, of course, is that covering pairs are not so easily read. On the other hand, comparallel pairs are still the ends of a polygonal monotonic path.

Perhaps the most important open question is this one posed in [2]. Let f(m) be the smallest integer such that any ordered set with maximum degree m has an f(m)–slope diagram. *Is* f(m) *finite? What about* f(2)*?* We know only that $f(m) \geq 2m - 1$. To see this let P_m be the ordered set whose covering graph is the complete, bipartite graph $K_{m,m}$ with bipartition consisting of m maximal and m minimal vertices. Now, in any diagram of P_m there is a "left–most" edge a covering b, say. Then a has m - 1 other lower covers and b has m - 1 other upper covers. It is easy to see that all of the line segments used to represent these covering relations must have different slopes, that is, 2m - 1 different slopes, at least. If we resetrict our attention to lattices then we have no such lower bound. For lattices $f(m) \geq m + 2$, for $m \geq 2$. Indeed, there is some reason to believe that lattices may be different from the general case, for, any lattice whose smallest cycle has 2m elements must have maximum degree at least m (cf. [2]).



A 5–slope diagram

Figure 6

Main Results

We shall make use of this auxiliary result which, however, seems to be of independent interest.

PROPOSITION 1. *The covering pairs of any finite ordered set of maximum degree* k *can be* k*–coloured such that if, either* a >— b *and* a >— c*, or else,* b ≥— a *and* c >—a*, then the edges* (a,b) *and* (a,c) *have different colours.*

Proof. Let P be any finite ordered set with elements $\{p_1,\ldots,p_n\}$ and of maximum degree k. Construct an (undirected) bipartite graph G as follows: the vertices of G are $\{p_1',\ldots, p_n, p_1'',\ldots, p_n''\}$; if p_j is an upper cover of p_i in P then there is an edge in G joining p_i'' and p_j'; G has no other edges. Moreover, every vertex in G has degree at most k. Thus, according to the well–known theorem about the edge colouring of bipartite graphs [ ], there exists a k–colouring of the edges of G, such that edges adjacent to a given vertex have different colours.

Now define a k–colouring of covering pairs of P by this rule. If b >— a, then the pair (a,b) gets the colour of the edge joining a" to b' in the graph G. It is easy to see that this colouring has the required property.

We can now prove our first main result.

THEOREM 2. *Every ordered set of maximum degree* k *has a one-bend, k-slope diagram.*

Proof. We split the proof according to the parity of k. First suppose that k is even, say $k = 2m$. Take any diagram D of P with vertices $p_1,\ldots, p_n$ and let $s_1,\ldots., s_r$ be all the slopes of lines formed by pairs of vertices in this diagram. Select m positive slopes $t_1,\ldots, t_m$ larger than all of the p_i and m negative slopes $t_{m+1},\ldots, t_k$ smaller than all of the s_i. We construct a one-bend diagram D* of P whose vertices are the same as in D and whose crooked edges follow the slopes t_i. Find a k–edge colouring of D, by Proposition 1 such a colouring exists. Let p_j >— p_i and denote by c the colour assigned to this covering pair. Draw the lower half–line with slope t_c and origin p_j and the upper half–line with slope t_{k+1-c} and origin p_i. By the choice of slopes t_i these lines must intersect in a point x not a vertex in D and no other vertex of the D lies on any of them. We plot the polygonal line p_i p_j as the crooked, one–bend edge joining the covering pair (p_i,p_j). It follows from the property of the k–colouring that for distinct upper covers of a vertex in P the respective crooked edges will not have common parts. This concludes the proof in the case that the maximum degree k is even.

If the maximum degree k is odd, the difficulty is with the "central slope". The construction outlined above must be modified. We are no longer able to keep the vertices of the newly constructed diagram in their original positions (the half–lines constructed above might not intersect). Nevertheless, we shall situate the vertices more conveniently and construct a one–bend, k–slope diagram in this case as well.

Let $k = 2m + 1$ and let $(p_1,\ldots, p_n)$ be any linear extension of the ordered set P. Take any set $t_1,\ldots, t_m$ of positive slopes between 1 and 2, as t_{m+1} take the vertical and, take any set $t_{m+1},\ldots,t_k$ of negative slopes between –2 and –1. For any $i \le n$ let I_j

be the unit horizontal interval with centre having coordinates $(0, 2j)$. The intervals I_j form "shelves" on which we will place the vertices of our crooked diagram: p_j will be placed on I_j.

First place all minimal elements on their shelves so that no two of them are on the same vertical. Once an element p_i is already placed consider an upper cover p_j. Let $c \leq k$ be the colour of the covering pair (p_i, p_j) from Proposition 1. If $c = m + 1$ (and hence $t_{k+1-c} = t_c$ is vertical), place p_j on the shelf I_j vertically above p_i and join the vertices p_i and p_j by a vertical segment. If $c \neq m + 1$, place p_j on the shelf I_j avoiding all verticals passing through previously constructed vertices. Draw the lower half–line with slope t_c and origin p_j and the upper half–line with slope t_{k+t-c} and origin p_i. As before, those lines must intersect on a point x and (by the choice of slopes) they do not intersect any shelf. Again p_i p_j is the new crooked, one–bend edge joining the ends of the covering pair (p_i, p_j). This concludes the proof.

In the case that the miximum degree is even, the argument presented above actually proves a stronger result.

THEOREM 3. *Let* D *be any diagram of a finite ordered set of even maximum degree* k. *Then there exists a one–bend,* k*–slope diagram* D* *of it whose vertices are the same as in* D.

Theorem 3 cannot be generalized to include the case of odd maximum degree k. Our next result provides an example of a diagram of an ordered set which cannot be corrected using one–bend, k slopes, as long as the vertices remain fixed. Once vertices may be rearranged, it can be done in light of Theorem 2.

THEOREM 4. *There exists a diagram* D *of a finite ordered set of maximum degree three such that no one–bend, three–slope diagram of it has the same vertices as* D.

Proof. Consider the bipartite graph illustrated in Figure 7.

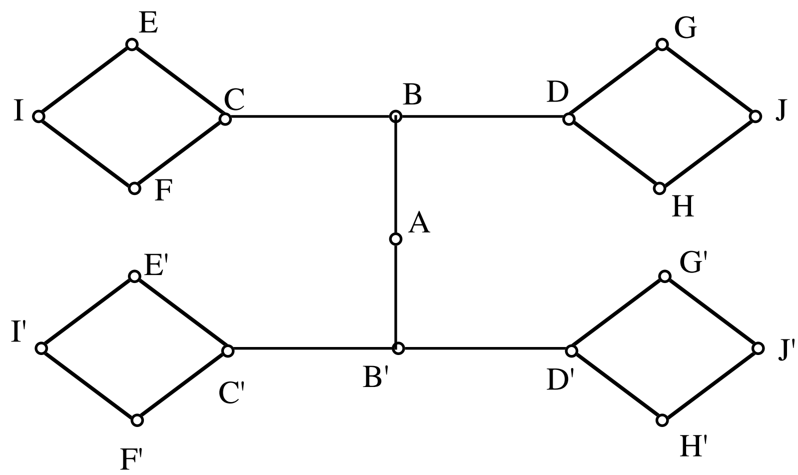


Figure 7

In order to prove the theorem, we consider the diagram illustrated in Figure 8, which is an orientation of this graph. (To keep the figure clear, the lines joining primed points are not drawn.)

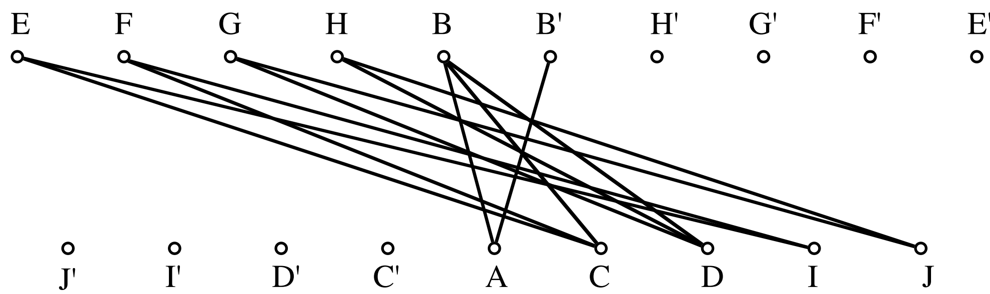


Figure 8

Suppose, by symmetry, that among the three available slopes descending from a point X, the middle one, according to Figure 9, is on the left–hand side of the vertical line passing through the point X.

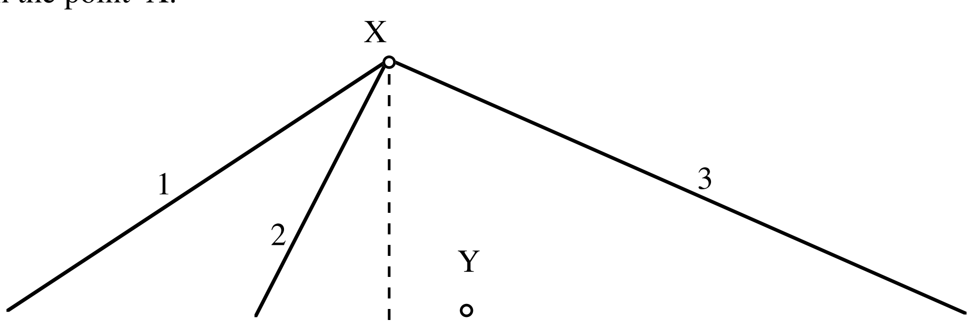


Figure 9

Obviously, any crooked edge using segments drawn down from X along the slopes 1 and 2 will reach only points situated inside the angle between the lines 1 and 2 drawn down from X. Consequently, in order to draw a one–bend, crooked edge joining A with B, (situated, as in Figure 9, between the lines 2 and 3) we have to use the slope 3 (either for the lower or the upper part of the crooked edge).

According to the observation above, when the segments BA, BC and BC are replaced by one–bend crooked edges, every such edge must use the slope 3 for its lower or upper part. The upper part may be drawn along slope 3 only once in these three cases, as all these crooked edges are joined in B. Hence one of the two crooked edges, one of which joins B with C, and the other joining B with D, must use for its lower part the slope 3.

Suppose that C is the vertex for which the part entering it is drawn along the slope 3. As the crooked edges replacing segments CE and CF must also use the slope 3, the upper part of of each of them must be drawn along the slope 3. The vertices E and F are joined with the point I and once more we have to use the slope 3 when plotting crooked edges. As the slope 3 going down from E and F is already occupied and the slope 3 going up from point I may be used just once, this is clearly impossible. This is a contradiction.

The same argument holds when it is the point D which must use the slope 3 for the lower part of the crooked edge joining D with B. We must then use the points G and H instead of E and F, and J instead of I.

In the symmetrical case, when the middle available slope 2 is situated on the right–hand side of the vertical line (as opposed to the case for Figure 3) an identical proof applies to the primed vertices from Figure 1 and Figure 2.

In view of the example above, it is impossible to generalize Theorem 2 to include the case of odd maximum degree, at least for arbitrary finite orders. Nevertheless, we are able to prove such a result from an important class of orders; dismantlable lattices — without restricting the parity of the maximum degree. Before we proceed, a short discussion of dismantlable lattices seems appropriate.

Let L be a finite lattice with at least two elements. Let t stand for the top element and b for the bottom element. Thus, every element x satisfies $b \le x \le t$. An element x satisfying $b < x < t$ has *degree two* just if it has precisely one lower cover $\underline{x}$ and precisely one upper cover x. In algebraic terms such an element is said to be *supremum–irreducible* (for, if $x = u+v$ then either $x = u$ or $x = v$) and, *infimum–irreducible* (for, if $x = u . v$ then either $x = u$ or $x = v$). In particular, $L - \{x\}$

is a sublattice of L, that is, for each $u, v \in L - \{x\}$, $u + v \in L - \{x\}$ and $u . v \in L - \{x\}$. Let D(L) stand for the set of all such elements of the lattice L. Say that L is *dismantlable* if its elements can be enumerated $L = \{x_1,x_2,\ldots,x_n\}$ in such a way that, for each $i \le n-2$,
$x_i \in D(L-\{ x_1,x_2,\ldots,x_{i-1}\})$, $x_i = b$, and, $x_i = t$. Thus, a dismantlable lattice can be decomposed, one element at a time, into a succession of sublattices, each with one less element that before, arriving finally at the two-element sublattice $\{b<t\}$.

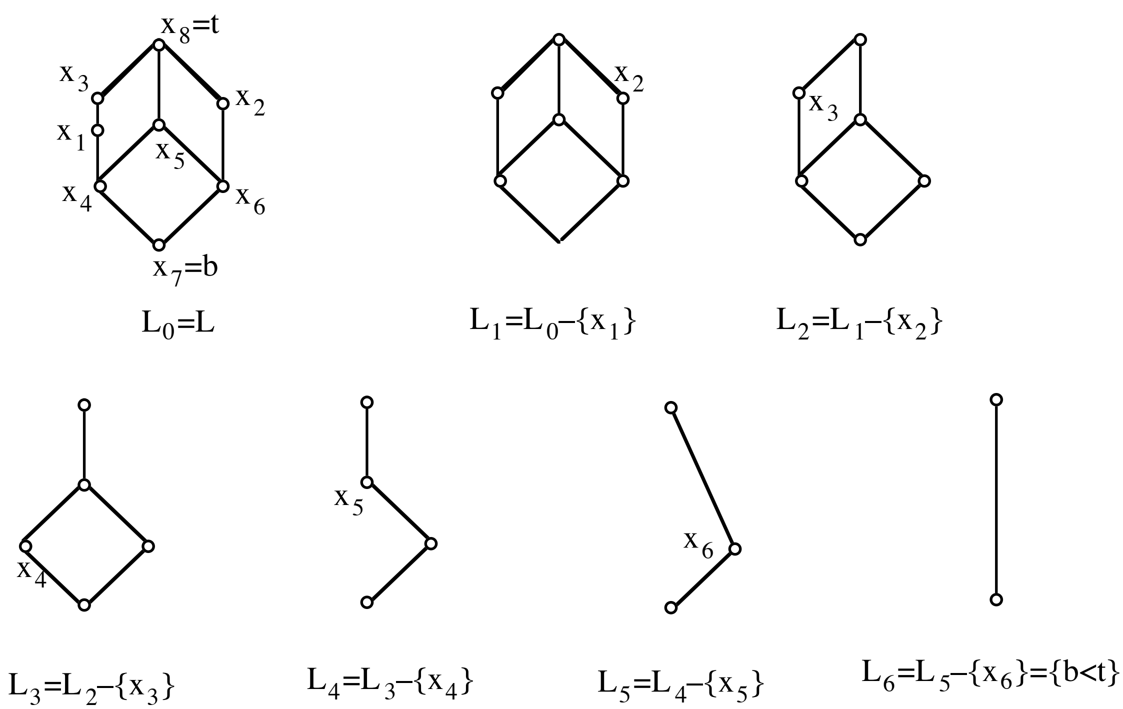


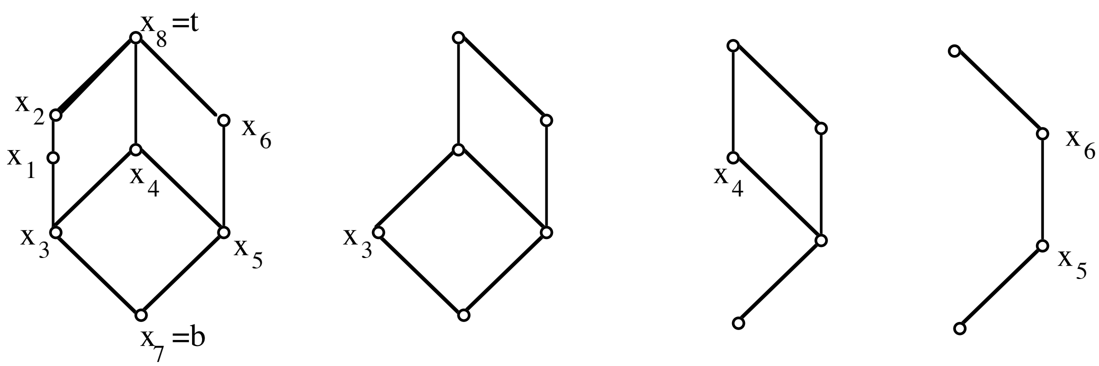
Figure 10

Notice too that the covering edges of any sublattice so obtained need not be actual covering edges of L. Thus in L_1, $x_3 >\!\!- x_4$ although not in L_0; in L_5, $x_8 >\!\!- x_6$ although not in L_4 (and hence not in $L= L_0$) and, of course, $t = x_8 >\!\!- x_7 = b$ at the very last step L_6, although not at any preceding stage.

We will need the following lemma from [ ].

LEMMA 5. *Let* L *be a finite dismantlable lattice. Then there is a a partition* C_1, $C_2,\ldots, C_k$ *of* L *such that, for each* $j \ge 1$, C_i *is a covering chain of* L *each of whose internal vertices has degree two in* $C_1 \cup \ldots \cup C_j$ *and* b, t *are vertices in* C_1.

The point is that the "dismantling" of L can be carried out in such a way that, at each stage, a sublattice is obtained whose diagram is indeed a subdiagram of the original diagram and, which is itself obtained by removing a chain of elements each of degree two, at that stage (see Figure 11). The chains are removed in order $C_k, C_{k-1}, \ldots C_1$.



$L_0 = L$ $L_1 = L_0 - \{x_1 \leq x_2\}$ $L_2 = L_1 - \{x_3\}$ $L_3 = L_2 - \{x_4\}$

Figure 11

THEOREM 5. *Let* D *be any diagram of a finite dismantlable lattice* L *of maximum degree* k. *Then there exists a one–bend,* k*–slope diagram* D* *of* L *whose vertices are the same as in* D.

Proof. Let D be a given diagram of L with vertices $p_1, \ldots, p_n$ and let $s_1, \ldots, s_r$ be all the slopes of lines formed by couples of vertices in this diagram. Choose as t_1 any positive slope larger than all of the $s_1, \ldots, s_r$ and as t_k any negative slope smaller than all of the $s_1, \ldots, s_r$. As $t_2, \ldots, t_{k-1}$ take any distinct slopes between t_1 and t_k, different from all s_i.

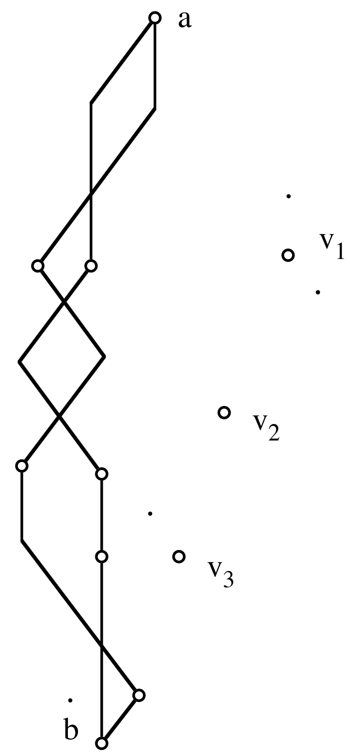


Figure 12

We construct the crooked diagram D^* by consecutively plotting the covering chains C_j between the vertices already placed. Suppose $C_1,\ldots,C_j$ are already plotted following the slopes t_i. We now have to plot the chain C_{j+1} with endpoints $a > b$ and internal vertices $v_1,\ldots,v_m$ already placed. Let t_i be a (possibly unique at this time) free slope downward from a. If t_i is positive, the lower half–line with slope t_i and origin a must intersect the upper half–line with origin v_i and slope t_1 in a point x. If t_i is negative, t_k instead of t_1 will do. Since all slopes from v_1 are free at this time, we can plot the polygonal line axv_1 as the crooked edge joining a and v_1. A similar argument works for the covering pair (b,v_m). For covering pairs (v_i,v_{i+1}) this is even simpler: since all slopes down from v_i and up from v_{i+1} are free, we can always use, say, the extremal slopes t_1 and t_k for each crooked edge. Thus all the chains C_j can be plotted inductively and the crooked diagram D^* is ready.

Although we are not able to produce a one–bend, k–slope diagram with the same vertices as in a given diagram D of an arbitrary ordered set P of maximum degree k, our next result shows that two–bend edges are always sufficent.

THEOREM 6. *Let* D *be any diagram of an ordered set* P *of maximum degree* k. *Then* *there exists a two–bend,* k*–slope diagram* D* *of* P *whose vertices are the same as in* D.

Proof. Let D be a given diagram of P. Choose slopes $t_1, \ldots, t_k$ as in the proof of Theorem 2. We plot the two–bend edges one by one. Suppose that $a > b$ is a covering pair and that the slopes t_1 downward from a and t_1 upward from b are "free". Let H_u' be the lower half–lines with slopes t_u and origin a and H_u'' the upper half–lines with slopes t_u and origin b. Denote by R the parallelogram with sides H_1', H_k', H_1'', H_k''. If the half–lines H_i' and H_j'' intersect in a point q inside R then plot the ploygonal line aqb (with one bend) as the crooked edge joining the covering pair (a,b). If not, (that is H_i' and H_j'' intersect outside of R or else are parallel), then either H_j'' intersects H_1' and H_i' intersects H_1'' (see Figure 13) or H_j'' intersects H_k' and H_i' intersects H_k''. In the first case there are infinitely many couples (x,y) of points such that $x \in H_i'$, $y \in H_j''$ and the line xy has slope t_1. In the second case there are infinietly many such couples with xy of slope t_k. In each case we choose x, y so that no vertex of the diagram lies on the line xy. The polygonal line axyb (with two bends) is as required and can be plotted as the crooked edge joining the covering pair (a,b). This concludes the proof.

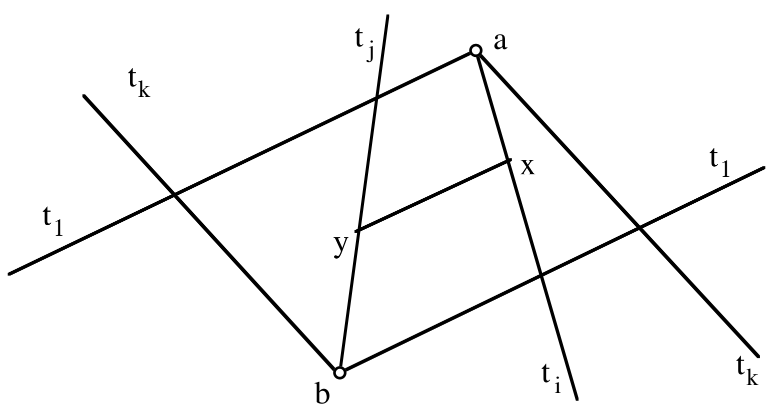


Figure 13

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