GEOMETRIC CONTAINMENT AND PARTIAL ORDERS

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Abstract

Given two solid geometric figures on the plane (eg. rectangles) we say that A fits in B (denoted $A < B$) if there is a translation, a rotation and (if needed) a reflection that maps A into B. Given a family $\mathcal{F}$ of geometric figures, we say that the relation "<" on the elements of $\mathcal{F}$ is reducible to vector dominance if there exists an n and a mapping $f:\mathcal{F} \to \mathbb{R}^n$ such that for $A,B \in F$ $A < B$ iff $f(A) < f(B)$ coordinate by coordinate. A recent result states that if $\mathcal{F}$ is the set of all rectangles, "<" is not reducible to a vector dominance relation regardless of the finite value of n. In this paper we extend this result to other families of geometric figures and to a partial order obtained from quadratic polynomials.

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1. Introduction

In recent years, the study of relationships between geometry and partial orders has attracted the attention and the interest of many researchers from different fields, as witnessed by the large number of results on the subject published in the last few years (for a survey, see [16]). Depending on whether the geometric objects under considerations are fixed in the plane (the "static" case) or can be moved in the plane through translation, rotation or even reflection (the "dynamic" case), different problems arise and have been studied. The majority of the investigations in both cases have focused on "simple" geometric figures such as rectangles [1,7,11,12,16], polygons [3,13,15,16], circles [2,14,15,16], angular regions [5,6,7,13,16], etc.

In this paper, we continue the investigation of the "dynamic" case started in [12]. Specifically, the following question will interest us here :

Given a class of geometric figures, how many real variables are required to parameterize the class in such a fashion that one figure from the class is contained in another (perhaps after translation, rotation and even reflection) if and only if the parameter values for the first figure are no greater than those for the second figure? In particular, when will finitely many parameters suffice?

or equivalently:

Is it possible to reduce geometric containment to vector dominance?

The problem of determining whether a figure is contained in another is of principal interest in computational geometry; thus, an answer to the above questions would be of immediate practical relevance due to the existence of efficient computational methods for determining dominance relationships among vectors [8,9]. Furthermore, reduction to vector dominance has already been successfully employed to solve other basic geometric problems [4,10,17]. Also, it is not hard to visualize applications of positive results to packing problems as well as others.

For some families **H** of figures, this reduction can be easily accomplished. For example, $f(A)$ defined as the area of A, will work for the family $\mathbf{P}_k$ of regular polygons with $k \geq 3$ sides, as well as for the family **C** of circles. A more interesting example is the family **E** of ellipses: for each $E_i \in \mathbf{E}$, define $f(E_i)=(x_i,y_i)$ where x_i and y_i denote the length of the minor and major axis of E_i, respectively; it is an easy to show that E_i can be contained in E_k if and only if $f(E_i) \leq f(E_k)$. Similiarly, two parameters (namely, the lengths of the diagonals) suffice also for rhombi.

In the opposite direction, it was shown that finitely many parameters do not suffice for plane rectangles [12]; this result also implies that a finite reduction does not exist for convex polygons with at least $k \geq 4$ sides. On the other hand, in the same paper it was also shown that a reduction is possible using a countable number of parameters.

Many families of figures commonly considered do have "natural" finite parameterizations, but these may not faithfully reflect the containment relation. Typically these parameterizations do possess natural monotonicity and homogeneity properties (example: length and width for rectangles). In section 2 parameterizations with these properties (denoted (M) and (H)) are studied, and used to provide an abstract version of the rectangle theorem; it is then shown how the proof of the (negative) rectangle theorem in [12] can be formulated as an instance of the abstract result. In section 3 the negative results are extended from rectangles to some other classes of figures, including right circular cylinders and isosceles triangles; in particular, several instances are given using a known low-dimensional negative result (for instance, rectangles) to obtain a new higher-dimensional one (for instance, cylinders). In section 4 the abstract theorem is used to show that a certain natural algebraic partially ordered set cannot be faithfully represented by finitely many parameters. Finally, section 5 echoes section 4 of [12], displaying a representation of the family of (congruence classes of) non-empty compact (= closed and bounded) sets in $\mathbb{R}^k$ by countably many parameters which are continuous in an appropriate sense.

2. <u>Preliminaries, and the abstract theorem.</u>

If $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ are vectors in $\mathbb{R}^I$ for some non-empty index set I, write $\mathbf{x} \leq \mathbf{y}$ provided $x_i \leq y_i$ for all $i \in I$; $\leq$ is a partial order known as "vector dominance" or "Pareto dominance".

Let $\underline{P} = (P, \preceq)$ be any partially ordered set (or poset). Thus P is a non-empty set and $\preceq$ is a transitive binary relation on P such that for elements $\mathbf{a}$, $\mathbf{b}$ of P, $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b} \preceq \mathbf{a}$ if and only if $\mathbf{a}=\mathbf{b}$. Any injection $j: P \to \mathbb{R}^I$ induces a partial order $\preceq_j$ on $j(P)=\{j(\mathbf{a}):\mathbf{a} \in P\}$ by the rule $j(\mathbf{a}) \preceq_j j(\mathbf{b})$ iff $\mathbf{a} \preceq \mathbf{b}$. If $\preceq_j$ coincides with the restriction of $\leq$ to j(P), we say that j reduces $\preceq$ to vector dominance in $\mathbb{R}^I$. It is easy to see that any partial order can be reduced to vector dominance in $\mathbb{R}^I$ for some (possibly infinite) index set I. Indeed, if $\underline{P} = (P, \preceq)$ is any poset, take I = P and define $j: P \to \{0,1\}^I \subseteq \mathbb{R}^I$ by $j(\mathbf{a})_i = 0$ if $\mathbf{a} \preceq \mathbf{i}$ and $j(\mathbf{a})_i = 1$ otherwise; clearly $\mathbf{a} \preceq \mathbf{b}$ implies $j(\mathbf{a}) \leq j(\mathbf{b})$, while if $\mathbf{a} \preceq \mathbf{b}$ is false, then $j(\mathbf{a})_{\mathbf{b}} = 1$ while $j(\mathbf{b})_{\mathbf{b}} = 0$, so $j(\mathbf{a}) \leq j(\mathbf{b})$ is false. Note that the same construction of j will work if instead of I = P we take I = S for any subset S of P which is separating in the sense that

a≠**b** implies {**i** □ S: **a** □**i**} ≠ {**i** □ S: **b** □ **i**} ; this observation will be of some use later.

Congruence is an equivalence relation on the class of non-empty subsets of $\mathbb{R}^k$. Typically, our poset P = (P,□) will be designed so that each element **a** of P is a congruence class of compact subsets of some fixed $\mathbb{R}^k$, and **a** □**b** means that some representative for **a** is a subset (in $\mathbb{R}^k$) of some representative for **b**; this usage differs slightly from that in [12], but would not if in [12] rectangles and ellipses had been defined so as to include their interiors (and not just their boundary curves). This partial order □will be denoted □ and called "containment" (for the family P). Thus our main problem can be stated as follows: When can the containment partial order be reduced to vector dominance in $\mathbb{R}^k$ for some finite k?

We shall not in this paper explore "containment" using various natural sub-equivalence relations of congruence obtained by starting with a closed subgroup G of the group of isometries of $\mathbb{R}^k$; for instance, one choice of G leads to orientation-preserving congruence (which changes nothing for most of the classes of figures we shall study).

We now state the abstract theorem. Fix a positive integer k and a cone K in

$$\mathbb{R}^k_+ = \{\mathbf{x} = (x_1,...,x_k) \ \square\ \mathbb{R}^k : x_i > 0 \ i=1,...,k\} \ .$$

(That K is a cone means that K ≠ ø and t **x** □ K whenever **x** □ K and t > 0). We consider a partial order □ on K which satisfies the following monotonicity and homogeneity hypotheses:

(M) **x** □ K, **y** □ K, **x** □ **y** together imply **x** □ **y**.

(H) **x** □ K, **y** □ K, **x** □**y**, t > 0 together imply t**x** □t**y**.

<u>Theorem 1.</u> Suppose that K is a cone in $\mathbb{R}^k$ and □ is a partial order on K that satisfies (M) and (H). Suppose that there are distinct points **z**, **w** in K and sequences ($\mathbf{x}^{(n)}$), ($\mathbf{y}^{(n)}$) in K such that:

(1.1) $\mathbf{x}^{(n)}$ □**w** and **z** □ $\mathbf{y}^{(n)}$ □ **w** for all n.

(1.2) $\mathbf{x}^{(n)}$ □$\mathbf{y}^{(n)}$ is false for all n.

(1.3) $\mathbf{x}^{(n)} \to \mathbf{z}$ and $\mathbf{y}^{(n)} \to \mathbf{w}$ in $\mathbb{R}^k$ as $n \to \infty$.

Then $\prec$ is not reducible to vector dominance in $\mathbb{R}^m$ for any finite m.

<u>Proof</u>. Suppose, to obtain a contradiction, that $j = (f_1,\ldots,f_m)\colon K \to \mathbb{R}^m$ reduced $\prec$ to vector dominance in $\mathbb{R}^m$ for some finite m. Each of the 2m functions $t \to f_i(t\mathbf{z})$, $t \to f_i(t\mathbf{w})$ is non-decreasing for $0 < t < \infty$ by (M), so they have a common point of continuity t_0. By (H) we may replace $\mathbf{x}^{(n)}$, $\mathbf{y}^{(n)}$, $\mathbf{z}$, $\mathbf{w}$ by $t_0\mathbf{x}^{(n)}$, $t_0\mathbf{y}^{(n)}$, $t_0\mathbf{z}$, $t_0\mathbf{w}$ respectively without affecting the hypotheses, so we may suppose $t_0 = 1$. Given a positive number ε, there is a positive number $\delta < 1$ such that $|t-1| \leq \delta$ implies $|f_i(t\mathbf{z}) - f_i(\mathbf{z})| < \varepsilon$ and $|f_i(t\mathbf{w}) - f_i(\mathbf{w})| < \varepsilon$ for all i. Thus by (M)

$$U = \{\mathbf{x} \in \mathbb{R}^k\colon (1-\delta)\mathbf{z} \leq \mathbf{x} \leq (1+\delta)\mathbf{z}\}$$

and

$$V = \{\mathbf{y} \in \mathbb{R}^k\colon (1-\delta)\mathbf{w} \leq \mathbf{y} \leq (1+\delta)\mathbf{w}\}$$

are neighborhoods of $\mathbf{z}$ and $\mathbf{w}$ respectively in $\mathbb{R}^k$ such that $\mathbf{x} \in U \cap K$ implies $|f_i(\mathbf{x}) - f_i(\mathbf{z})| < \varepsilon$ for all i, and $\mathbf{y} \in V \cap K$ implies $|f_i(\mathbf{y}) - f_i(\mathbf{w})| < \varepsilon$ for all i. In other words, $\mathbf{z}$ and $\mathbf{w}$ are points of continuity of each f_i as a function of a (k-dimensional) variable from K.

By (1.1) and the properties of j,

$$f_i(\mathbf{x}^{(n)}) \leq f_i(\mathbf{w}) \text{ and } f_i(\mathbf{z}) \leq f_i(\mathbf{y}^{(n)}) \leq f_i(\mathbf{w})$$

for all i and n. Since $\mathbf{z} \neq \mathbf{w}$, $j(\mathbf{z}) \neq j(\mathbf{w})$, so there is a non-empty set A of indices in $\{1,\ldots,m\}$ such that

$$f_i(\mathbf{z}) < f_i(\mathbf{w}) \qquad \forall\ i \in A$$

$$f_i(\mathbf{z}) = f_i(\mathbf{w}) \qquad \forall\ i \in \{1,\ldots,m\}\backslash A.$$

By this and (1.1)

$$(1.4)\quad f_i(\mathbf{x}^{(n)}) \le f_i(\mathbf{w}) = f_i(\mathbf{y}^{(n)}) = f_i(\mathbf{z}) \qquad i \in \{1,\dots,m\}\backslash A$$

holds for every n. Let $2\varepsilon = \min\{|f_i(\mathbf{w}) - f_i(\mathbf{z})|: i \in A\} > 0$. By (1.3) and continuity of each f_i at $\mathbf{z}$ and at $\mathbf{w}$, if n is large enough we have

$$|f_i(\mathbf{x}^{(n)}) - f_i(\mathbf{z})| < \varepsilon \text{ and } |f_i(\mathbf{y}^{(n)}) - f_i(\mathbf{w})| < \varepsilon \quad \forall\, i,$$

so if $i \in A$ then

$$f_i(\mathbf{x}^{(n)}) - f_i(\mathbf{y}^{(n)}) < (f_i(\mathbf{z}) + \varepsilon) - (f_i(\mathbf{w}) - \varepsilon) = 2\varepsilon - (f_i(\mathbf{w}) - f_i(\mathbf{z})) \le 0,$$

so $f_i(\mathbf{x}^{(n)}) < f_i(\mathbf{y}^{(n)})$; with (1.4) this shows that $j(\mathbf{x}^{(n)}) \le j(\mathbf{y}^{(n)})$, that is, $\mathbf{x}^{(n)} \preceq \mathbf{y}^{(n)}$ for large n, contradicting (1.2).

$\square$

We shall now outline the proof of the rectangle containment theorem in [12] in such a way that Theorem 1 is the central device; this will provide a model for our proofs below. By a "rectangle" we mean the congruence class $\mathbf{R}$ of $\{(x,y) \in \mathbb{R}^2: 0 \le x \le W,\ 0 \le y \le L\}$ for some point $j_{\mathcal{R}}(R) = (W,L) \in \mathbb{R}^2_+$ with $W \le L$; $\mathcal{R}$ denotes the set of all rectangles. The map $j_{\mathcal{R}}: \mathcal{R} \to \mathbb{R}^2$ converts containment $\sqsubseteq$ on $\mathcal{R}$ into a partial order $\preceq_{\mathcal{R}} = \preceq_{j_{\mathcal{R}}}$ on the cone $K = \{(W,L) \in \mathbb{R}^2_+ : W \le L\}$, and $\preceq = \preceq_{\mathcal{R}}$ clearly satisfies (M) and (H). It only remains to provide points in K which satisfy (1.1)-(1.3). This is accomplished by analyzing the "containment curve" of a square. In our language, take any positive number S and let $\mathbf{z} = ((\sqrt{2}-1)S, S)$, $\mathbf{w} = (S,S)$. Among the points $\mathbf{x} \in K$ which satisfy $\mathbf{x} \preceq_{\mathcal{R}} \mathbf{w}$ but $\mathbf{x} \ne \mathbf{w}$, the $\preceq_{\mathcal{R}}$-maximal ones turn out to be the points $\mathbf{x}_u = (u, \sqrt{2}S - u)$ for $0 < u < (\sqrt{2}-1)S$, and in fact $\mathbf{x} \preceq_{\mathcal{R}} \mathbf{w}$ is equivalent to $\mathbf{x} \le \mathbf{x}_u$ for some u or $\mathbf{x} \le \mathbf{w}$. In particular, if $0 < u^{(n)} < (\sqrt{2}-1)S \le v^{(n)} < S$, $u^{(n)} \to (\sqrt{2}-1)S$ and $v^{(n)} \to S$, we may take $\mathbf{x}^{(n)} = (u^{(n)}, \sqrt{2}S - u^{(n)})$ and $\mathbf{y}^{(n)} = (v^{(n)}, S)$ to get (1.1)-(1.3).

3. <u>Application to other figures.</u>

We wish to apply Theorem 1 to several additional classes of figures (besides rectangles), each exhibiting somewhat different features. We begin with cylinders.

<u>Theorem 2</u>. Containment for (congruence classes of) right circular cylinders in $\mathbb{R}^3$ is not reducible to vector dominance in $\mathbb{R}^m$ for any finite m.

Proof. The theorem will follow from the stronger result, which we will state more formally and prove below, that containment for rectangles and containment for cylinders induce the same partial order on the cone $K = \{(W,L) \square \mathbb{R}^2_+ : W \leq L\}$. By a "right circular cylinder" we mean the congruence class C of $\{(x,y,z) \square \mathbb{R}^3: x^2 + y^2 \leq (W/2)^2,\ |z| \leq L/2\}$ for some point $J_\square(C) = (W,L) \square R^2_+$; $\square$ is the set of such C. The map $j_\square : \square\square\ \mathbb{R}^2$ converts containment $\square$ on $\square$ into a partial order $\square_\square = \square_j$ on $\mathbb{R}^2_+$, and $\square = \square_\square$ satisfies (M) and (H). We shall prove:

(2.1) The restriction of $\square_\square$ to K coincides with $\square_\square$.

Let $(W_i, L_i) \square$ K be given (i=1,2).

First, suppose $(W_1, L_1) \square_\square (W_2, L_2)$. Let $\square$ be a representative for $j_\square^{-1} (W_i, L_i)$ (i=1,2) such that $\square_1 \square \square_2$. We may assume that the central axes of $\square_1$ and $\square_2$ both lie in planes parallel to
the xy-plane. Let $\square_i$ be the set of points in the xy-plane obtained by projecting all the points of $\square$ perpendicularly onto the xy-plane (i=1,2). Then $\square_1 \square \square_2$ and $\square_i$ is a representative for $j_\square^{-1} (W_i, L_i)$, so $(W_1, L_1) \square_\square (W_2, L_2)$.

Conversely, suppose $(W_1, L_1) \square_\square (W_2, L_2)$. We may suppose that (W_1, L_1) is $\square$-maximal in K with respect to this property, that is, that $(W,L) \square$ K, $(W,L) \square_\square (W_2, L_2)$ and $(W_1, L_1) \square (W, L)$ imply $(W, L) = (W_1, L_1)$. From [12] $W_1 \leq W_2$. If $(W_1, L_1) = (W_2, L_2)$ there is nothing to prove, so we may assume $(W_1, L_1) \neq (W_2, L_2)$, hence $\square$-maximality of (W_1, L_1) forces $L_1 > L_2$, then $W_1 < W_2$. Let $\square_i$ be a representative of $j_\square^{-1} (W_i, L_i)$ (i=1,2) such that $\square_1 \square \square_2$. We may assume that the four vertices of $\square_2$ lie at $(\pm W_2/2, \pm L_2/2)$. By [12] the interior of each edge of $\square_2$ contains one vertex of $\square_1$, hence $\square_1$ (like $\square_2$) is centered at the origin. We may suppose that the midpoints of the short (length W_1) sides of $\square_1$ lie in (the interior of) the first and third quadrants, so on a line $l_\square = \{(x,y) \square \mathbb{R}^2: y \cos \square = x \sin\square\}$ for some $\square$,
$0 < \square < \square/2$. Let $\square_1$ be obtained by rotating $\square_1$ about $l_\square$, and $\square_2$ by rotating $\square_2$ about the y-axis. Then $\square$ is a representative for $j_\square^{-1} (W_i, L_i)$, and we need only check that $\square_1 \square \square_2$. While this may seem geometrically obvious, it does deserve a proof, since it genuinely depends on the particular nature of the inclusion $\square_1 \square \square_2$.

First, computing the x-coordinate of the right-most vertex of $\square_1$ and the y-coordinate of the top-most vertex of $\square_1$ gives the equations

$$(L_1/2)\cos\theta + (W_1/2)\sin\theta = W_2/2$$

$$(L_1/2)\sin\theta + (W_1/2)\cos\theta = L_2/2.$$

Solving for $\cos\theta$ and $\sin\theta$ gives

$$(2.2)\quad \cos\theta = \frac{W_2 L_1 - W_1 L_2}{L_1^2 - W_1^2}\ , \qquad \sin\theta = \frac{L_1 L_2 - W_1 W_2}{L_1^2 - W_1^2}\ .$$

A typical point of C_1 has the form

$$(x,y,z) = (t\cos\theta + u\sin\theta,\ t\sin\theta - u\cos\theta,\ v)$$

where

$$(2.3)\quad |t| \le L_1/2, \quad u^2 + v^2 \le (W_1/2)^2.$$

To prove that $(x,y,z) \in C_2$ amounts to showing that if (2.3) holds then

$$(2.4)\quad |t\sin\theta - u\cos\theta| \le L_2/2,$$

$$(2.5)\quad (t\cos\theta + u\sin\theta)^2 + v^2 \le (W_2/2)^2.$$

$|t\sin\theta - u\cos\theta| \le |t|\sin\theta + |u|\cos\theta \le (L_1/2)\sin\theta + (W_1/2)\cos\theta = L_2/2$ gives (2.4). For (2.5), begin with

$$(t\cos\theta + u\sin\theta)^2 + v^2 \le (|t|\cos\theta + |u|\sin\theta)^2 + (W_1/2)^2 - u^2$$

$$= t^2\cos^2\theta + 2|t||u|\cos\theta\sin\theta - u^2\cos^2\theta + (W_1/2)^2$$

$$\le (L_1/2)^2\cos^2\theta + 2(L_1/2)|u|\cos\theta\sin\theta - u^2\cos^2\theta + (W_1/2)^2.$$

Varying $|u|$, this increases until $|u| = (L_1/2)\tan\theta$, that is (using (2.2)) until $|u| = (L_1/2)[(L_1L_2-W_1W_2)/(W_2L_1-W_1L_2)]$, which is at least as great as $W_1/2$. Since $|u| \le W_1/2$ by (2.3), we have

$$(t \cos□ + u \sin□)^2 + v^2 \leq$$

$$(L_1/2)^2 \cos^2□ + 2(L_1/2)(W_1/2) \cos□ \sin□ - (W_1/2)^2 \cos^2□ + (W_1/2)^2 =$$

$$((L_1/2) \cos□ + (W_1/2) \sin□)^2 = (W_2/2)^2$$

proving (2.5), hence (2.1) and the theorem.

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It is possible to prove the theorem with a little more economy by not proving the full strength of (2.1). We shall now consider triangles.

Theorem 3. Containment for (congruence classes of) isosceles triangles in $\mathbb{R}^2$ is not reducible to vector dominance in $\mathbb{R}^m$ for any finite m.

Proof. By an "isosceles triangle" we mean the congruence class of T = {(x,y) □ $\mathbb{R}^2$: $0 \leq y \leq H$, □x□ $\leq (W/2)(1- y/H)$} for some point $j_□(T) = (W,H)$ □ $\mathbb{R}^2_+$; □ is the set of isosceles triangles, and $j_□$: □□ $\mathbb{R}^2$ converts containment □ on □ into a partial order $□_□ = K_{j□}$ on $\mathbb{R}^2_+$ which satisfies (M) and (H). We shall show that $□_□$ satisfies the hypotheses of Theorem 1, so is not reducible to vector dominance in any $\mathbb{R}^m$, by examining the "containment curve" for an equilateral triangle.

Fix $S > 0$ and let $\mathbf{w} = (S, (\sqrt{3}/2)S)$, so $j_□^{-1}(\mathbf{w})$ is the congruence class of equilateral triangles of side S. For $0 < W \leq S$ let $h(W) = \max \{H : (W,H)\ □_□\ \mathbf{w}\}$, so $(W, h(W))\ □_□\ \mathbf{w}$ and $(\sqrt{3}/2) S \leq h(W) \leq S$. Let $W_0 = \sqrt{3}\, S \tan □/18$. Lengthy but unenlightening computations show that h(W) has one of the following two forms, depending on the value of W:

(3.1) If $W = u \leq W_0$ then

$$h(W) = (\sqrt{3}/2)\, S \cos□/\cos(□/6 - 2□)$$

where □ = □(u) satisfies $0 <$ □ $\leq$ □/18 and

$u = \sqrt{3}\, S \sin □/\cos (□/6 - 2□)$.

(3.2) If $W = v \geq W_0$ then $h(W) = (\sqrt{3}/2)S$.

If $W = W_0$ the two formulae for $h(W)$ agree. In case (3.1) □(u) is half the "odd" angle of $j_□^{-1}$ (W,h(W)), and increases with W=u. (The form of any inclusion $□_w$ □ $□_S$ of a representative for $j_□^{-1}$(W,h(W)) in a representative for $j_□^{-1}$(**w**) is easily specified: the "odd" vertex of $□_W$ must coincide with a vertex of $□_S$, and its opposite side in case (3.2), or one of its adjacent sides in case (3.1), must lie along a side of $□_S$).

Let $\mathbf{x}_u$ = (W,h(W)) in case (3.1), let $\mathbf{y}_v$ = (W,h(W)) in case (3.2), and let $\mathbf{z} = \mathbf{x}_{W0} = \mathbf{y}_{W0}$. Assuming $u \neq W_0$, one verifies that $\mathbf{x}_u\ □_□\ \mathbf{x}\ □_□\ \mathbf{w}$ iff $\mathbf{x} = \mathbf{x}_u$ or $\mathbf{x} = \mathbf{w}$ or $u = W_0$ and $\mathbf{x} = \mathbf{y}_v$ for some v, and that $\mathbf{y}_v\ □_□\ \mathbf{x}\ □_□\ \mathbf{w}$ iff $\mathbf{x} = \mathbf{y}_{v'}$, for some $v' \geq v$. Thus taking sequences $(u^{(n)})$, $(v^{(n)})$ with $0 < u^{(n)} < W_0 \leq v^{(n)} < S$, $u^{(n)}$ □ W_0, and $v^{(n)}$ □ S, we may take $\mathbf{x}^{(n)} = \mathbf{x}_{u(n)}$ and $\mathbf{y}^{(n)} = \mathbf{y}_{v(n)}$ to get (1.1)-(1.3).

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We shall close this section with some remarks on lifting negative results from low dimensions to higher dimensions. For example, does the fact that containment of rectangles is not reducible to vector dominance in any $\mathbb{R}^m$ imply a corresponding result for rectangular boxes in $\mathbb{R}^3$? The simplest approach to the box problem seems to use the "local" character of Theorem 1 and the rectangle result: only a small portion of the cone $K = \{(W,L)$ □ $\mathbb{R}^2_+ : W \leq L\}$ is actually needed, and the widths W in this portion are bounded away from 0.

<u>Proposition</u>. Let □ be a non-empty family of congruence classes of sets in $\mathbb{R}^k$ for some finite k. Suppose there is a positive number □ such that, for every line L in $\mathbb{R}^k$, every representative of every member of □ contains a segment of length □ parallel to L. Fix d, 0 < d < □. For F □ □ let F denote the congruence class of $□_F$ [0,d] in $\mathbb{R}^{k+1}$, where $□_F$ is any representative for F. Then if F_i □□ (i=1,2), we have F_1 □ F_2 iff F_1 □ F_2.

The point here is that a rectangle with sides d and □ can be placed in one with sides d and □>□ only with the sides of length □ lying along those of length □. This forces any embedding of a representative of F_1 in a representative of F_2 to be induced in the obvious way by an embedding of a represeentative of F_1 in a representative of F_2. It is clear how the proposition permits one to prove the box result from the rectangle theorem.

4. <u>An example: quadratic polynomials</u>.

Not all interesting consequences of Theorem 1 involve geometric containment. A natural partial order □ is given on the set of all polynomials with real coefficients by declaring that P_1 □ P_2 provided $P_1(x) \leq P_2(x)$ for all non-negative real numbers x. It is easy to see that the restriction of □ to the class of linear polynomials is reducible to vector dominance in $\mathbb{R}^2$: associating the point j(P) = (A,B) to P(x) = Ax + B does the job. It is perhaps surprising that this result does not extend one step further to the class Q of quadratic polynomials $P(x) = Ax^2 + 2Bx + C$ with real coefficients A, B, C.

<u>Theorem 4</u>. The restriction of the partial order □ to the set Q of quadratic polynomials with real coefficients is not reducible to vector dominance in $\mathbb{R}^m$ for any finite m.

<u>Proof</u>. We prove the stronger result that □ restricted to the set Q_+ of polynomials in Q with strictly positive coeeficients A, 2B, C is not so reducible. The map $j_Q : Q_+$ ∅ $\mathbb{R}^3$ which takes $P(x) = Ax^2 + 2Bx + C$ into j_Q (P) = (A,B,C) □ $R^3{}_+$ induces a partial order $□_Q = □_{jQ}$ on $\mathbb{R}^3{}_+$ which satisfies (M) and (H). It is easy to see that (A_1, B_1, C_1) $□_Q$ (A_2, B_2, C_2) precisely if the following two conditions hold:

(4.1) $A_1 \leq A_2$ and $C_1 \leq C_2$; and

(4.2) Either $B_1 \leq B_2$ or $(B_1 - B_2)^2 \leq (A_2 - A_1)(C_2 - C_1)$.

Now let **z** = (1, 1, 1), **w** = (2, 1, 1) (any pair of points of $\mathbb{R}^3{}_+$ which agree in two coordinates will do here). Let (□$^{(n)}$) be a sequence of positive numbers such that □$^{(n)}$ < 1/2 and □$^{(n)}$ □ 0, and let □$^{(n)}$= (□$^{(n)}$(1 - □$^{(n)}$)) $^{1/2}$. Set $\mathbf{x}^{(n)}$ = (1 + □$^{(n)}$, 1 + □$^{(n)}$, 1 - □$^{(n)}$), and $\mathbf{y}^{(n)}$ = (2 - □$^{(n)}$, 1, 1). A short computation shows that (1.1)-(1.3) hold for □ = $□_Q$, completing the proof.

This theorem can be interpreted as a result about the inclusion relationship for the family of plane sets $E_{A,B,C} = \{(x,y) \in \mathbb{R}^2 : x \geq 0, \quad y \leq Ax^2 + 2Bx + C\}$; it says that this inclusion partial order is not reducible to vector dominance in any $\mathbb{R}^m$.

5. Containment for compact sets.

In section 4 of [12] it was shown that the family $\square$ of congruence classes of plane rectangles can be mapped into the space $\mathbb{l}_2$ of square-summable sequences of real numbers in a manner that converts containment to vector dominance, and that this can be accomplished continuously, if $\mathbb{l}_2$ is equipped with the usual metric

$$d_{\mathbb{l}}(\mathbf{u}, \mathbf{v}) = [\sum_{n\geq 1} (u_n - v_n)^2]^{1/2}$$

for $\mathbf{u} = (u_n)_{n\geq 1}$ and $\mathbf{v} = (v_n)_{n\geq 1}$ in $\mathbb{l}_2$. We shall now indicate how to extend this result substantially.

Fix a positive integer k and endow $\mathbb{R}^k$ with its usual Euclidean metric d_k. Let $\square$ denote the family of non-empty compact subsets of $\mathbb{R}^k$, ~ congruence for subsets of $\mathbb{R}^k$, F the congruence class of $F \in \square$, $\square = \{ F : F \in \square \}$, $tF = \{t\mathbf{x} : \mathbf{x} \in F\}$ and $tF = (tF)^{\sim}$ for $F \in \square$ and $t > 0$. For $F_i \in \square$ (i=1,2) set

$$\square_1 (F_1, F_2) = \sup_{\mathbf{x} \in F_1} [\inf_{\mathbf{y} \in F_2} d_k (\mathbf{x}, \mathbf{y})],$$

$$\square_1 (F_1, F_2) = \inf \{\square_1 (F_1,F_2): F_2 \sim F_2\} ,$$

$$\square(F_1, F_2)=\square_1 (F_1, F_2) + \square_1 (F_2, F_1),$$

$$\square(F_1, F_2) = \inf \{\square(F_1, F_2) : F_2 \sim F_2\} .$$

Thus ($\square$, $\square$) and ($\square$, $\square$) are metric spaces, and $\square$ (respectively $\square$) is a good measure of closeness of (congruence classes of) compact sets. (Note that, in general, $\square(F_1, F_2) \neq \square_1 (F_1, F_2) + \square_1(F_2,F_1)$; one need only consider a circle of radius r without interior and a segment of length 2r to see this). Clearly $\square_1 \leq \square$ and $\square_1 \leq \square$. Let $(L_n)_{n\geq 1}$ be an enumeration of all non-empty finite unions of boxes $[a_1, b_1]$ ... $[a_k, b_k]$ with a_i, b_i rational and $a_i < b_i$. Define g_n : □□ $\mathbb{R}$ by $g_n(F) =$ □$_l(F, L_n)$ and let $f_n = 2^{-n/2} \tan^{-1} g_n$. Let $\mathbf{f} = (f_n)$:□ □ $\mathbb{l}_2$. The crucial facts about g_n are these:

$g_n(F) \geq 0$ with equality iff F □ L_n;

□$g_n(F_1) - g_n(F_2)$ □$\leq \max \{$□$_l(F_1, F_2)$,□$_l(F_2, F_1)\} \leq$ □(F_1, F_2);

if F_1 □ F_2 then $g_n(F_1) \leq g_n(F_2)$; and

if it is false that F_1 □ F_2, then for some n, $g_n(F_1) > 0 = g_n(F_2)$.

From these facts it follows not only that **f** is continuous and converts K on □ into vector dominance in $\mathbb{l}_2$, but also that **f** satisfies the following Lipschitz condition:

$$\text{(L)}\quad d_l(\mathbf{f}(F_1), \mathbf{f}(F_2) \leq (F_1, F_2).$$

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