Graham Triangulations and Triangulations With a Center Are Hamiltonean

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Abstract

Let P be a point set with n elements in general position. A triangulation T of P is a set of triangles with disjoint interiors such that their union is the convex hull of P, no triangle contains an element of P in in its interior, and the vertices of the triangles of T are points of P. Given T we define a graph $G(T)$ whose vertices are the triangles of T, two of which are adjacent if they share an edge. We say that T is hamiltonean if $G(T)$ has a hamiltonean path. We prove that the triangulations produced by Graham's Scan are hamiltonean. Furthermore we prove that any triangulation T of a point set which has a point adjacent to all the points in P (a center of T) is hamiltonean.

Key words: Graham's Scan, triangulation, hamiltonean, point sets.

1 Introduction

Let P be a point set with n elements on the plane in general position. A *triangulation* T of P is a set of closed triangles satisfying the following conditions:

1. The vertices of the triangles of T are points of P
2. No triangle in T contains a point of P in its interior

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3. The union of the triangles of T is the convex hull of P; see Figure 2.

If two points $p, q \in P$ are vertices of a triangle of T, we will say that they are adjacent in T, and the edge joining them will be denoted by $p - q$.

Given a triangulation T, we define the graph $G(T)$ as the graph whose vertices are the triangles of T, two of which are adjacent if they have an edge in common. If $G(T)$ has a hamiltonean path (cycle), we say that T has a hamiltonean path (cycle). In most of this paper when we say that T is hamiltonean, we mean that $G(T)$ has a hamiltomean path, otherwise we will mention explicitly that we have a hamiltonean cycle.

The study of triangulations of point sets on the plane has received considerable attention, among other reasons, for their applications in numerous areas. In [1] the problem of calculating hamiltonean triangulations of point sets is considered. The existence of a Hamiltonian path allows faster rendering on a graphics screen via pipelining [1, 4, 2]. In [1] the authors present an easy algorithm for calculating such triangulations. A straightforward implementation of the algorithm presented in [1] runs in $O(n^2)$ time, and as the authors mention, using ham-sandwich type algorithms, it can be implemented in $O(n \ln n)$ time. Such an implementation however, uses balanced partition type algorithms for point sets, whose implementations, while not overly complicated, are not straightforward either.

Our main goal in this paper is to show that the triangulations produced by Graham's Scan are indeed hamiltonean. We present a straightforward way to obtain a hamiltonean path from them, and if P is such that it has at least one point in the interior of its convex hull, we can easily modify Graham's Scan to obtain a triangulation containing a hamiltonean cycle. Our algorithms run in $O(n \ln n)$ time.

2 Graham's triangulations

We review briefly Graham's Scan [5, 6].

2.1 Graham's Scan

One of the most widely used algorithms to calculate the convex hull, and a triangulation of a point set, is Graham's algorithm. It works as follows:

1. Given a point set P, first find the point p_0 of P with the smallest x-coordinate, and sort the remaining elements of P with respect to the slope of the line segments connecting

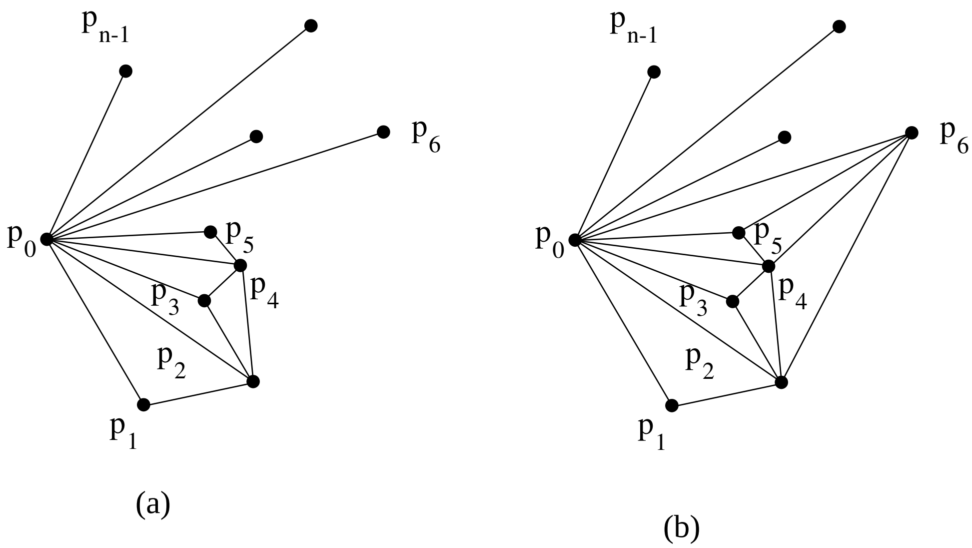

Figure 1:

them to p_0; see Figure 1(a). Relabel the elements of $P - \{p_0\}$ $p_1, ..., p_{n-1}$ according to this order.

2. Once the convex hull of $\{p_0, ..., p_{k-1}\}$ has been calculated, calculate the convex hull of $\{p_0, ..., p_{k-1}, p_k\}$ recursively as follows:
   Draw the line segments connecting p_k to the elements of P in the convex hull of $\{p_0, ..., p_{k-1}\}$ visible from p_k, that is the elements in the convex hull of $\{p_0, ..., p_{k-1}\}$ such that the line segments connecting them to p_k do not intersect the interior of the convex hull of $\{p_0, ..., p_{k-1}\}$; see Figure 1(b)

Our main objective in this section is to prove:

Theorem 1 *Let P be a point set on the plane in general position. The triangulation produced by applying Graham's Scan to P is hamiltonean.*

Some results will be necessary to prove our result.

2.2 Separating triangles

Given a triangulation T of a point set P, a *separating triangle* S of T is a triangle with vertices $p_i, p_j, p_k \in P$ such that edges $p_i - p_j$, $p_i - p_k$, and $p_j - p_k$ are edges of T, and at

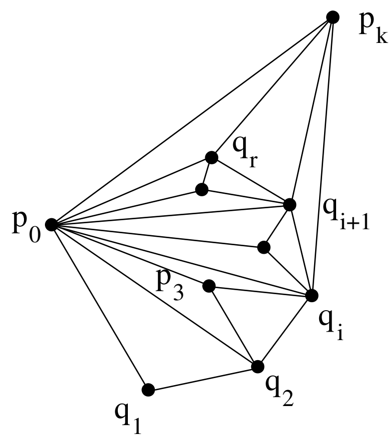

Figure 2:

least one element of P lies in the interior of S. For example in Figure 1(b) the triangles with vertices $\{p_0, p_2, p_4\}$, and $\{p_0, p_2, p_6\}$ are separating triangles. Given a point set P, the triangulation produced by Graham's Scan will be denoted by $GT(P)$. Observe that if S is a separating triangle of a triangulation T, S is not one of the triangles of T, as by definition the triangles in T cannot contain elements of P in their interior. Since in $GT(P)$ vertex p_0 is adjacent to all the vertices of $GT(P)$, the following lemma is clear.

Lemma 1 *Let S be a separating triangle in $GT(P)$. Then one of the vertices of S is p_0.*

Given a separating triangle S of T, T_s will denote the subtriangulation of T defined by all the triangles of T contained in S. Let e_1, e_2, and e_3 be the edges of S, and let t_1 (respectively t_2, and t_3) be the triangle of S' that has e_1 (respectively e_2 and e_3) as an edge. To make our proofs easier, we will say that a hamiltonean path of T_S that starts at t_i and ends in t_j *enters* S in e_i and *exits* it at e_j. The following lemma is the basis of our main result.

Lemma 2 *Let S be a separating triangle of $GT(P)$, with edges e_1, e_2, e_3. Then for any $\{i, j\} \subset \{1, 2, 3\}$ there is a hamiltonean path of T_S that enters S in e_i and exits it in e_j.*

Proof: Observe first that all separating triangles of $GT(P)$ with vertices p_0, p_i, p_k, $i < k$ are created during the execution of Step 2 of Graham's Scan when we join vertex p_k to the vertices of the convex hull of $\{p_0, ..., p_{k-1}\}$ visible from p_k. For example when p_6 is joined to p_4 and to p_2 in Figure 1(a), two separating triangles are created.

Suppose the result is true for separating triangles S of $GT(P)$ with vertices p_0, p_i, p_j, $i, j < k$. We prove it for all triangles with vertices p_0, p_i, p_k.

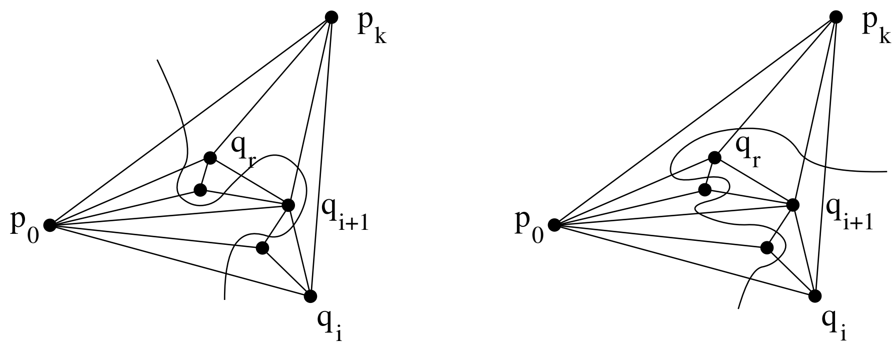

Figure 3:

Let us relabel the vertices *on the convex hull* of $\{p_0, ..., p_{k-1}\}$ counter-clockwise starting at p_0 by $q_0 = p_0, q_1, ..., q_r = p_{k-1}$ for some $r \leq k-1$, and assume that in Step 2 of Graham's Scan p_k is joined to $q_r, .., q_s$, for some $s \geq 1$.

Suppose that when we join p_k to q_i for some i, $1 \leq i < r$ we create a separating triangle S with vertices p_0, q_i, p_k, see Figure 2. We now prove that for any two of the edges of S, there is a hamiltonean path of T_S that enters and exits S in these edges.

Suppose that triangles S' and S" with vertices p_0, q_i, q_{i+1} and p_0, q_{i+1}, p_k respectively are separating triangles of $GT(P)$ (our result follows easily if one or both of S' or S" are not separating triangles). Assume by induction that the lemma is valid for these triangles. We now show that the result follows for S. We show how to construct a hamiltonean path of T_S that enters and exits S in any pair of edges of S, e.g. a hamiltonean path that enters S by edge $p_0 - q_i$ and exits it by edge $p_0 - p_k$. The other cases follow similarly.

By induction there is a hamiltonean path P_1 in the subtriangulation of $GT(P)$ induced by S' that enters it in edge $p_0 - q_i$ and exits it in edge $q_i - q_{i+1}$. Similarly there is a hamiltonean path P_2 that enters S'' in $q_{i+1} - p_k$ and exits it by edge $p_0 - p_k$. By first traversing P_1, then the triangle with vertices q_i, q_{i+1}, p_k and then P_2, we obtain a hamiltonean path that traverses all the triangles contained in S. The path enters S in $p_0 - q_i$ and exits it in $p_0 - p_k$.
■

To prove our main result we proceed as follows: Let $q_0 = p_0, q_1, ..., q_r$ be the vertices of the convex hull of P labeled in the counterclockwise order starting at the leftmost point p_0 of P. For $i = 1, ..., r-1$ let $S(i)$ be the triangle with vertices p_0, q_i, q_{i+1}. Clearly $GT(P)$ contains the edges of $S(i)$, $i = 1, ..., r-1$. By Lemma 2 for each $1 \leq i < r$ there is a hamiltonean path P_i of $T_{S(i)}$ that enters $S(i)$ by $e_i = p_0 - p_i$ and exits it by $e_{i+1} = p_0 - p_{i+1}$. Concatenating $P_1, ..., P_{r-1}$ we obtain a hamiltonean path of $GT(P)$. ■

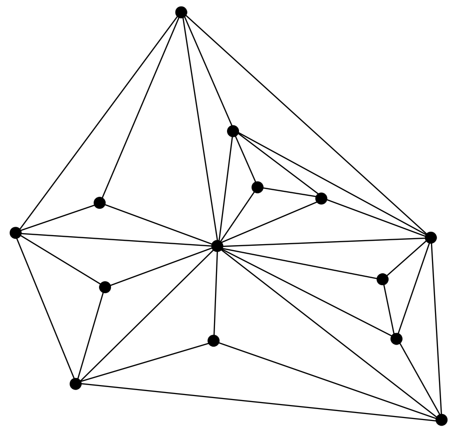

Figure 4: A triangulation with a center.

3 Triangulations with a central point

If a triangulation T of P has has a vertex v adjacent to all the vertices of G we call such a vertex a *center* of T. In this section we generalize our previous result and prove:

Theorem 2 *A triangulation with a center is hamiltonean.*

First we prove our result for triangulations T of point sets P such that:

- the convex hull of P is a triangle
- The center of T, labeled p_0, lies on the convex hull of P,

see Figure 5. Call these pseudo-triangulations *triangulations with an external center.*

We will need the following. Let P be a simple polygon with vertices labelled $\{p_1, ..., p_{n-1}\}$ clockwise around its boundary, and T_P a triangulation of P; that is, a set of closed triangles contained in P with disjoint interiors such that their vertices are vertices of P and their union is P, see Figure 6. Edges of these triangles that are not edges of P will be called the diagonals of P in T. Associate to each diagonal $e = p_i - p_j$ of P the interval $\{i, i+1, ..., j\}$; $i < j$. Define a partial order $\mathcal{P}$ in the set of diagonals of T as follows: an edge $e = p_i - p_j$ is smaller that $e' = p_k - p_m$ iff $\{i, i+1, ..., j\} \subset \{k, k+1, ..., m\}$. It is easy to see that the diagonals of T can be labeled $e_1, ..., e_{n-4}$ in such a way that if e_i is smaller than e_j in $\mathcal{P}$ then $i < j$. Such a labeling is called a consistent labeling with respect to $\mathcal{P}$; see Figure 6.

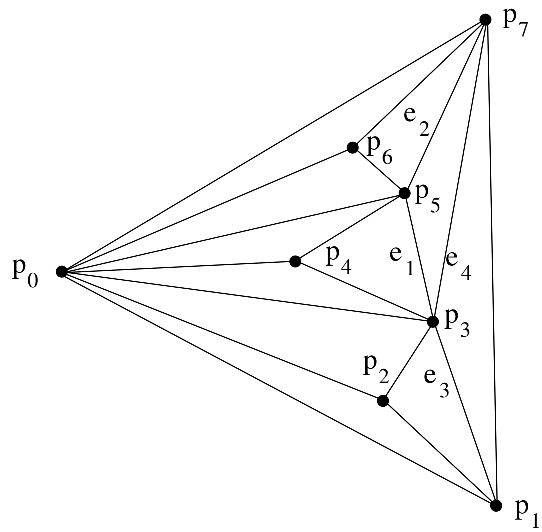

Figure 5:

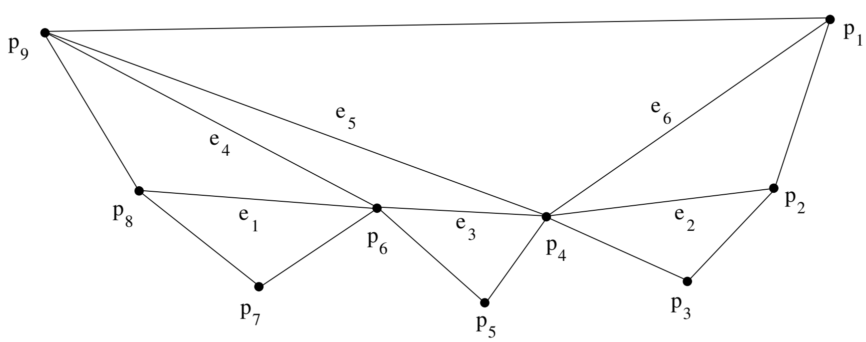

Figure 6: A triangulation of a polygon with a consistent labeling of its diagonals.

Lemma 3 *Let P be such that its convex hull is a triangle and T be a triangulation with an external center p_0. Then T has a hamiltonean path that enters and leaves T through any two of the external edges of T,*

Proof: Assume that the vertices of T are labeled $p_0, p_1, ..., p_{n-1}$ in such a way that if $i < j$, then the slope of the line segment joining p_i to p_0 is smaller than the slope of the segment joining p_j to p_0; see Figure 5. Observe that if we remove vertex p_0 together with the edges and triangles incident to it from T we obtain a triangulated polygon P with vertices $p_1, ..., p_{n-1}$.

Clearly all separating triangles of T have p_0 as one of their vertices and a diagonal of P as one of their edges. Let $\{e_1, ..., e_{n-4}\}$ be a consistent labeling of the diagonals of P. Let $S(i)$ be the separating triangle of T whose vertices are the vertices of a diagonal e_i of P and p_0, e.g. in Figure 5 $S(3)$ has vertices p_0, p_1, p_3.

We now claim that, as in the proof of Theorem 1, for every pair of edges of $S(i)$ there is a hamiltonean path of $T_{S(i)}$ that enters and exits $S(i)$ at these edges, $i = 1, ..., n-3$. The proof proceeds by induction on i, $i = 1, ..., n-4$. For $i = 1$ our result follows. Suppose it is true for $T_{S(1)}, ..., T_{S(k-i)}$.

Several cases arise, the hardest is when $T_{S(k)}$ is the union of two sub-triangulations $T_{S(i)}$, $T_{S(j)}$ of T and the triangle with edges $\{e_i, e_j, e_k\}$ for some pair of indexes i, j, $1 \leq i < j < k$. For example in Figure 5, $T_{S(4)}$ contains $T_{S(1)}$, $T_{S(2)}$ and the triangle with vertices $\{p_3, p_5, p_7\}$. Using the same reasoning as in Lemma 2, it follows that for any two edges e and f of $S(i)$ there is a hamitonian path that enters and leaves $S(i)$ in e and f respectively. The remaining cases follow in a similar way. ∎

The proof of Theorem 2 now follows in a similar way to that of Theorem 1. Observe, however, that if the center vertex of T is in the interior of the convex hull of P, we obtain a hamitonean cycle. ∎

4 Conclusions

We have proved that for any point set P in general position, the triangualtion T of P produced by Graham's Scan has a hamiltonean path. Furthermore we proved that any triangulation of P with a central vertex also has a hamiltonean path. If the central vertex of T is in the interior of the convex hull of P, then T has a hamiltonean cycle. It is straightforward to see that our methods yield algorithms with $O(n \ln n)$ time complexity to obtain such triangulations and hamiltonean paths and cycles. To conclude, we would like to mention the following graph theoretical implication.

A plane graph G is called a pseudo-triangulation if all of its faces except at most one, which we call *the external face* of G, are triangles (that is, are bounded by three edges). The external face of G may also be a triangle. It is well-known that such a graph G has a plane immersion on the plane in which

1. all its vertices are represented by point sets in general position,
2. the edges of G are represented by non-crossing open straight line segments, and
3. the outer face of G is mapped to a convex polygon; see Figure 4.

The pseudo-dual graph $D(G)$ of a pseudo-triangulation G is the graph whose vertices are the faces of G, except for its external face, two of which are adjacent if they share an edge. From our results it follows that if G has a central vertex, then $D(G)$ has a hamiltonean path.

Moreover if a central vertex of G is not a vertex of the external face of G, then $D(G)$ has a hamiltonean cycle. In this case we can find the hamiltonean path or cycle in linear time.

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