# ILLUMINATING RECTANGLES AND TRIANGLES IN THE PLANE*

by

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**Abstract**

A set S of light sources, idealized as points, illuminates a collection F of convex sets if each point in the boundary of the sets of F is visible from at least one point in S. For any n disjoint plane isothetic rectangles, $\lfloor (4n+4)/3 \rfloor$ lights are sufficient to illuminate their boundaries. If in addition, the rectangles have equal width, then n+1 lights always suffice and n-1 are occasionally necessary. For any family of n plane triangles, $\lfloor (4n+4)/3 \rfloor$ light sources are sufficient. For collections of n homothetic triangles, n+1 light sources are always sufficient and n-1 are occasionally necessary.

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## 1.- Introduction

Let $F$ be a collection of n disjoint compact convex sets in the plane. A set $L$ of light sources, idealized as points in the plane, is said to illuminate $F$ if every point in the boundary of each set in $F$ is visible from at least one point of $L$; that is, if for each point x in the boundary of every set in $F$, there is a point y of $L$, such that the segment xy meets the union of the sets in $F$ exactly in {x}.

How many light sources are sufficient to illuminate $F$ ? This question is closely related to the classical Art Gallery Theory, where a typical problem is to determine how many guards are sufficient to protect objects on the n walls of a polygonal art gallery. For a survey of results in the Art Gallery Theory the reader may wish to consult [4].

In [2], L. Fejes Toth proved that 4n-7 lights are always sufficient to illuminate n disjoint compact convex sets in the plane. In this article we consider the particular case where the sets are isothetic rectangles, as well as the case in which the sets are triangles.

We show that $\lfloor (4n+4)/3 \rfloor$ lights are always sufficient to illuminate n disjoint isothetic rectangles. If, in addition, the rectangles are required to have equal width, then n+1 lights suffice and n - 1 are occasionally necessary. For n triangles, we prove that $\lfloor (4n+4)/3 \rfloor$ light sources are always sufficient and for any collection of n pairwise homothetic triangles, n + 1 lights suffice and n - 1 are occasionally needed. The bounds presented in this article for the general cases of arbitrary isothetic rectangles and arbitrary triangles are not tight. The best lower bounds that we now, coincide with the ones given for the restrictive cases of isothetic rectangles with equal width and homothetic triangles.

Two sets A and B are homothetic if there is a positive constant t and a point x, such that $B = x + tA$. A set of rectangles is said to be isothetic if all their sides are parallel to the coordinate axis.

## 2.- Illuminating Rectangles

Consider a collection $F = \{R_1,...,R_n\}$ of n disjoint isothetic rectangles in the plane. Assume they are contained in a big rectangle **R**. Let $R'_1,...,R'_n$ be maximal rectangles with pairwise disjoint interiors such that $\mathbf{R} \supseteq R'_i \supseteq R_i$ for i=1,2,...,n. See Figure 1. The rectangles $R'_1,...,R'_n$ induce a partition $\pi = \pi(\mathbf{R}, R'_1,...,R'_n)$ of **R**. Notice that, in addition to the rectangles $R'_1,...,R'_n$, the partition $\pi$ may contain some rectangular regions $R'_{n+1},...,R'_{n+h}$, none of which includes any of the rectangles in $F$. We shall call these regions holes of $\pi$.

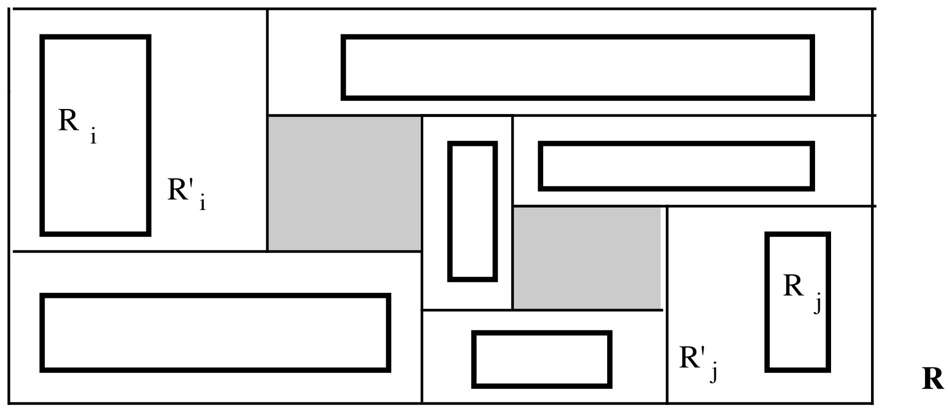


Figure 1

Let us define a graph $G(\pi)$, associated with the partition $\pi$ in the following way: the vertices of $G(\pi)$ are the corners of every region in $\pi$. Two vertices $u$ and $v$ are adjacent in $G(\pi)$ if they are joined by a line segment $s(uv)$, contained in the boundary of some region $R_i$, and such that no other vertex of $G(\pi)$ is included in $s(uv)$. Notice that the number of vertices in $G(\pi)$ is $2n + 2h + 2$.

**Theorem 1**. Let $F = \{R_1,...,R_n\}$ be a collection of n disjoint isothetic rectangles. Let $\mathbf{R}$, $R'_1,...,R'_n$, $\pi = \pi(\mathbf{R}, R'_1,...,R'_n)$ and $G(\pi)$ be defined as above. If the partition $\pi$ contains no holes, then $F$ can be illuminated with at most $\lfloor (4n+4)/3 \rfloor$ lights.

**Proof -** Starting at any corner of $\mathbf{R}$, the graph $G(\pi)$ can be dismantled by deleting, one at a time, vertices of degree at most two. This shows that $G(\pi)$ is a 3-vertex colourable graph. Take a 3-colouring of $G(\pi)$ and place a light at each vertex in the two less popular chromatic classes. There are at most $\lfloor (4n+4)/3 \rfloor$ such vertices.

Each edge of $G(\pi)$ has a light in at least one of its end points. Let S be a side of a rectangle $R_i$ and let S' be the corresponding side of $R_i'$. At least one edge e of $G(\pi)$ is contained in S' and there is a light placed at least at one end point of e. Clearly this light illuminates S.

If $\pi = \pi(\mathbf{R}, R'_1,...,R'_n)$ contains h holes, the graph $G(\pi)$ has $2n + 2h + 2$ vertices. In this case, the proof of Theorem 1 would give a set of $\lfloor(4n+4h+4)/3\rfloor$ lights that illuminates $R_1,...,R_n$. In Theorem 2, we adapt this proof by eliminating the holes in $\pi$ to form a partition $\pi'$, whose corresponding graph $G(\pi')$ has $2n + 2$ vertices.

An extension of theorem 1 is the following stronger result.

**Theorem 2**. For any collection $F$ of n disjoint isothetic rectangles, $\lfloor(4n+4)/3\rfloor$ lights are sufficient to illuminate $F$.

**Proof -** Let $F = \{R_1,...,R_n\}$ be a collection of n disjoint isothetic rectangles. Let $\mathbf{R}$, $R'_1,...,R'_n$, $\pi = \pi(\mathbf{R}, R'_1,...,R'_n)$ and $G(\pi)$ be defined as above. Let h denote the number of holes in $\pi$.

Each hole $H=H(R'_{i_1},R'_{i_2},R'_{i_3},R'_{i_4})$ of $\pi$ is bounded by four regions, say $R'_{i_1}$, $R'_{i_2}$, $R'_{i_3}$ and $R'_{i_4}$, such that for j = 1, 2, 3 and 4, the region $R'_{i_j}$ has a vertex $v(R'_{i_j})$ that lies in the boundary of $R'_{i_{j+1}}$. A rectangle $R_{i_j}$, in a hole H, is said to be exposed with respect to H if the straight line that contains one of its sides crosses H. Otherwise $R_{i_j}$ is retracted with respect to H. In the holes illustrated in Figure 2, $R_{i_2}$ is retracted, while $R_{i_1}$, $R_{i_3}$ and $R_{i_4}$ are exposed.

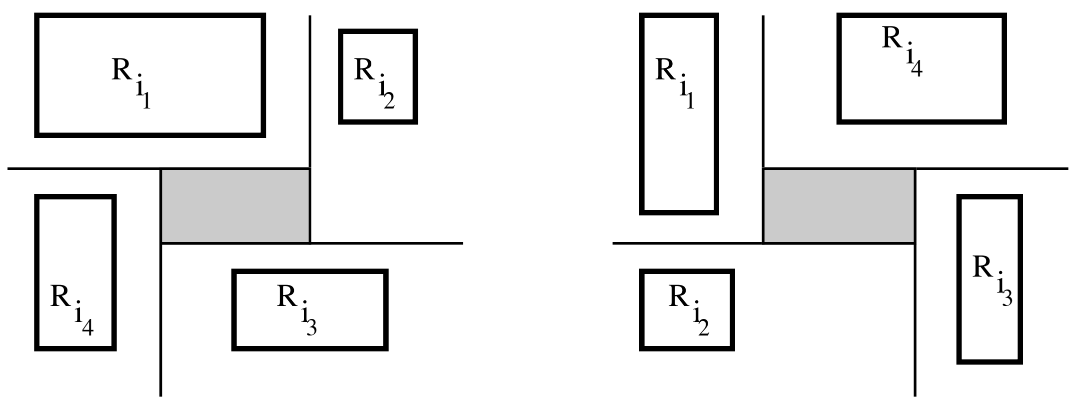


Figure 2

If every hole is eliminated, by inserting one of its two diagonals and deleting the corresponding edges as in Figure 3, then $\pi$ is modified to a partition $\pi^* = \pi^*(\mathbf{R}, R_1^*,...,R_n^*)$ of $\mathbf{R}$ into n polygonal regions $R_1^*,...,R_n^*$, not necessarily convex, such that for i=1,2,...,n, $\mathbf{R} \supseteq R_i^* \supseteq R_i' \supseteq R_i$.

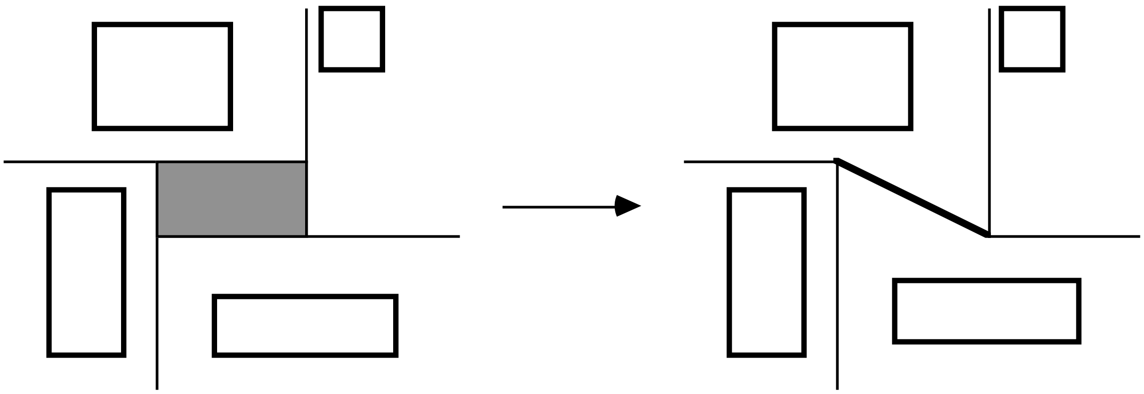

Figure 3

A graph $G(\pi^*)$ may be defined in the same way as $G(\pi)$. Independently of the choice of the diagonal used to eliminate each hole, the graph $G(\pi^*)$ contains $2n + 2$ vertices and is 3-vertex colourable.

We want the diagonals to be such that every rectangle $R_i$ is illuminated whenever lights are placed such that every edge of $G(\pi^*)$ has a light in at least one of its vertices. To assure this we have the following rules to chose the diagonal to eliminate a hole $H = H(R'_{i_1}, R'_{i_2}, R'_{i_3}, R'_{i_4})$.

a) If one, but not all of the rectangles $R_{i_1}, R_{i_2}, R_{i_3}$ or $R_{i_4}$ is exposed with respect to $H$, then there is an exposed rectangle, say $R_{i_1}$, followed by a retracted rectangle $R_{i_2}$. Insert the diagonal of $H$ whose end points are $v(R'_{i_2})$ and $v(R'_{i_4})$ and delete the segments joining $v(R'_{i_1})$ to $v(R'_{i_2})$ and $v(R'_{i_3})$ to $v(R'_{i_4})$; see Figure 4. Notice that, since each hole is bounded by exactly four regions, whenever the two pairs of regions that bound a hole correspond to an exposed rectangle followed by a retracted one, the same diagonal is to be inserted; therefore rule (a) is well defined.

b) If the rectangles $R_{i_1}, R_{i_2}, R_{i_3}$ or $R_{i_4}$ are all exposed or all retracted, insert any of the two diagonals and delete the corresponding edges.

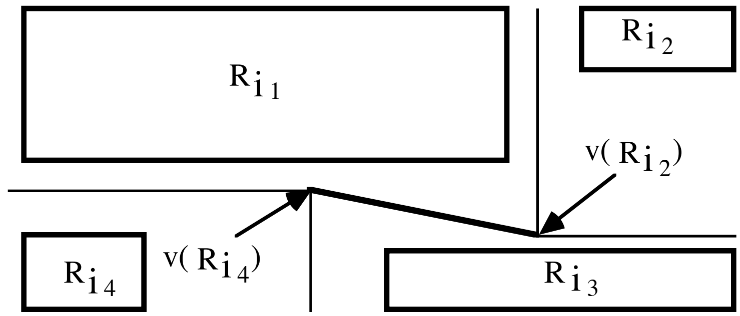


Figure 4

Modify $\pi$ to $\pi^*$, by eliminating the holes in $\pi$, using rules (a) and (b) and let $G(\pi^*)$ be the corresponding graph. Take a 3-colouring of $G(\pi^*)$ and place a light at each vertex in the two less popular chromatic classes. Since $G(\pi^*)$ has $2n+2$ vertices, then the number of lights is at most $\lfloor(4n+4)/3\rfloor$ We claim that every rectangle $R_i$ is entirely illuminated.

Suppose that for some $R_{i_j}$, one of its sides, say S, is not entirely visible from both end points of any edge of $G(\pi^*)$ that is contained in the boundary of $R_{i_j}*$. Without loss of generality assume S is the top side of $R_{i_j}$. Let L be the horizontal side of $R_{i_j}*$ that lies above S. At least one of the end points of L must be obstructed from S, say $u$ ; see Figure 5.

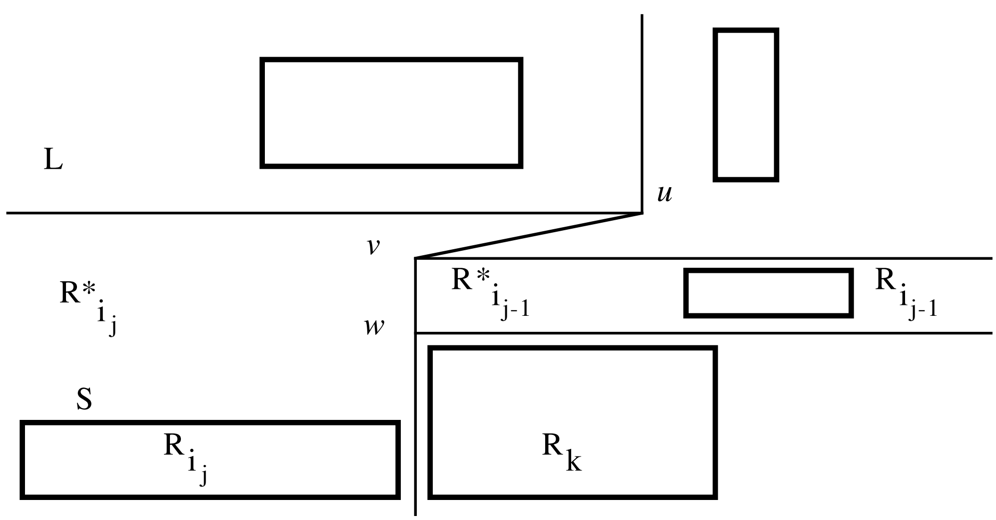


Figure 5

The point $u$ must be an end point of a diagonal $uv$ inserted when a hole H was eliminated, otherwise S would be entirely visible from $u$ . Moreover, $R_{i_j}$ must be retracted with respect to H and by (a) and (b), the rectangle $R_{i_{j-1}}$ that precedes $R_{i_j}$ around the hole H, must also be retracted. Then $R_{i_{j-1}}$ cannot be an obstruction between $u$ and S, therefore such obstruction must come from another rectangle $R_k$, as in Figure 5. Notice that the lower side of the region $R_{i_{j-1}}{}^*$ must then intersect $R_{i_j}{}^*$ in a point $w$ that lies above S. This leads to a contradiction since now S would be entirely visible from the two end points of the edge *vw*. This ends the proof.

If in addition, the rectangles are required to have equal width, then we can improve the result to the following:

**Theorem 3 -** For any collection of n disjoint isothetic rectangles with the same width, n+1 lights are sufficient to illuminate them.

**Proof -** Let $R = \{R_1,\ldots,R_n\}$ be a collection of n disjoint isothetic rectangles with the same width, and let L be a line above all of the rectangles in $R$.

Extend the vertical sides of each $R_j$, upwards, until they either reach L or they reach another rectangle in $R$. For each $R_j$ a polygonal region $S_j$ is defined; it is bounded by the top side of $R_j$, some segments of bases of other rectangles in $R$ and some segments of the extended sides. It is easy to see that since all the rectangles $R_j$ have the same width, then all of the regions $S_j$ are star-shaped; see Figure 6.

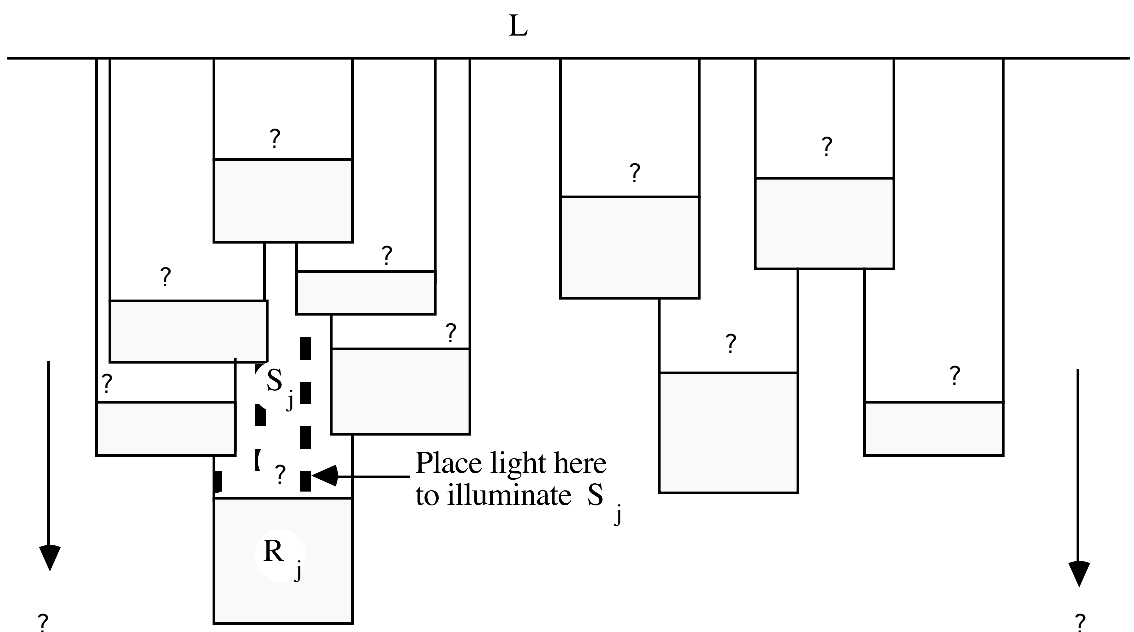


Figure 6

To illuminate each of the star-shaped regions $S_j$, a single light source is sufficient, placed near the top side of $R_j$, directly below a point in the top horizontal side of $S_j$, see Figure 6. At this point, all points in the boundaries of $R_1,\ldots,R_n$, contained in the boundary of some $S_j$, are illuminated. The reader may verify that two extra lights suffice to illuminate all the boundary points of $R_1,\ldots,R_n$ which are not located in any of the star-shaped regions $S_j$. These two lights should be placed far enough below, one of them to the left of the collection *R* and the other to the right.

Observe that at least one of the $n+2$ lights used to illuminate the collection may be saved in the following way. Let $t$ be the number of rectangles whose top sides are completely exposed from above; if $t \geq 2$ then all of the $t$ highest lights may be replaced by a single one placed far enough above L. When $t = 1$, let $R_{i_1}$, $R_{i_2}$ and $R_{i_3}$ be the rectangles with the highest top sides, in that order. The three lights used to illuminate the star-shaped regions $S_{i_1}, S_{i_2}$ and $S_{i_3}$ may be replaced by two lights: one placed in the line L, far enough to the left, and the other placed on the line that supports the top side of $R_{i_3}$, far enough to the right.

The following example illustrates that occasionally $n-1$ lights are required to illuminate $n$ isothetic rectangles with equal width.

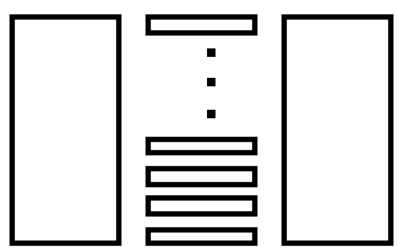

Figure 7

**3.- Illuminating Triangles**

In this section we consider collections of arbitrary plane triangles and collections of pairwise homothetic plane triangles. We shall prove the following results.

**Theorem 4 .-** Any family H of n disjoint plane triangles can always be illuminated with at most $\lfloor (4n+4)/3 \rfloor$ lights.

**Theorem 5 -** For any collection of $n$ disjoint pairwise homothetic triangles in the plane, $n+1$ lights are always sufficient to illuminate them.

The main idea in our proofs is to create a convex partition $\pi$ of the complement of the union of the triangles, such that a large number of disjoint pairs $\{R_i, R_j\}$ of adjacent regions of $\pi$ may be matched. The triangles can be illuminated by placing a light source in $R_i \cup R_j$ for each pair $\{R_i, R_j\}$ of matched regions and one source of light for each unpaired region. Two well known results in matching theory will be used in the proofs.

**Theorem N** ( T . Nishizeki, [3]).**-** If G is a planar 2-connected graph with m vertices and minimum degree at least three, then for all $m \geq 14$, G has a matching of size at least $\lceil (m+4)/3 \rceil$ and for $m<14$, G has a matching of size $\lfloor m/2 \rfloor$

**Theorem T** ( W. T. Tutte, [5]).- Let G be a graph with $2m+1$ vertices. If for every subset S of vertices, the number of connected components of G-S, with an odd number of vertices, is at most $|S|+1$, then G has a matching of size m.

**Proof of theorem 4**. Let $H=\{T_1,...,T_n\}$ be a family of $n$ disjoint triangles. Add three triangles $T_{n+1}, T_{n+2}$ and $T_{n+3}$, together with three rays $L_{3n+1}$, $L_{3n+2}$ and $L_{3n+3}$ and six line segments $L_{3n+4},...,L_{3n+9}$, as shown in Figure 8; they are chosen such that for $i=1,2,...,n$, each $T_i$ lies within the hexagonal region bounded by $T_{n+1}, T_{n+2}, T_{n+3}, L_{3n+7}, L_{3n+8}$ and $L_{3n+9}$. The reader may notice later that these triangles are added to facilitate the use of Nishizeki's theorem.

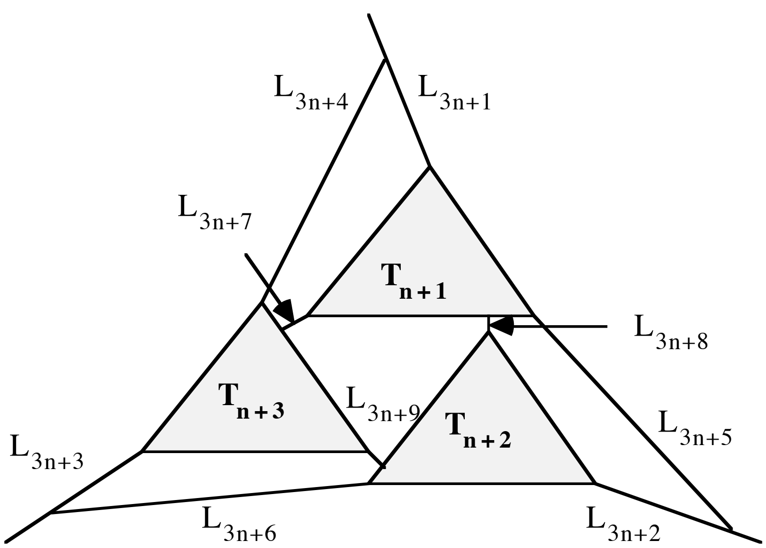

Figure 8

Let F denote the collection $\{T_1,...,T_{n+3}\}$ and let **T** be the union of all triangles in F. A convex partition $\pi = \pi(F, L_1,...,L_{3(n+3)})$ of $R^2/\mathbf{T}$ may be obtained as follows: one at a time, consider the vertices of the triangles $T_1,...,T_n$. When the vertex $p_i$ of the triangle $T_s$ is being considered, draw a line segment $L_i$ with an end point at $p_i$, within the open angular region $Ang(p_i)$ which is defined by the extension of the sides of $T_s$ incident in $p_i$. The line segment $L_i$ extends until it either reaches another triangle $T_r$ or it reaches a previously drawn line segment $L_j$. See Figure 9.

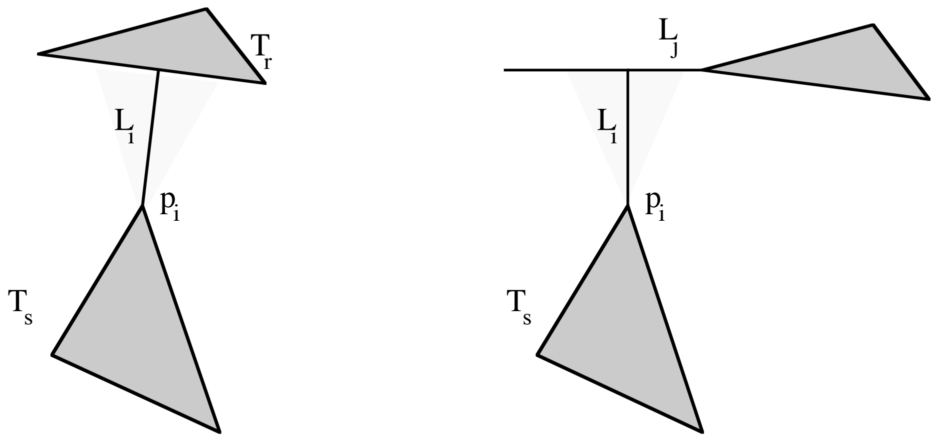


Figure 9

The partition $\pi = \pi(F, L_1,...,L_{3(n+3)})$ depends on $L_1,...,L_{3(n+3)}$. In all cases, there are $2(n+3) + 1$ regions $R_1,...,R_{2(n+3)+1}$ in $\pi$. Observe that two adjacent edges of a region $R_i$ may be collinear, nevertheless, they are considered as different edges. Let us define an adjacency graph D $= D(\pi)$ in a natural way: there is a vertex $v_i$ in D for each region $R_i$ and an edge $v_iv_j$ whenever the boundaries of $R_i$ and $R_j$ have an edge in common.

The triangles $T_1,..., T_{n+3}$ are disjoint, hence regardless of the choice for the line segments $L_1,...,L_{3n}$, if a region $R_i$ of $\pi$ has $m \geq 5$ edges, then $R_i$ shares an edge with at least three other regions and thus has degree at least three in $D(\pi)$. We shall show that the line segments $L_1,...,L_{3n}$ can be chosen in such a way that each region $R_i$ has degree at least three in $D(\pi)$. Some definitions will be useful.

For $m=1,2,...,3n$, let $\pi_m$ denote the partition of $R^2/\mathbf{T}$ obtained when the line segments $L_1,...,L_m$, have been drawn. For a vertex p of a triangle T, we say that a triangle T', an edge E of a triangle T' or a line segment $L_t$ is *blocking* $p$ *in* $\pi_m$, if every point in Ang(p), which is visible from p, lies in T', E or $L_t$, respectively. A line segment $L_t$, with $t \leq m$ is a *special line segment*, if its origin is a vertex p that is blocked in $\pi_m$ by an edge of a triangle. Notice that, for any pair of triangles $T_i$ and $T_j$, if an edge of $T_j$ blocks a vertex of $T_i$, then no edge of $T_i$ blocks any vertex of $T_j$.

Suppose the line segments $L_1,...,L_{i-1}$ have been chosen in such a way that the corresponding partitions $\pi_1,..., \pi_{i-1}$, satisfy the following conditions:

i) No region of $\pi_j$ has exactly three edges.

ii) If a region $R_i \in \pi_j$ has exactly four edges, then $R_i$ has an edge in common with three other regions in $\pi_j$.

iii) For each $j=1,2,...i-1$, either $L_j$ is a special line segment or there is a line segment $L_m$ with $m<i$ such that one of the line segments $L_j$ and $L_m$ hits the other .

We may assume also that no line segment $L_j$, with $j<i$, reaches a triangle at a vertex. Let p be a vertex of a triangle T such that no line segment $L_i$ has yet been drawn from p. To chose $L_i$, several possibilities will be considered:

**Case 1 -** Throughout this case, no point y visible from p, with $y \in Ang(p)$, is contained in any line segment $L_t$, such that $t < i$.

**a)** If p is blocked in $\pi_{i-1}$ by an edge E of a triangle T', then let x be a point in $E \cap Ang(p)$, x visible from p, and let $L_i$ be the line segment joining p and x; see Figure 10.

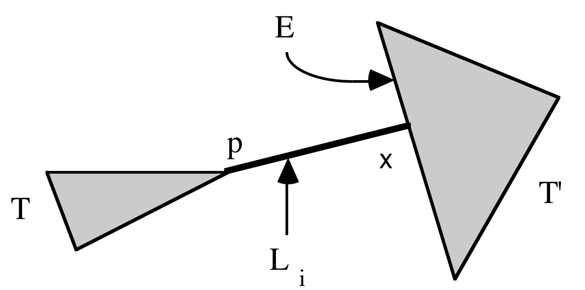


Figure 10

**b)** If p is blocked in $\pi_{i-1}$ by a triangle T' but is not blocked by any edge of T', then there is a vertex q of T', with $q \in Ang(p)$ which is visible from p. Since T' blocks p, then no line segment $L_t$ has been drawn from q. Let $L_{i+1}$ be a line segment drawn from q, within Ang(q), such that $L_{i+1}$ does not reach T; this is possible since T cannot block q. Now let $L_i$ be a line segment joining p with a point $x \in L_{i+1} \cap Ang(p)$, x visible from p; see Figure 11.

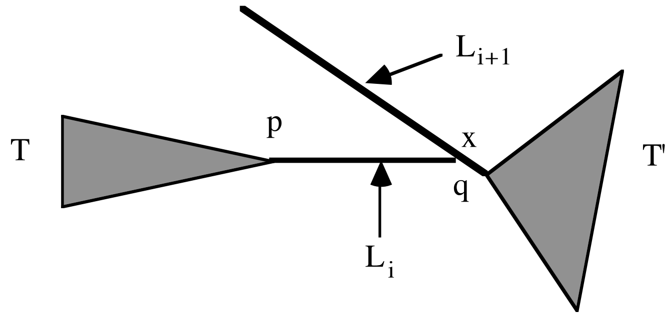


Figure 11

**c)** If p is not blocked in $\pi_{i-1}$ by any triangle in *F*, let q be a vertex of a triangle T', closest to p with the following properties: the point q is visible from p, $q \in Ang(p)$ and q is not blocked in $\pi_{i-1}$ by T. Such a vertex exists since $T_1,...T_n$ are surrounded by $T_{n+1}$, $T_{n+2}$, $T_{n+3}$ and $L_{3n+1},...,L_{3n+9}$. Let $L_i$ be any line segment drawn from p, going through a small neighborhood of q, and let $L_{i+1}$ be a line segment drawn from q, within Ang(q) that reaches $L_i$; see Figure 12.

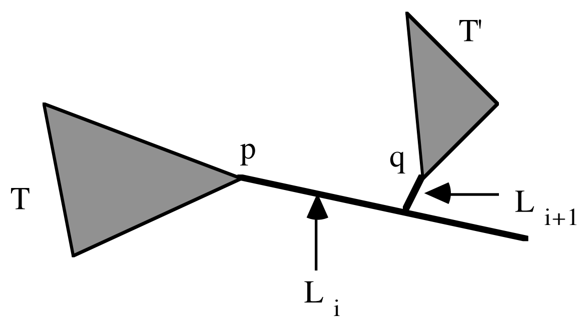


Figure 12

**Case 2 -** Some line segment $L_t$, with $t<i$, has a point $y \in Ang(p)$, y visible from p.

**a)** If p is blocked in $\pi_{i-1}$ by $L_t$, then let $L_i$ be a line segment drawn from p, within Ang(p), and such that $L_i$ reaches $L_t$; see Figure 13.

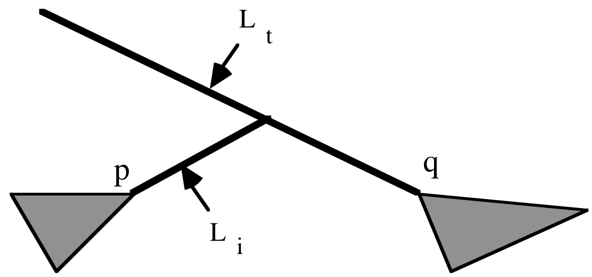


Figure 13

**b)** If $p$ is not blocked in $\pi_{i-1}$ by $L_t$ and $L_t$ does not reach T, then proceed as in Case 2a.

**c)** Every line segment $L_t$ that has a point $y$, visible from $p$, with $y \in Ang(p)$, is such that $L_t$ reaches T. Let $L_s$ be the line segment in $L_1, ..., L_{i-1}$ that has the closest point in $Ang(p) \cap (L_1 \cup ,..., \cup L_{i-1})$ that is visible from p.

**c')** If $L_s$ is not blocked in $\pi_{i-1}$ by T, then $L_s$ is not a special segment, by (iii) there must be at least one other line segment $L_m$ such that $L_m$ reaches $L_s$ in a point $r \neq q$. Let $x$ be a point in $L_s$ between $q$ and $r$; note that $x$ must be visible from $p$. Let $L_i$ be the line segment $px$; see Figure 14.

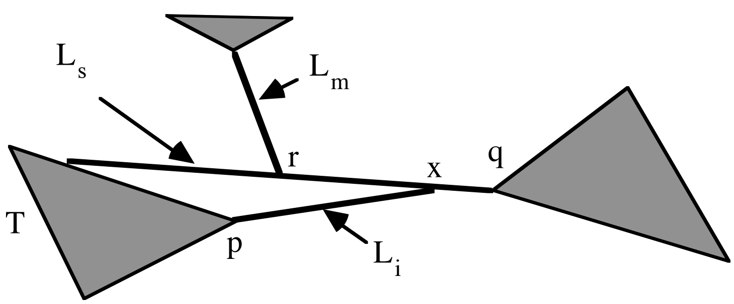


Figure 14

**c'')** If T blocks $L_s$ in $\pi_{i-1}$, let T' be the triangle where $L_s$ originates. Observe that T' must lie completely within Ang(p), in particular another vertex q' of T' is visible from p; proceed as in cases (1c) or (2a), whichever applies with q' in place of q.

Cases (1a), (1b), (1c), (2a), (2b), (2c') and (2c'') cover all possibilities. In each case, the line segment $L_i$ is either a special line segment, it reaches a line segment $L_a$ or is reached by a line segment $L_b$. When $L_{i+1}$ is chosen together with $L_i$, one of them reaches the other. No region with only three edges is created and if any region of $\pi_i$ has exactly four edges, then it has three neighbors in $\pi_j$.

By induction all line segments $L_1, L_2, ..., L_{3n+3}$ may be chosen such that they satisfy (i), (ii) and (iii).

Let $D = D(\pi)$ be defined as above. D has minimum degree at least three; it is clear that D is 2-connected and planar. By Nishizeki's result, D has a matching M of size at least $\lceil((2(n+3)+1)+4)/3\rceil = \lceil(2n+11)/3\rceil$.

For each pair $\{R_i, R_j\}$ of regions matched by M, place a light source in $R_i \cap R_j$. Add one source of light for each unmatched region. This gives a set of $(2(n+3)+1) - \lceil(2n+11)/3\rceil = \lfloor(4n+10)/3\rfloor$ light sources that entirely illuminates the regions $R_1,...,R_{2n+7}$. In particular, the boundary of each triangle in F is illuminated.

Finally, notice that at least two lights are placed within the closure of the three unbounded regions of $\pi$; since this regions do not meet the hexagonal region bounded by $T_{n+1}, T_{n+2}, T_{n+3}, L_{3n+7}, L_{3n+8}$ and $L_{3n+9}$, then at least two of the $\lfloor(4n+10)/3\rfloor$ lights are not needed to illuminate the original collection.

For collections of homothetic triangles we make a similar construction; there we can find a matching of size n+3 in the corresponding graph $D(\pi)$ by using Tutte's theorem.

**Proof of theorem 5 -** Let $T_1,...,T_n$ be disjoint pairwise homothetic triangles. Add three homothetic triangles $T_{n+1}, T_{n+2}$ and $T_{n+3}$, together with six rays $L_{3n+1}, L_{3n+2}, ..., L_{3n+6}$ and three line segments $L_{3n+7}, L_{3n+8}$ and $L_{3n+9}$, as shown in Figure 15. They are chosen such that for $i=1,2,...,n$, each $T_i$ lies within the hexagonal region bounded by $T_{n+1}, T_{n+2}, T_{n+3}, L_{3n+7}, L_{3n+8}$ and $L_{3n+9}$.

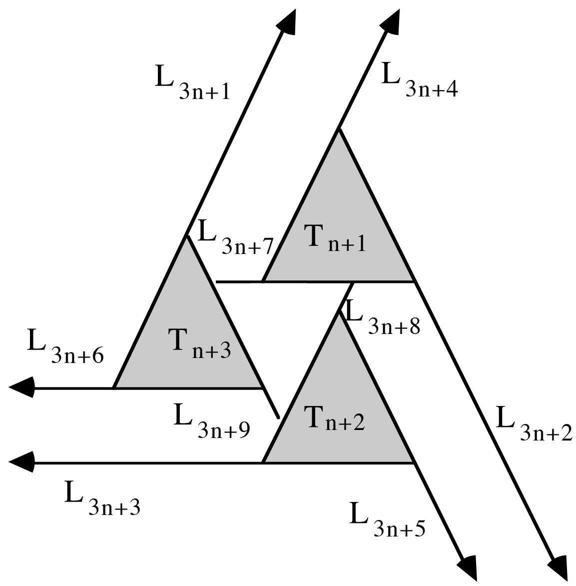


Figure 15

Let **T** be the union of $T_1,...,T_{n+3}$. A convex partition $\pi = \pi(T_1,...,T_{n+3})$ of $R^2\backslash\mathbf{T}$ may be obtained as follows: consider the vertices of the triangles $T_1,...,T_n$ one at a time. When a vertex $p_i$ of a triangle $T_t$ is being considered, draw a directed line segment $L_i$ with an end point in $p_i$ and extending one of the sides of the triangle that meet at $p_i$. See figure 16.

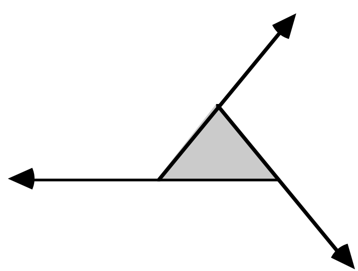

Figure 16

Each side is extended until it either reaches another triangle or it reaches a previously extended side. See figure 17.

The partition $\pi$ contains exactly $2(n+3)+1$ regions $R_1,R_2,.....,R_{2n+7}$. Observe that two adjacent edges of a region $R_i$ may be collinear, nevertheless, they are considered as different edges. Define the adjacency graph $D = D(\pi)$ as in theorem 4. We claim that D satisfies the conditions on Tutte's theorem. In fact, we shall prove that the total number of connected components of D-S is at most $|S|+1$. For this purpose, we use a counting argument which was used in a similar connection in [1].

For each set $U = \{v_1,...,v_u\}$ of vertices of D let $\mathbf{U} = \{R_1,...,R_u\}$ be the corresponding set of regions in $\pi$. A component C of D-U, containing some vertices $v_{i_1}, v_{i_2},..., v_{i_r}$ of D corresponds to the connected region **C** formed by the union of the corresponding regions $R_{i_1}, R_{i_2},....., R_{i_r}$. We shall call **C** a component of R/**T.**

Let us specify certain vertices and edges of the regions and components. A ***corner*** of a region $R_i$ (or component **C**) is a vertex of $R_i$ (of **C**), where two directed line segments $L_s$ and $L_t$ meet in opposite directions, as seen from inside $R_i$ (from **C**). Note that the point where a line $L_s$ meets a triangle $T_j$ is not a corner . For instance, in Figure 10, $u$ is a corner of $R_i$ but is not a corner of $R_k$, while $v$ is a corner of $R_h$ but not of $R_i$

A ***side*** of a region $R_i$ (or component **C**) is an edge of $R_i$ (of **C**), completely contained in a line segment or ray $L_s$ and such that none of its end points is a corner of $R_i$ (of **C**). Note that the edges of $T_1,...,T_{n+3}$ are not part of any sides. In Figure 10, $l$ is a side of $R_i$ but is not a side of $R_h$, while $m$ is a side of both $R_i$ and $R_j$.

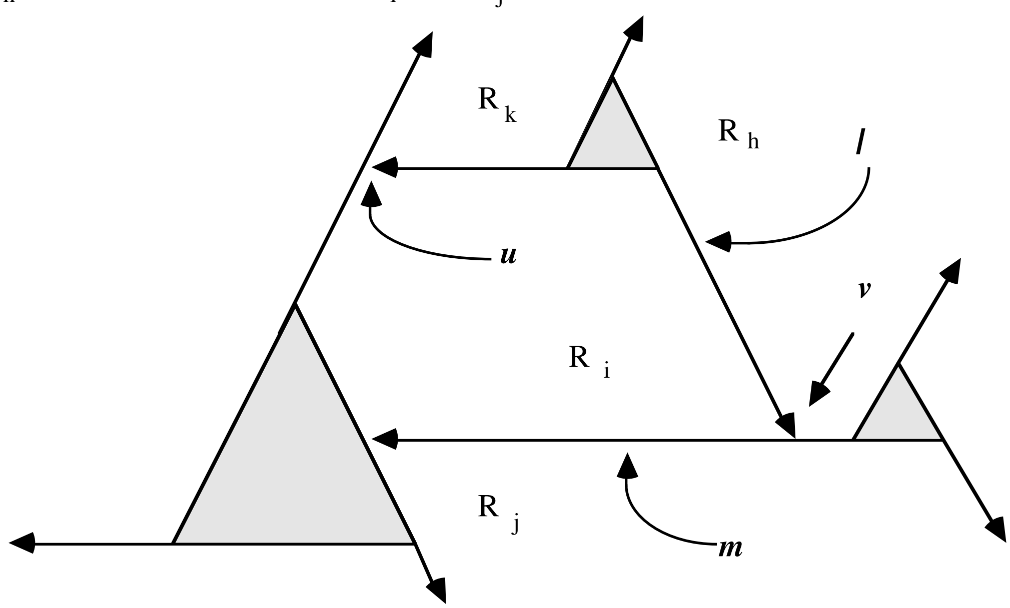


Figure 17

Each of the unbounded regions $R_{i_1}$, $R_{i_2}$ and $R_{i_3}$ of $\pi$, identified in Figure 18, has two sides and no corners. Each other region $R_i$ contains exactly three sides or corners. Each component **C**, which is not by itself one of the regions $R_{i_1}$, $R_{i_2}$ or $R_{i_3}$, contains at least three sides or corners.

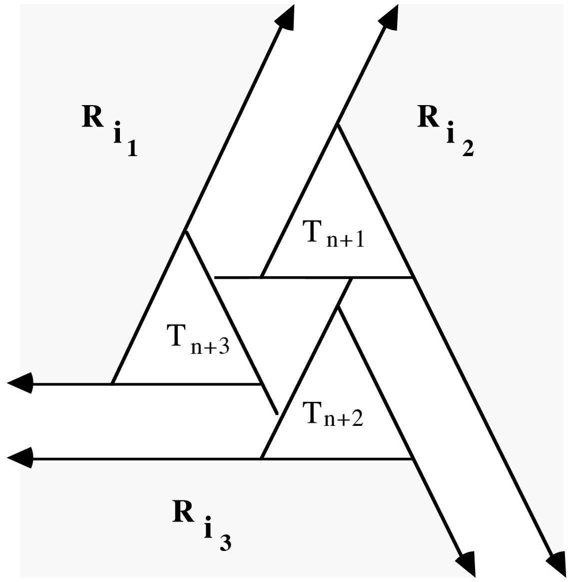


Figure 18

Let $S$ be a set of $s$ of the regions $R_1,...,R_{2n+1}$; without loss of generality, we may assume $S = \{R_1,...,R_s\}$. Delete, from $R^2/\mathbf{T}$, the regions in $S$ and denote by $W_0$ the set of components hereby obtained. The total number $\beta_0$ of sides and corners among the elements of $W_0$ is at least $3t - 3$, where $t = |W_0|$. Replace the regions $R_1,...,R_s$ one at a time. For
$i = 1,2,...,s$, let $W_m$ denote the set of components obtained after the regions $R_1,...,R_m$ have been replaced and let $\beta_m$ be the number of corners and sides among the elements of $W_m$.

Notice that at the $m^{th}$ step, several components may be joined by $R_m$ but the difference between $\beta_{m-1}$ and $\beta_m$ does not exceed 3 since at most one corner (side) is lost for each corner (side) of $R_m$.

Clearly $\beta_s = 0$ since $W_s = \{R^2/\mathbf{T}\}$. Thus $s$ must be large enough so that $3s \geq 3t - 3$ and therefore $t \leq s+1$ as claimed.

Let $M$ be a matching of $D$ with size $n+3$. For every pair $\{R_i, R_j\}$ of regions matched by $M$, place a light source in any point in $R_i \cap R_j$. This light illuminates both regions since they are convex. Complete the set of lights by placing a source inside the sole unmatched region. Notice that at least two lights are placed within the closure of the three unbounded regions of $\pi$; since this regions do not meet the hexagonal region bounded by $T_{n+1}$, $T_{n+2}$, $T_{n+3}$, $L_{3n+7}$, $L_{3n+8}$ and $L_{3n+9}$, then at least two of the $n + 3$ lights are not needed to illuminate the original collection.

We end this article by describing a collection of n pairwise homothetic triangles for which at least n-1 lights are required:

Let $T_1, T_2$ and $T_3$ be mutually tangent triangles. Insert a triangle $T_4$ in the gap bounded by $T_1, T_2$ and $T_3$. Three gaps are now created; in each gap insert a triangle so as to create nine new gaps.

Continue inserting triangles until $3^k$ gaps are created. In the final step, insert triangles $S_1, S_2, \ldots, S_{3^k}$, one in each gap and add three triangles $S_{3^k+1}, S_{3^k+2}$ and $S_{3^k+3}$ outside $T_1, T_2$ and $T_3$. Finally, shrink all triangles by an amount, small enough, so that no light source may illuminate more than two of the $3(3^k+3)$ edges of the triangles $S_1, S_2, \ldots, S_{3^k+3}$. The number of triangles in the collection is $n = 3+1+3+\cdots+3^{k-1}+(3^k+3) = (3^{k+1}+11)/2$, and the number of lights needed is $m \geq (3(3^k+3))/2 = n-1$. Figure 19 illustrates the collection with k=2.

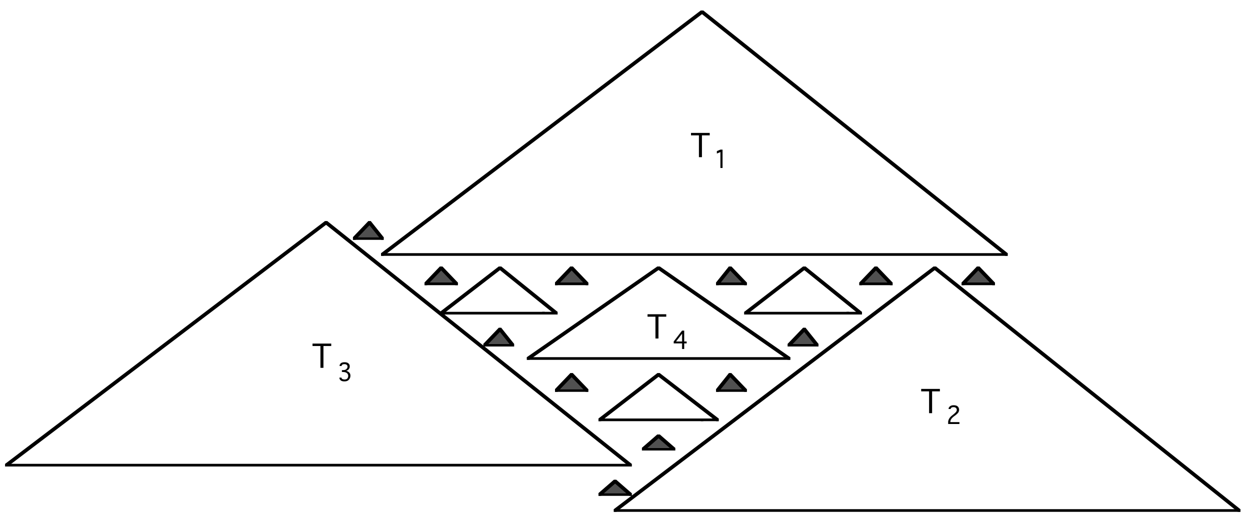


Figure 19

**4.- Conclusions and remarks.**

The bounds presented in this article for the general cases of arbitrary isothetic rectangles and arbitrary triangles are not tight. We believe that there are constants $c_1$ and $c_2$ such that $n+c_1$ lights are always sufficient to illuminate a collection of n isothetic disjoint rectangles, and $n+c_2$ are always sufficient to illuminate n disjoint triangles. The best lower bounds that we now, coincide with the ones given for the restrictive cases of isothetic rectangles with equal width and homothetic triangles.

Our definition disallows illumination by grazing contact. An alternative definition would permit a point x to illuminate a point y if the line segment xy intersects the boundary of some set in F, but does not meet the interior of any set in F. Clearly all the results in this article remain valid under this alternative definition; nevertheless, it is posible that with this definition tighter bounds may be found.

**Acknowledgments.**

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