LATTICES CONTAINED IN PLANAR ORDERS ARE PLANAR

by

Richard Nowakowski[1],

Ivan Rival[2],

and

Jorge Urrutia[2].

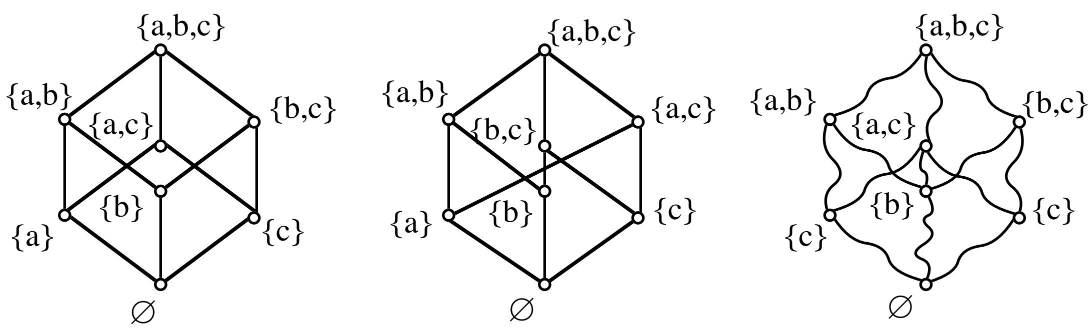
Abstract. The covering graph of a lattice which is contained in an ordered set with a planar covering graph is itself planar.

AMS subject classifications (1980). 06A10

Key words. Covering graph, diagram, planar, lattice.

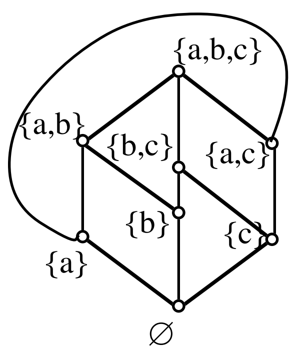
[1] Department of Mathematics, Dalhousie University, Halifax, Canada.
[2] Department of Computer Science, University of Ottawa, Ottawa, Canada, K1N 9B4.

Although it is not yet well understood, the most commonly used graphical scheme to represent an ordered set is a *diagram.* For elements a and b in an ordered set P, say that a *covers* b or b is *covered by* a, and write a>-b, if a > b and, for each x in P, a > x ≥ b implies x=b. A *diagram* of P is a pictorial representation of P on the plane in which small circles, corresponding to the elements of P, are arranged in such a way that, for a and b in P, the circle corresponding to a is higher than the circle corresponding to b whenever a > b and a line segment is drawn to connect the two circles just if a covers b. Although any diagram determines the corresponding ordered set, there is considerable variation possible in the pictorial realization of it. In fact, we may even (and often do) relax the requirement that straight line segments connect circles, by using monotonic curves, that is, curves on which no two distinct points have the same second coordinate (see Figure 1(c)).



(a) (b) (c)

Diagrams of the lattice $\mathbf{2}^3$ of all subsets of a three-element set,ordered by set inclusion.

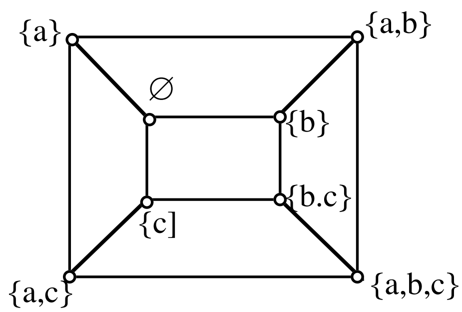


(d)

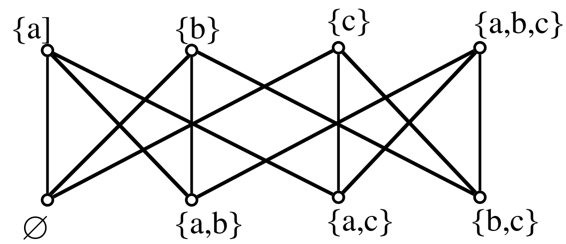
Not a diagram of the lattice $\mathbf{2}^3$.

Figure 1

Closely related to the diagram of P is its *covering graph* which is the graph whose vertices are the elements of P and with an edge joining two vertices a and b just if a covers b or b covers a. A covering graph may have many "orientations', that is, there may be (and usually are) many different ordered sets with the same covering graph. Unlike the diagram there are no monotonicity constraints at all on the edges joining vertices in the covering graph. (See Figure 2.)



A drawing of the planar covering graph of $\mathbf{2}^3$.



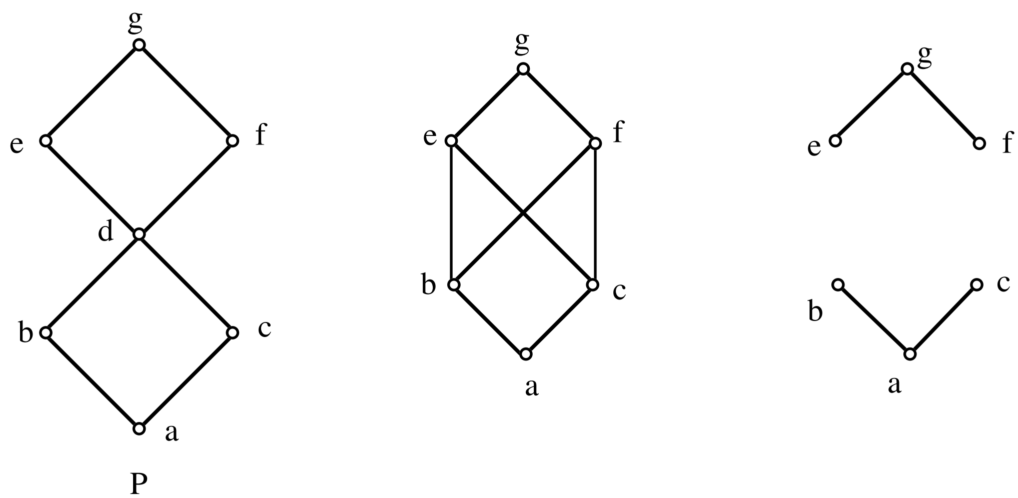
A diagram of an orientation (not a lattice) of the covering graph of $\mathbf{2}^3$.

Figure 2

As for diagrams, it is common to identify a pictorial representation of the covering graph with the covering graph itself. Thus, a (covering) graph is *planar* if it has a representation (drawing) in which no edges intersect, except possibly at their endpoints.

It is a major unsolved problem to characterize the covering graphs of ordered sets [9], although it is now known that the problem of recognizing graphs which are diagrams is NP-complete [J. Nesetril and V. Rödl (1986)]. The problem to enumerate or describe the orientation of a covering graph is also unsolved. A specific variation of this problem is to describe an *invariant* of the covering graph, that is, a (nontrivial) property which, if satisfied by an ordered set is also satisfied by any orientation of its covering graph (cf. [3], [8]). Our main result draws a connection between a property of the covering graph and a property of the order.

For an ordered set P, we consider order subsets Q of P, that is, subsets of P with the ordering induced from P. We also say Q is an order *contained in* P, or even just "subset" of P. This containment relation for orders does not, however, correspond to the usual (induced) containment relation for graphs.



An ordered set P. A subset of P. A subgraph of the covering graph of P.

Figure 3

The few known advances bearing on covering graphs concern lattices, that is, ordered sets in which, for each pair a, b of elements, both the supremum a+b and the infimum a·b exist (cf. [1], [4], [10]). A frequent sufficient condition is that the ordered set be a *truncated lattice*, that is, after adjoining a top and a bottom element, it becomes a lattice. Of course, any lattice is itself a truncated lattice. Here is our main result.

Theorem. *Let* P *be a finite ordered set with planar covering graph and let* cov(P) *be a planar representation of it. Let* L *be a truncated lattice contained in* P*. Then the covering graph of* L *has a representation* cov(L) *in which, whenever* b *covers* a *in* L*, the edge joining* a *to* b *may be taken as a curve following some covering chain in* P *from* a *to* b*. Moreover, this representation* cov(L) *is planar, too.*

Our main result has several striking consequences.

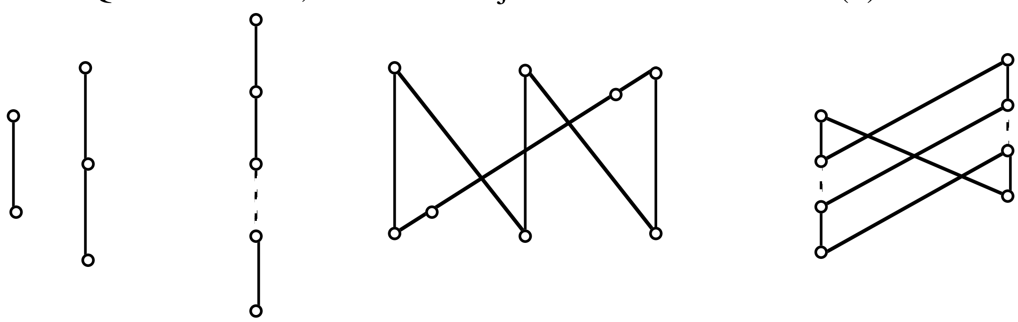
Quite some time ago D. Kelly announced the following, apparently similar, result (cf. [5]). A diagram is *planar* if no edges joining covering pairs intersect except possibly at their endpoints.

Corollary 1. *Let* P *be a finite ordered set. If* P *has a planar diagram then any truncated lattice contained in* P *has a planar diagram too.*

On the other hand, this result fails for arbitrary ordered subsets contained in P, for the ordered set {a, b, c, e, f, g} of Figure 3 is a subset of the planar ordered set P yet it is not planar.

We hasten to mention that a graph with a planar representation also has a planar representation in which all edges are straight line segments [I. Fáry (1948)]; [K. Wagner (1936)]. Similarly an ordered set with a planar diagram also has a planar diagram in which all edges are straight line segments too [D. Kelly (1987)].

We came to these results from an apparently different direction, a problem initiated recently by R. Nowakowski and I. Rival (1987) (cf.[11]) about graphical data structures for ordered sets. Given an ordered set P, what is the smallest number k(P) such that there is an ordered set Q of width k such that a diagram of P is a subdiagram of a diagram of Q? For instance, if P is the disjoint sum of chains then k(P)=1.



P Q P Q

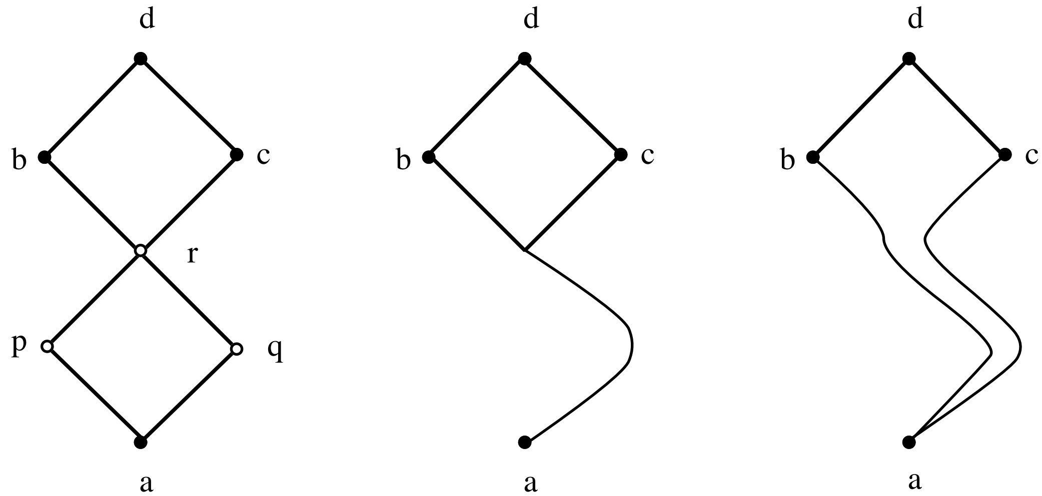
k(P) = 1 k(P) = 2

Figure 4

We shall verify, for instance, that k(P)=2 whenever the diagram of P is a binary tree.Moreover, any dimension two ordered set is contained in an ordered set Q satisfying k(Q)=2. Nowakowski and Rival (1987) have shown that *every finite ordered set* P *has a subdivision which is a subdiagram of an ordered set of width at most three*.

Corollary 2. *There are finite ordered sets* P *for which any ordered set* Q *containing* P *cannot be a subdiagram of an ordered set of width two*.

Proof of the Theorem. Let a∈L. We consider the set of all covering chains *in* P joining a to its upper covers *in* L. Then successively delete edges from the collection of the covering edges in these covering chains until the removal of any further edge would "disconnect" a from one of its upper covers in L. In this case, this union of the remaining covering chains is a tree with root a. By splitting the paths slightly we can produce a tree T_a in which for any distinct upper covers x, y of a the path from a to x does not meet the path from a to y, except in a (see Figure 5).



The shaded elements form L in P.

The collection of covering chains with unnecessary edges removed.

The 'tree' structure for each element of L with paths splitted where needed.

Figure 5

Now, consider a planar representation of cov(P) and distinguish the vertices of P which correspond to L. For the edge joining a covering pair x >- a of L choose the unique path in T_a from a to x. If two such edges cross but not at an endpoint, say a to x and b to y (b -< y in L) then the crossing point must correspond to a point of P, obviously not in L. As these paths were generated by covering chains in P it follows that y > a and x > b, too. Since L is a lattice a,b < a+b < x,y and, in particular, y cannot cover b nor x can cover a. Therefore, such edges do not cross at all and this representation of cov(L) induced by cov(P) is planar too. This completes the proof.

The ordered set P in Figure 5 shows that we cannot expect to use edges following *any* path from a in L to its upper covers b and c for the paths a < p < r < c and a < q < r < b will surely intersect, even after any small splitting. An example illustrating the Theorem is drawn in Figure 6.

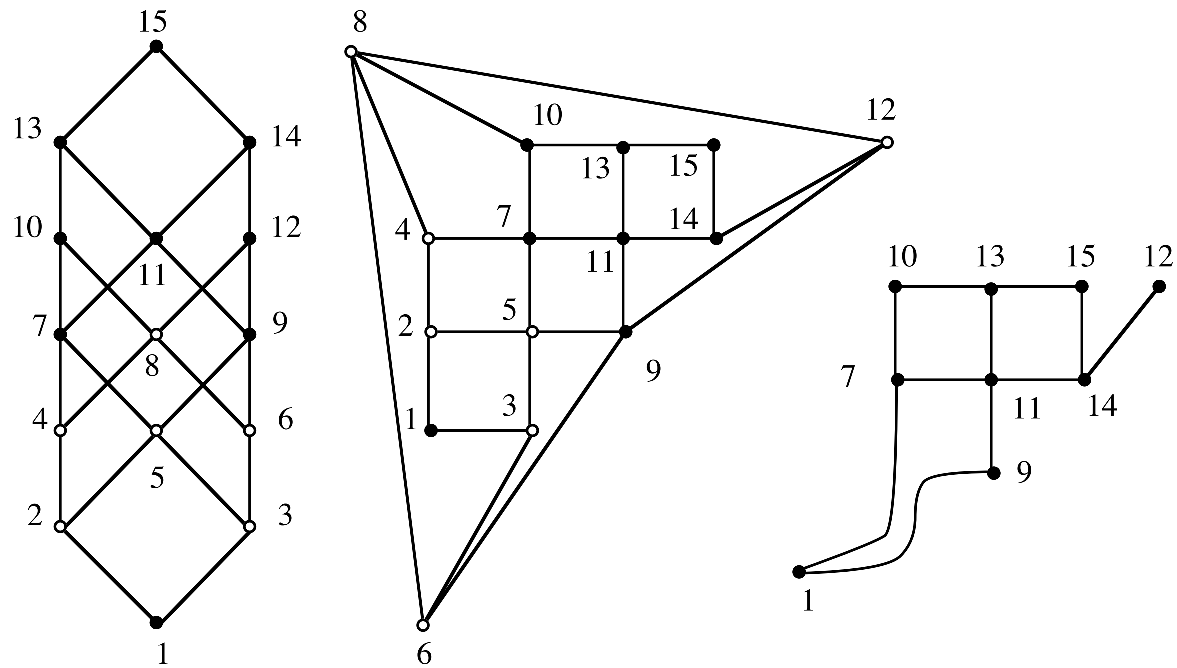
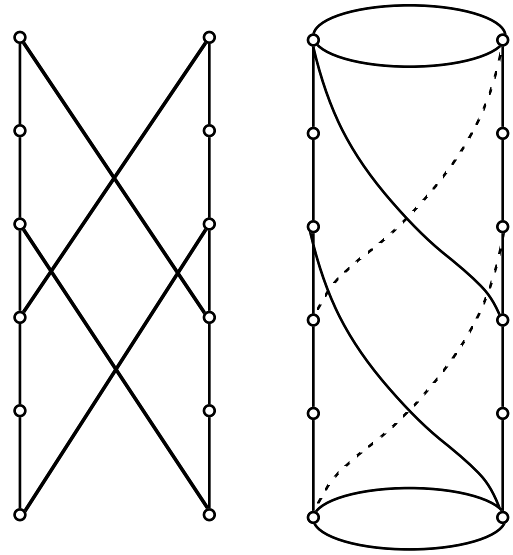


Figure 6

The proof of Corollary 1 is now obvious for, if we start with a planar diagram of P then all paths in P constructed from covering chains in P are actually monotonic arcs. Therefore, the edges chosen for the planar representation of cov(L) in the Theorem will be monotonic too. This yields a planar representation of the diagram of L.

The proof of Corollary 2 is slightly more intricate. Let Q be an ordered set of width two. We may visualize any such ordered set as consisting of two vertical chains with edges joining them, some with positive slope, never intersecting, and the others with negative slope, also never intersecting (see Figure 7).



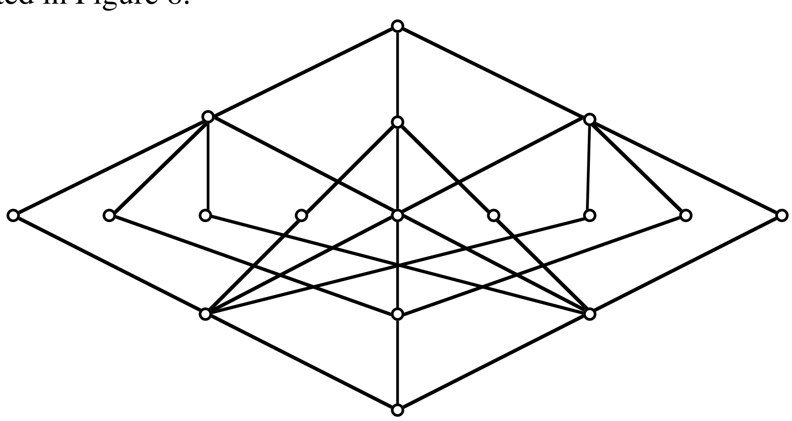
An ordered set of width two.

A "cylindrical" embedding.

Figure 7

We may furthermore draw this same diagram on a vertical cylinder such that the two vertical chains form parallel stripes on either side of the cylinder, that the edges with positive slope are all drawn on the back of the cylinder, and that the edges with negative slope are all drawn on the front of the cylinder, say. As any graph which can be embedded on a cylinder is planar it follows that any ordered set of width two must have a planar covering graph. Moreover, any ordered set whose diagram is a subdiagram of an ordered set of width two must be planar too.

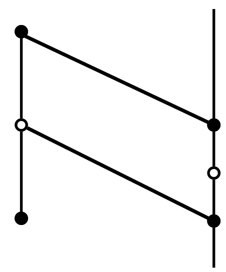
Now, let P be any finite lattice with a nonplanar covering graph, for instance, the lattice illustrated in Figure 8.



A lattice whose covering graph contains $K_{3,3}$.

Figure 8

Note that while any subdiagram of a width two ordered set cannot contain elements with three or more upper or lower covers, an ordered set of width two may well contain subdiagrams with elements which do cover, or are covered by, three or more elements (cf. Figure 9).

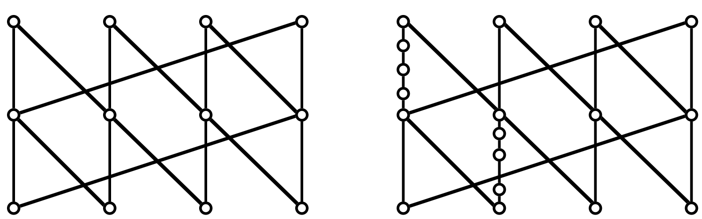


A subdiagram of an ordered set of width two containing an element with three lower covers.

Figure 9

Suppose there were an ordered set Q containing the lattice P illustrated in Figure 8 which in turn were a subdiagram of an ordered set of width two. Then cov(Q) must be planar and, according to Theorem 1, cov(P) would be planar, too. This is a contradiction.

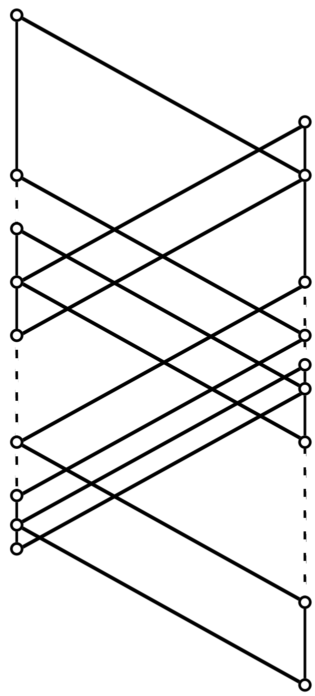
We conclude this note with a construction scheme to produce ordered sets P satisfying k(P) = 2. Let P(m,n) be the ordered set with m levels and with n elements at each level such that each consecutive pair of levels consists of a 2n-cycle (cf. Figure 10). Next construct Q(m,n) from



P(3,4) Q(3,4)

Figure 10

P(m,n) by subdividing one particular edge at each level, m times. Then k(Q(m,n)) = 2 (cf. Figure 11).



k(Q(3,4))=2.

Figure 11

Any two-dimensional grid is contained in P(m,n), for sufficiently large m and n — in the upset generated by a minimal element of P(m,n). As every two-dimensional order is contained in a two-dimensional grid it is a subset of an ordered set which is a subdiagram of one of width two. A binary tree too is embeddable in a two-dimensional grid. At the same time it is easy to verify that P(n-1,n) contains the ordered set of singletons and (n-1)-element subsets of an n-element set, whence the dimension of P(n-1,n) is at least n, yet there are ordered sets Q containing P(n-1,n) such that k(Q) = 2.

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