Universal Measuring Boxes with Triangular Bases

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Abstract

Measuring cups are everyday instruments used to measure the amount of liquid required for many common household tasks such as cooking. A measuring cup usually has gradations marked on its sides. In this paper we study measuring boxes without gradations which can nevertheless measure any integral amount, say liters, of liquid up to their full capacity. These boxes will be called *universal measuring boxes*. We study two types of measuring boxes with triangular bases, and for each type determine its maximum possible capacity.

1 Introduction

A traditional Japanese device used to measure from 1 to 6 liters is a lidless rectangular box of capacity 6. By tilting the box to align the surface of the liquid with its edges and vertices, 1, 3, and 6 liters can be measured as shown in Figure 1. If a shop clerk needs to give a customer 4 liters of liquid, he would first fill the box by immersing it in the large shop container. He would then pour 3 liters into the customer's container. Next he would pour liquid back into the shop container until 1 liter was left, which he would then pour into the customer's container. It is easy to see that a similar procedure could be carried out to obtain any amount of k liters, $1 \le k \le 6$.

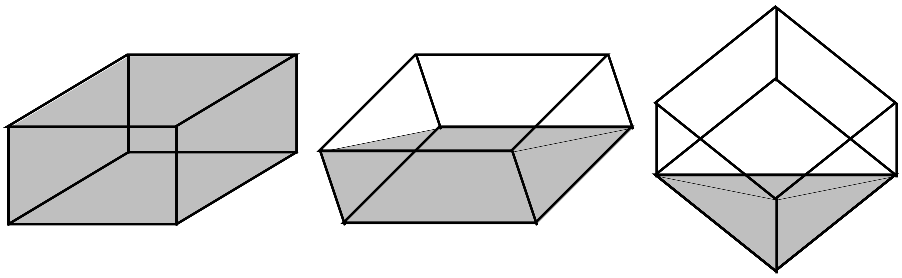

Figure 1:

Motivated by this, we proceed to study the problem of transferring some exact amount of liquid from a large container A filled with liquid to another container B using some measuring box $\mathcal{M}$ under the following conditions:

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(C1) *We are allowed to take liquid out of container A by filling $\mathcal{M}$ only once, but we may pour back some liquid from $\mathcal{M}$ to A as many times as we need, and may transfer liquid from $\mathcal{M}$ to container B as many times as we need. However, we are not allowed to pour liquid back from container B to $\mathcal{M}$.*

(C2) *In measuring the volume of liquid, we may use only vertices of $\mathcal{M}$ as markers; in particular, when measurement is carried out, the plane formed by the surface of the liquid in $\mathcal{M}$ should contain at least three of its vertices.*

A measuring box $\mathcal{M}$ that allows us to measure from 1 to k liters, where k is the capacity of $\mathcal{M}$, is called a *universal measuring box.* In this paper we consider measuring boxes with shapes other than rectangular. We study two types of universal measuring boxes with triangular bases as shown in Figure 2, and determine the dimensions they should have to maximize their capacity. In the first case, we require the boxes to have *sides orthogonal* to their bases. We call these *orthogonal measuring boxes.* We show that the capacity of an orthogonal measuring box is at most 41, and present two boxes with this capacity. We then drop the orthogonality condition, and show that in this case, the maximum capacity of such universal measuring boxes is 127. Measuring boxes with this capacity are also presented. Since all measuring boxes considered here have triangular bases, we will refer to them in the remainder of this paper simply as "measuring boxes."

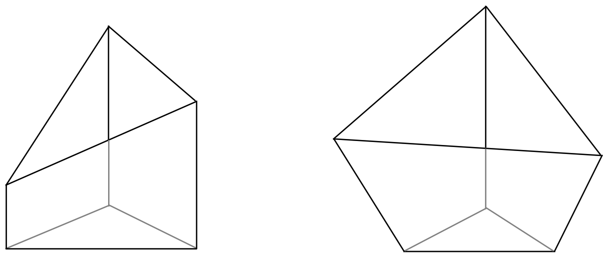

Figure 2:

2 Orthogonal Measuring Boxes

Let $\mathcal{M}$ be an orthogonal measuring box. Let $h_1 \leq h_2 \leq h_3$ be the heights of the vertical edges of $\mathcal{M}$. To simplify the arithmetic, we will assume from now on that the area of the base of $\mathcal{M}$ is 3. The objective of this section is to prove the following theorem.

Theorem 1 *The largest capacity of an orthogonal universal measuring box $\mathcal{M}$ is* 41. *Moreover there are two different measuring boxes with this volume. Their heights are* $\{12, 13, 16\}$ *and* $\{4, 18, 19\}$.

We begin by proving the following theorem.

Theorem 2 *Let $\mathcal{M}$ be a box with base area 3, and heights $h_1 < h_2 < h_3$. Then we can measure the following amounts of liquid: $h_1,\ h_2,\ h_3,\ h_1+h_2,\ h_1+h_3,\ h_2+h_3,\ h_1+h_2+h_3$.*

Proof: Let the vertices of $\mathcal{M}$ be labelled $a_1,\ a_2,\ a_3,\ a_1',\ a_2',\ a_3'$ as shown in Figure 3. Call the distance between a_i and a_i' h_i, $i = 1, 2, 3$. Recall that the volume of a tetrahedron with base of area A and height h is $\frac{Ah}{3}$. Since the area of the base of this measuring box is 3, it follows immediately that the volume of the tetrahedron with vertices $a_1,\ a_2,\ a_3,\ a_i'$ is h_i, $i = 1, 2, 3$.

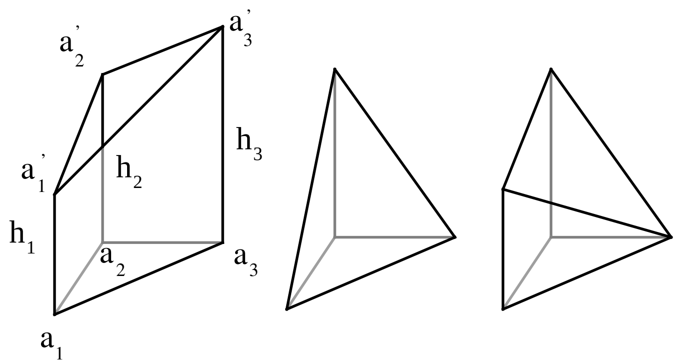

Figure 3: The volumes of the polyhedra shown here are $h_1 + h_2 + h_3$, h_2 and $h_1 + h_2$ respectively.

We now show that the volume of the polyhedron with vertices $a_1,\ a_2,\ a_3,\ a_i'$, and a_j' is precisely $h_i + h_j$, $i \neq j,\ i,\ j \in \{1, 2, 3\}$.

Given a triangular cylinder $\mathcal{T}$, assume that the area of the triangle obtained by cutting $\mathcal{T}$ along any plane perpendicular to its edges is 3. We observe that if we choose two points $p,\ q$ on one of its edges, and any two points r and s, one on each of the remaining edges of $\mathcal{T}$, then the volume of the tetrahedron with vertices $p,\ q,\ r,\ s$ is equal to the distance between p and q; see figure 4.

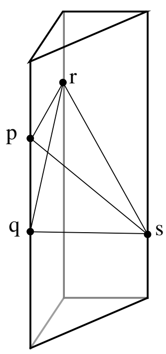

Figure 4: Once p and q are chosen, the volume of the tetrahedron with vertices $p,\ q,\ r$, and s remains constant regardless of the positions of r and s.

Suppose w.l.o.g. that $i = 1,\ j = 2$. Let us now dissect the polyhedron with vertices $a_1,\ a_2,\ a_3,\ a_1',\ a_2'$ into two tetrahedra with vertices $a_1,\ a_2,\ a_3,\ a_1'$ and $a_2,\ a_3,\ a_1',\ a_2'$ respectively; see Figure 5. The

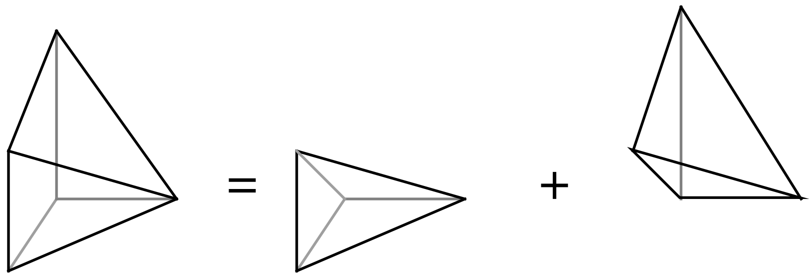

Figure 5:

volume of the first tetrahedron is h_1, and by the above observation, the area of the second tetrahedron is the distance between a_2 to a'_2, which is h_2.

Using similar arguments, we can now show that the volume of $\mathcal{M}$ is $h_1 + h_2 + h_3$, which proves the theorem. ∎

Let us relabel the amounts that can be measured using $\mathcal{M}$ by $l_1, \ldots, l_7$ such that $l_i < l_j$, $i < j$, and let $l_0 = 0$. Define $D = \{d_i = l_i - l_{i-1} : i = 1, 2, \ldots, 7\}$.

For example, let $\mathcal{M}_1$ be the orthogonal measuring box with $h_1 = 12$, $h_2 = 13$, and $h_3 = 16$. By Theorem 2, using $\mathcal{M}_1$ we can measure the following amounts:

$$l_0 = 0,\ l_1 = h_1 = 12,\ l_2 = h_2 = 13,\ l_3 = h_3 = 16,\ l_4 = h_1 + h_2 = 25$$
$$l_5 = h_1 + h_3 = 28,\ l_6 = h_2 + h_3 = 29,\ l_7 = h_1 + h_2 + h_3 = 41.$$

This corresponds to the set $D = \{12, 1, 3, 9, 3, 1, 12\}$.

If we choose a few terms from the sequence D, for example d_7, d_6, d_4, d_1, we can observe the following:

$$d_7 + d_6 + d_4 + d_1 = (l_7 - l_6) + (l_6 - l_5) + (l_4 - l_3) + (l_1 - l_0)$$
$$= (l_7 - l_5) + (l_4 - l_3) + l_1 = (41 - 28) + (25 - 16) - 12 = 34$$

Careful observation indicates that 34 liters can be measured with $\mathcal{M}_1$ as follows:

1. Fill $\mathcal{M}_1$, then pour some liquid from $\mathcal{M}_1$ into B until 28 liters are left in $\mathcal{M}_1$.
2. Now pour some liquid from $\mathcal{M}_1$ to A until 25 liters of liquid are left in $\mathcal{M}_1$, then pour liquid from $\mathcal{M}_1$ into container B until 16 liters are left in $\mathcal{M}_1$.
3. Finally, pour back some of the remaining liquid from $\mathcal{M}_1$ to A until 12 liters of liquid are left, then pour the remaining 12 liters into B.

Since this process can be carried out for any subset of the set D arising from any measuring box, we have proved the following lemma.

Lemma 1 *The set of all possible values of the volume of liquid that can be transferred to container B using an orthogonal measuring box $\mathcal{M}$ coincides with the set S of all numbers representable as sums of distinct terms appearing in the sequence D.*

The reader may now verify that $\mathcal{M}_1$ is indeed universal, i.e. that for any integer $1 \leq i \leq 41$ there is a subset of D such that the sum of its elements is i. In Section 2.2 we give a very efficient method to decide if a container is universal, from which the universality of $\mathcal{M}_1$ follows easily.

2.1 The Maximum Capacity of an Orthogonal Universal Measuring Box

We now proceed to show that the largest capacity of an orthogonal measuring box is 41. The amounts of liquid measured by $\mathcal{M}$ can be arranged in non-decreasing order in one of the following two ways:

$$h_1 \leq h_2 \leq h_3 \leq h_1 + h_2 \leq h_1 + h_3 \leq h_2 + h_3 \leq h_1 + h_2 + h_3$$
$$h_1 \leq h_2 \leq h_1 + h_2 \leq h_3 \leq h_1 + h_3 \leq h_2 + h_3 \leq h_1 + h_2 + h_3.$$

Suppose first that

$$h_1 \leq h_2 \leq h_3 \leq h_1 + h_2 \leq h_1 + h_3 \leq h_2 + h_3 \leq h_1 + h_2 + h_3.$$

In this case,

$$D = \{h_1, h_2 - h_1, h_3 - h_2, h_1 + h_2 - h_3, h_3 - h_2, h_2 - h_1, h_1\}.$$

Let $a = h_1$, $b = h_2 - h_1$, $c = h_3 - h_2$, and $d = h_1 + h_2 - h_3$, and observe that $d = a - c$. Since a, b, c appear twice in D and d appears only once in D, and the elements of the set S are representable as sums of some of these terms; they can be expressed in the form

$$U(i, j, k, l) = ia + jb + kc + ld$$

where $i, j, k \in \{0, 1, 2\}$ and $l \in \{0, 1\}$, not all of i, j, k, l equal to 0. Therefore, we can conclude that $|S| \leq 3 \times 3 \times 3 \times 2 - 1 = 53$. However, since $c = a - d$, if $d \geq 1$ and $a \leq 1$, we have

$$U(i, j, k, l) = U(i + 1, j, k - 1, l - 1),$$

where equality holds only if $i \in \{0, 1\}, j \in \{0, 1, 2\}, k \in \{1, 2\}, l = 1$. Then at least $2 \times 3 \times 2 \times 1 = 12$ combinations of the 53 generate repetitions. Therefore $|S| \leq 53 - 12 = 41$.

In the second case, where

$$h_1 \leq h_2 \leq h_1 + h_2 \leq h_3 \leq h_1 + h_3 \leq h_2 + h_3 \leq h_1 + h_2 + h_3,$$

$D = \{h_1, h_2 - h_1, h_1, h_3 - (h_1 + h_2), h_1, h_2 - h_1, h_1\}$.

Observe that h_1 appears four times, $h_2 - h_1$ twice, and $h_3 - (h_1 + h_2)$ once. Using similar arguments to those above, we obtain that in this case using subsets of D we can generate at most $5 \times 3 \times 2 = 30$ different integers. Thus we have proved that the largest capacity of an orthogonal universal measuring box with triangular base is 41. Theorem 1 is now proved.

2.2 Generating Sets

We now give an easy test to verify whether a measuring box is universal. From the results of the previous section it follows that $\mathcal{M}$ is universal if any integer between 1 and $h_1 + h_2 + h_3$ can be expressed as the sum of the elements of a subset of D.

A set of integers D that satisfies the property that for any integer i between 1 and $k = \sum_{j \in D} j$ there is a subset $S \subset D$ such that $i = \sum_{j \in S} j$ will be called a generating set. The best generating sets are the ones corresponding to powers of 2, i.e. $\{1, 2, 4, \ldots, 2^k\}$. In this section we develop a linear-time test to decide whether a set of integers is a generating set.

Assume that $D = \{d_1, \ldots, d_m$ such that $d_i \leq d_j$, $i < j\}$.

Theorem 3 *The set D is a generating set if and only if* $d_1 = 1$, *and for* $i > 1$, $d_i - 1 \leq s_1 + \cdots + d_{i-1}$, $i = 2, \ldots, m$.

Proof: The result is clearly true if D has one element. Suppose then that it is true for any set D' with $k-1$ elements and let D be a set with k elements such that $d_i - 1 \leq d_1 + \ldots + d_{k-1}$, $i = 1, \ldots, k$. By induction assume that any integer m between 1 and $d_1 + \ldots + d_{k-1}$ can be expressed as the sum of some subset S_m of $\{d_1, \ldots, d_{k-1}\}$. Then any integer of the form $m + d_i$, $0 \leq m \leq d_1 + \ldots + d_{i-1}$ can be also expressed as the sum of the elements of $S_m \cup \{d_k\}$ which is a subset of $\{s_1, \ldots, s_{i-1}, s_k\}$.

If for some i we have $d_i - 1 > d_1 + \ldots + d_{i-1}$ then D is not a generating sequence since $d_1 + \ldots + d_{i-1} + 1$ can not be generated. The result follows. ■

Using this theorem we can now show that $\mathcal{M}_1$ is universal. Indeed, the set D of differences generated by $\mathcal{M}_1$ consists of the elements $\{12, 1, 3, 9, 3, 1, 12\}$. If we sort the elements of D we get $D = \{1, 1, 3, 3, 9, 12\}$. Clearly this sequence of integers satisfies the theorem, proving that $\mathcal{M}_1$ is universal. In a similar way we can prove that a measuring box with heights $4, 18$, and 19 is universal. By computer search we found that no other orthogonal universal box exists.

3 Measuring Boxes with Sides not Orthogonal to the Base

In this section we prove the following theorem.

Theorem 4 *There is a universal measuring box with triangular base of volume* 127*. This is the best possible.*

Some preliminary results will be needed to prove the theorem.

Let $T(a,b,c)$ be the tetrahedron whose vertices are the origin and the points $A = (a\sqrt[3]{6},0,0), B = (0,b\sqrt[3]{6},0), C = (0,0,c\sqrt[3]{6})$, $a,b,c > 0$. Observe that the volume of $T(a,b,c)$ is $(a \times b \times c)$. In particular, the volume of $T(1,1,1)$ is 1. Let $\mathcal{P}(a,b,c)$ be the polyhedron obtained from $T(a,b,c)$ by cutting $T(1,1,1)$ away from it; $a,b,c > 1$; see Figure 6. We also observe that the volume of $\mathcal{P}(a,b,c)$ is $(a \times b \times c) - 1$. Let $P = (\sqrt[3]{6},0,o)$, $Q = (0,\sqrt[3]{6},0)$ and $R = (0,0,\sqrt[3]{6})$.

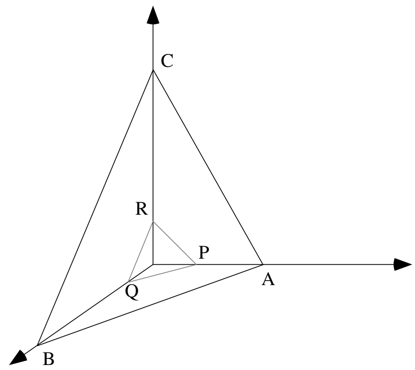

Figure 6:

Our measuring box $\mathcal{M}(a,b,c)$ will consist of all the faces of $\mathcal{P}(a,b,c)$ except the triangle with vertices A,B,C. We now prove the following theorem.

Theorem 5 *Using $\mathcal{M}(a,b,c)$ the following amounts of liquid can be measured:* $a-1$, $b-1$, $c-1$, $(a \times b)-1$, $(a \times c)-1$, $(b \times c)-1$ *and* $(a \times b \times c)-1$.

Proof: Consider the tetrahedron T_A with vertices P,Q,R, and A; see Figure 7(a). Define T_B and T_C in a a similar way. The volume of T_A is $(a \times 1 \times 1) - 1 = a - 1$. Similarly, the volumes of T_B and T_C are $b-1$ and $c-1$ respectively.

Let $T_{A,B}$ be the polyhedron with vertices P,Q,R,A and B; see Figure 7(b). The volume of $T_{A,B}$ is one unit less than the volume of the tetrahedron with vertices A,B,R and the origin, or $(a \times b \times 1) - 1 = (a \times b) - 1$. In a similar way the volumes of $T_{A,C}$ and $T_{B,C}$ are $(a \times c) - 1$, and $(b \times c) - 1$.

Since the volume of $\mathcal{M}(a,b,c)$ is $(a \times b \times c) - 1$, and the volumes that can be measured with $\mathcal{M}(a,b,c)$ are the volumes of $T_A, T_B, T_C, T_{A,B}, T_{A,C}, T_{B,C}$ and $\mathcal{M}(a,b,c)$, the result follows. ∎

Let us now consider $\mathcal{M}(2,4,16)$. By Theorem 5. using this container we can measure $1,3,7,15,31,63$, and 127 liters. Using Lemma 1, it can be seen that the set of values that can be measured are those

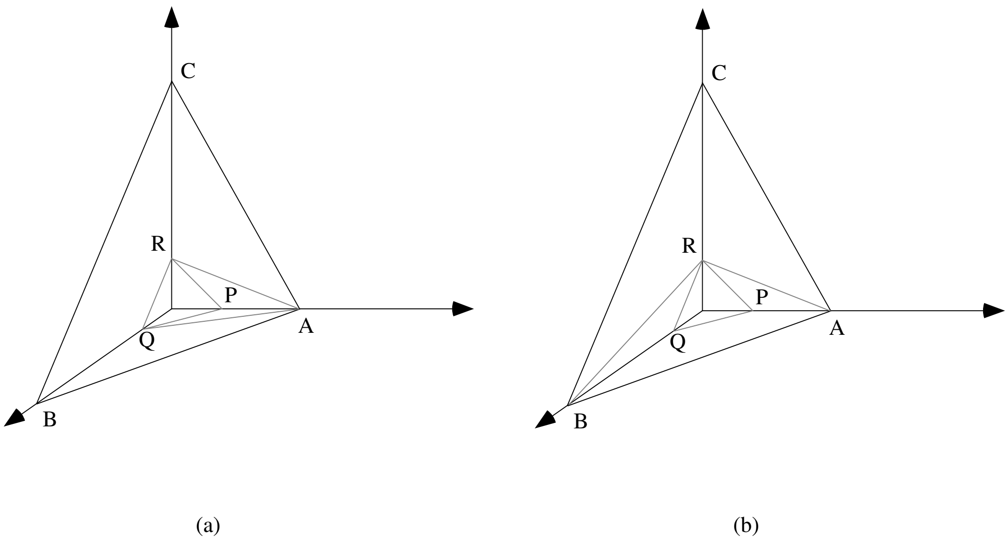

Figure 7:

numbers that can be written as combinations of

$$D = \{1, (3-1), (7-3), (15-7), (31-15), (63-31), (127-63)\} = \{1, 2, 4, 8, 16, 32, 64\},$$

which are the numbers between 1 and 127. Since this is clearly optimal, Theorem 4 follows. ∎

4 Related Problems

We can also consider universal measuring boxes with trapezoidal sides perpendicular to a non-triangular base. In [1, 2], we discuss such universal measuring boxes with a rectangular base; see Figure 8. In [2] it is shown that there is a universal measuring box with a rectangular base of area 6 which has capacity 858. The box has heights $\{130, 132, 156, 169\}$ which appear in this order along the sides of the box. This result was obtained using a C++ program in which the heights of the boxes are restricted to be at most 212 using the test derived from Theorem 3 in this paper. An improvement on the bound 858 may still be possible with further work and a more powerful computer.

It would be also interesting to investigate measuring boxes whose bases are various convex polygons. The following problems are suggested for further investigation.

Open Problem 1 *Find the universal measuring boxes of maximum capacity with rectangular bases.*

Open Problem 2 *For various convex polygonal bases, determine the universal measuring boxes of maximum capacity.*

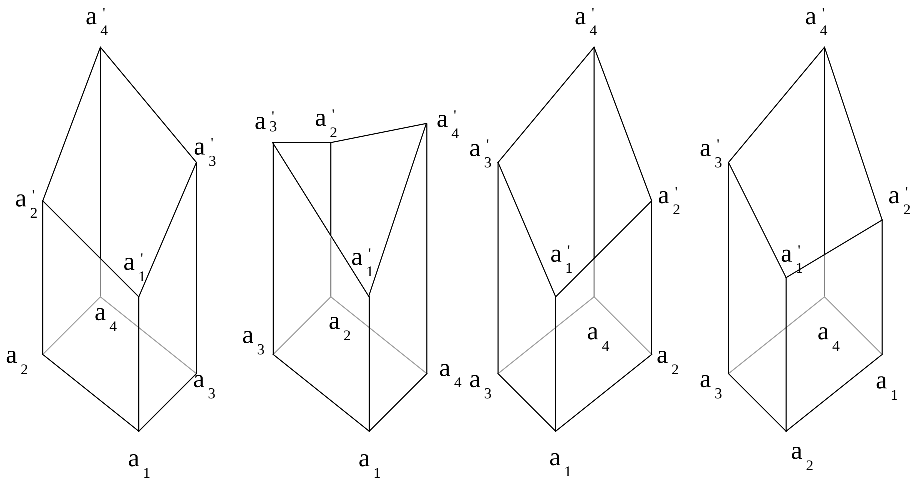

Figure 8:

If we remove the condition imposed on side faces to be perpendicular to the base, the problems become more challenging.

Open Problem 3 *For various convex polygonal bases, determine the universal measuring boxes of maximum capacity whose side faces need not be perpendicular to the base.*

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