On Modem Illumination Problems

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Resumen

In this paper we review recent results on a new variation of the *Art Gallery* problem. A common problem we face nowadays, is that of placing a set of wireless modems in a building in such a way that a computer placed anywhere within the building receives a signal strong enough to connect to the Web. In most buildings, the main limitation for this problem is not the distance of a computer to a wireless modem, but rather the number of walls that separate them. We study variations of the following problem: Let P be a simple polygon with n vertices. How many points $p_1, \ldots, p_k$ (representing wireless modems) are always sufficient such that for any other point p in P, there is a p_i such that the line segment joining p to p_i crosses at most k edges of P? The parameter k represents the *strenght* of the signal emited by the modems. We study variations of this problem for families of line segments, families of lines, orthogonal polygons, and sets of horizontal or vertical disjoint segments, or sets of lines.

1. Introducción

Let $p_0, \ldots, p_{n-1}$ be a set of points on the plane, and e_i the line segment joining p_i to p_{i+1}, $i = 0, \ldots, n-1$, addition taken *mod n*. We say that $e_1, \ldots, e_{n-1}$ form a simple polygon P if e_i and e_j do not intersect, except perhaps at a common end point, $i \neq j$. The elements of $\{e_0, \ldots, e_{n-1}\}$ are called the edges of P. All polygons considered here are simple. Abusing a bit our terminology, in this paper a polygon will refer to the closed region of the plane bounded by its edges. We say that two points p and q of a polygon P are visible if the line segment joining them is contained in P. A set of points $g_1, \ldots, g_k$ guards P if any point in P is visible from at least one g_i. In 1975, V. Klee posed the following problem, known as *The Art Gallery Problem*:

How many guards are always sufficient to guard any simple polygon with n vertices?

The Art Gallery problem was solved by V. Chvatal [3], he proved that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to guard any polygon with n vertices.

Since then, there has been a plethora of very interesting results and variations to the original problem, the interested reader can consult [8, 11, 12]. A new variation in which the edges of our *polygons* are allowed to be arcs of convex curves has been studied in and [2, 6, 7]. In this paper we deal with a new variation of the Art Gallery Problem arising from the following everyday and practical problem: How to place wireless modems in a building in such a way that at any point within the building a laptop with a wireless card receives a signal strong enough to have a stable connection to navigate in the Web?

Experience dictates that in most buildings, the distance of our laptop to a wireless modem is not a limiting factor to obtaining a good signal, the main limiting factor seems to be the number of walls that separate us from a wireless modem.

This inspired us to study the following problem which we cast as follows: Let $k \geq 0$ be an integer, and P a polygon with n vertices. We say that a k-modem $\mathcal{M}$ represented by a point on the plane covers a point $p \in P$ if the line segment joining p to $\mathcal{M}$ *crosses* the boundary of P at most k times.

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In this paper we review some of the results known in this area, and present some new results. In section 3 we study several variations of the following problem:

The k-modem Art Gallery Problem: *How many k-modems are always sufficient to cover all the points of a polygon P with n vertices?*

In Figure 1 we show a polygon for which we need two 2-modems to cover it, one is not sufficient.

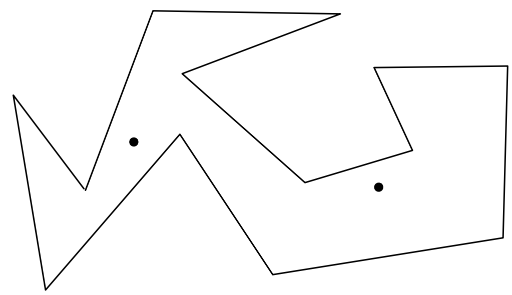

Figura 1: Two 2-modems placed at the points shown as small circles cover the polygon shown here.

In section 2 we study the following problem: Suppose then that we have a set of obstacles represented by a set $L = \{l_1, \ldots, l_n\}$ of n disjoint line segments. As above, we say that a point p is covered by a k-modem $\mathcal{M}$ if the line segment connecting p to $\mathcal{M}$ intersects at most k elements of L.

The k-modem Covering Problem of the Plane: *How many k-modems are always sufficient to cover all the points of the plane in the presence of n obstacles, represented by a set of n disjoint lines?*

In section 3 we review briefly the known results on covering polygons with modems.

In section 4 we study covering problems for polygons, orthogonal polygons, and sets disjoint horizontal or vertical line segments using few modems.

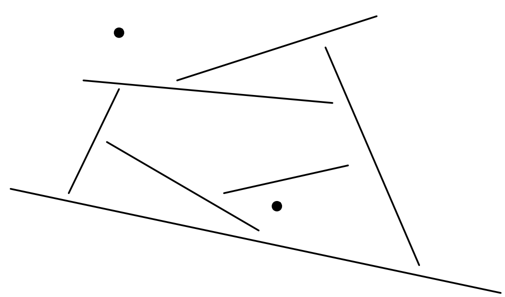

Figura 2: Two 2-modems located at the points shown as small circles are sufficient to cover the plane.

2. Arrangements of Lines

In this section we consider a variation of our problem involving families of lines. Let $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ be a set of lines such that no three of them intersect at a point, and containing no parallel lines, see Figure 10. The elements of $\mathcal{A}$ divide the plane into a set of faces bounded by the elements of $\mathcal{A}$. The set $\mathcal{A}$ together with the set of faces they produce is known as an *arrangements of lines*, and will be simply denoted as $\mathcal{A}$. For technical reasons we will assume that the faces of $\mathcal{A}$ are open.

We say that a k-modem $\mathcal{M}$ coves a face $\mathcal{F}$ of $\mathcal{A}$ if any line joining $\mathcal{M}$ to any point in $\mathcal{F}$ crosses at most k elements of $\mathcal{A}$. In this case, we also say that $\mathcal{M}$ is at distance at most k from $\mathcal{F}$. In general we will assume that the modems are located in the interior of the faces of $\mathcal{A}$. In a few instances we will allow them to be located on the lines of $\mathcal{A}$. In such case we will assume that a ray emanating from a point on a line ℓ does not cross ℓ, and the bounds obtained will drop by one. A set of k-modems $\mathcal{H} = \{\mathcal{M}_1, \ldots, \mathcal{M}_i\}$ *covers an arrangement* $\mathcal{A}$ if every face of $\mathcal{A}$ is covered by at least one of the modems. In this section we study the following problem:

Arrangement Illumination Problem: *How many k-modems are always sufficient to cover any arrangement $\mathcal{A}$ with n lines?*

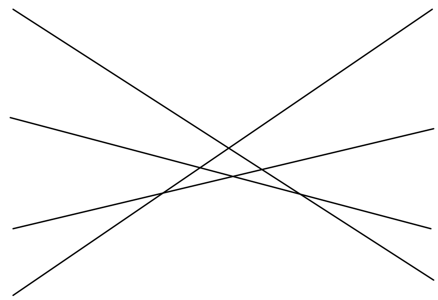

Figura 3: An arrangement of lines.

It is clear that a n-modem $\mathcal{M}$ placed anywhere on the plane always covers all the faces of an arrangement with n lines. It is also easy to see that if a k-modem $\mathcal{M}$ covers all the faces of $\mathcal{A}$, then $k \geq \lceil \frac{n}{2} \rceil$. On the other hand, it is easy to construct arrangements with n lines such that no $\lceil \frac{n}{2} \rceil$-modem covers them. For example if we have an arrangement $\mathcal{A}$ with three lines $\ell_1, \ldots, \ell_3$, then a modem placed in the interior of the bounded face of the arrangement will necessarily cross two lines to reach some faces of $\mathcal{A}$. A modem placed in the interior of any other face of $\mathcal{A}$ has to cross three lines to reach some faces of $\mathcal{A}$.

Replace now each of the lines $\ell_1, \ldots, \ell_3$ of $\mathcal{A}$ by an arrangement of n lines $\mathcal{A}_i$ such that:

1. all the lines of $\mathcal{A}_i$ have a slope within ϵ from that of l_i, $i = 1, 2, 3$,
2. all of the elements of $\mathcal{A}_i$ pass within an δ distance from the middle point of the segment contained in ℓ_i whose endpoints are the intersection points of ℓ_i with the other elements of $\mathcal{A}$, ϵ and δ small enough.

It is easy to see that the arrangement $\mathcal{A}^* = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_3$ is such that any modem placed in the interior of any face of $\mathcal{A}^*$ that covers it has power at least $\lceil \frac{2n}{3} \rceil$, see Figure 4. In this section we give a new proof of the next result first proved in [5], we also review their proof.

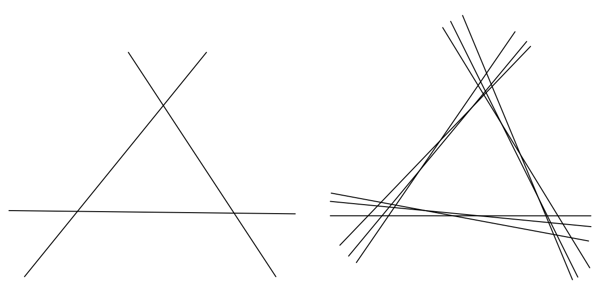

Figura 4: An arrangement of lines.

Theorem 2.1. *[5] For every arrangement of n lines on the plane, there is a point p such that a $\lceil \frac{2n}{3} \rceil$-modem placed at p covers the plane, $\lceil \frac{2n}{3} \rceil$ power is sometimes necesary.*

Let p be a point in the interior of a face of an arrangement $\mathcal{A}$ of n lines. The depth of p is the minimum number of lines in $\mathcal{A}$ that we have to cross to reach an unbounded face of $\mathcal{A}$. Rousseew and

Hubert [9] proved that for any arrangement of n lines there is always a point p with depth at least $\frac{n}{3}$, that is any ray emanating from p crosses at least $\frac{n}{3}$ lines of $\mathcal{A}$. It is now straightforward to see that if we place a $\frac{2n}{3}$ modem at p, it will cover $\mathcal{A}$.

We develop now some Voronoi-like properties on arrangements of lines that among other things, will give us a new proof of Theorem 2.1. Some preliminary results will be needed. Let $\mathcal{F}$ and $\mathcal{F}'$ be two faces of an arrangement $\mathcal{A}$. We say that $\mathcal{F}'$ is at distance k from $\mathcal{F}$ if a line segment joining a point in $\mathcal{F}$ to a point in $\mathcal{F}'$ crosses k elements of $\mathcal{A}$, the distance from $\mathcal{F}$ to $\mathcal{F}'$ will be denoted as $d(\mathcal{F}, \mathcal{F}')$. It is easy to see that $d(\mathcal{F}, \mathcal{F}')$ is well defined.

Let $\mathcal{A}$ be an arrangement of n lines. Observe that $\mathcal{A}$ has $2n$ unbounded faces. Suppose that we label these faces $\mathcal{F}_0, \ldots, \mathcal{F}_{2n-1}$ in the clockwise direction starting at any of these faces. Let $\mathcal{F}_i$ be an infinite face of $\mathcal{A}$. We call $\mathcal{F}_{i+n}$ the face opposite to $\mathcal{F}_i$, addition *mod* $2n$. It is clear that $\mathcal{F}_i$ and $\mathcal{F}_{i+n}$ are separated by all the lines of $\mathcal{A}$.

We say that a face $\mathcal{Q}$ of $\mathcal{A}$ is equidistant to two faces $\mathcal{F}_i$ and $\mathcal{F}_j$ of $\mathcal{A}$ if $d(\mathcal{F}_i, \mathcal{Q}) = d(\mathcal{Q}, \mathcal{F}_j)$. Suppose that the distance between two faces $\mathcal{F}_i$ and $\mathcal{F}_j$ of an arrangement $\mathcal{A}$ is even. We define the *bisector* of $\mathcal{F}_i$ and $\mathcal{F}_j$, to be the union of the set of faces of $\mathcal{A}$ (together with their boundaries) equidistant to $\mathcal{F}_i$ and $\mathcal{F}_j$, call this set by $Bis(\mathcal{F}_i, \mathcal{F}_j)$, see Figure 5(a).

It is worth noticing that if $d(\mathcal{F}_i, \mathcal{F}_j)$ is odd, then there is no face of $\mathcal{A}$ equidistant to $\mathcal{F}_i$ and $\mathcal{F}_j$. In this case it is easy to see that the set of points equidistant from $\mathcal{F}_i$ and $\mathcal{F}_j$ are points on a polygonal line contained in the lines of $\mathcal{A}$, see Figure 5(b). To obtain our results, we will be mainly concerned with bisectors of faces at even distance, and thus the set of faces equidistant to them will always be non-empty.

We now study some properties of the bisector of $Bis(\mathcal{F}_i, \mathcal{F}_j)$.

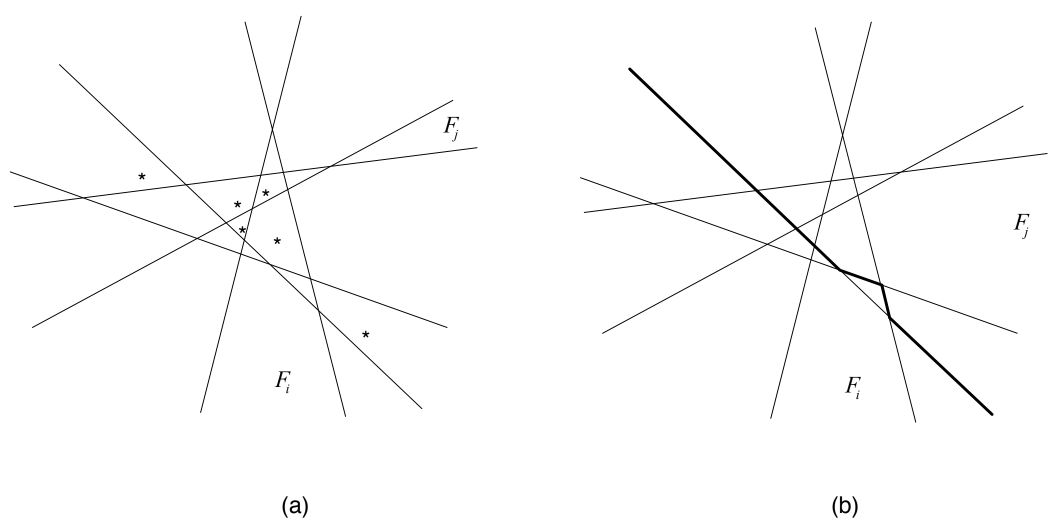

Figura 5: The set of faces equidistant to $\mathcal{F}_i$ and $\mathcal{F}_j$ in (a) are marked with an *.

Let $\mathcal{A}$ be an arrangement with an even number of lines, and let $\mathcal{F}_i$ and $\mathcal{F}_j$ be the unbounded opposite faces of $\mathcal{A}$ such that all the lines in $\mathcal{A}$ are above $\mathcal{F}_i$, and all the lines in $\mathcal{A}$ are below $\mathcal{F}_j$. In this case, it is easy to see that $Bis(\mathcal{F}_i, \mathcal{F}_j)$ contains what is known as the mid-level of the arrangement $\mathcal{A}$, that is $Bis(\mathcal{F}_i, \mathcal{F}_j)$ contains all the faces of $\mathcal{A}$ such that for any point in a face in $Bis(\mathcal{F}_i, \mathcal{F}_j)$, there are exactly half of the lines of $\mathcal{A}$ below it, and half of the elements of $\mathcal{A}$ above it. For this reason, when $\mathcal{F}_i$ and $\mathcal{F}_j$ are infinite and opposite faces of an arrangement $\mathcal{A}$ with an even number of lines, the bisector $Bis(\mathcal{F}, \mathcal{F}')$ will be refered too as to the *mid-level* of $\mathcal{A}$ with respect to $\mathcal{F}_i$ and $\mathcal{F}_j$, see Figure 6.

Suppose now that the unbounded faces of an arrangement $\mathcal{A}$ with $n = 2m$ lines are labeled $\mathcal{F}_0, \ldots, \mathcal{F}_{4m-1}$, in the counterclockwise order, and that $\mathcal{F}_0$ is the region below all the lines of $\mathcal{A}$. Then faces $\mathcal{F}_m$ and $\mathcal{F}_{3m}$ are in $Bis(\mathcal{F}_0, \mathcal{F}_{2m})$. What will be more important to our purposes, is the fact that there is a simple curve $f_{0,2m}$ (which in fact can be chosen so as to be the graph a real function

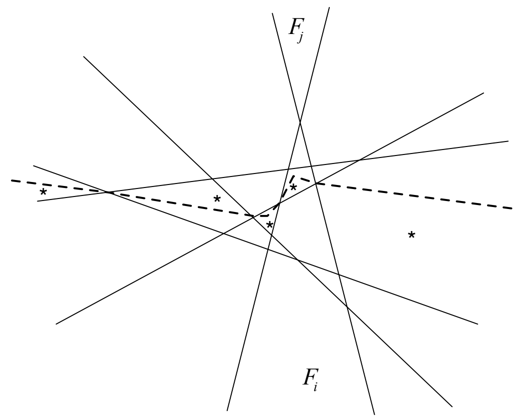

Figura 6: The set of faces equidistant to $\mathcal{F}_i$ and $\mathcal{F}_j$ are marked with an *.

$f : R \to R$) starting in $\mathcal{F}_{3m}$ and ending in $\mathcal{F}_m$ contained in $Bis(\mathcal{F}_0, \mathcal{F}_{2m})$, see Figure 6. Thus any point on $f_{0,2m}$ is equidistant to $\mathcal{F}_0$ and $\mathcal{F}_{2m}$. Clearly for each pair of opposite faces $\mathcal{F}_i$ and $\mathcal{F}_{i+2m}$ of $\mathcal{A}$, we can choose one such a curve, which we will denote as $f_{i,i+2m}$, addition taken *mod* $4m$.

Suppose now that $\mathcal{F}$ and $\mathcal{F}'$ are faces in $\mathcal{A}$ at even distance, but that they are not necessarily opposite faces in $\mathcal{A}$, nor necessarily unbounded. Split the lines of $\mathcal{A}$ into two subsets, S_1 containing all the lines of $\mathcal{A}$ that separate $\mathcal{F}$ and $\mathcal{F}'$, and S_2 the remaining elements of $\mathcal{A}$. Let $\mathcal{A}'$ be the arrangement defined by the elements of S_1. Clearly $\mathcal{F}$ and $\mathcal{F}'$ are opposite faces in $\mathcal{A}'$. The following lemma describes the union of the set of faces equidistant from $\mathcal{F}$ and $\mathcal{F}'$.

Lemma 2.2. *The union of the faces of $\mathcal{A}$ equidistant from $\mathcal{F}$ and $\mathcal{F}'$ is exactly the mid-level of $\mathcal{A}'$ with respect to $\mathcal{F}$ and $\mathcal{F}'$.*

Proof: Let ℓ_i be an element of S_2. When we add ℓ_i to S_1, it will create some new faces in the arrangement defined by $S_1 \cup \{\ell_i\}$. Let p be a point in one such face $\mathcal{F}_j$. If p belongs to the mid-level of $\mathcal{A}'$ with respect to $\mathcal{F}$ and $\mathcal{F}'$, then any line segment ℓ' joining p to a point r in $\mathcal{F}$ crosses the same number of lines in $\mathcal{A}'$ than any line segment ℓ'' joining p to a point in $\mathcal{F}'$. Since $\ell_i \in S_2$, it does not separate $\mathcal{F}$ and $\mathcal{F}'$, either both of ℓ' and ℓ" cross ℓ_i, or both of them do not cross ℓ_i. It follows that p remains equidistant to $\mathcal{F}$ and $\mathcal{F}'$. In a similar way, we can prove that if no point in $\mathcal{F}_j$ belongs to the mid-level of $\mathcal{A}'$ with respect to $\mathcal{F}$ and $\mathcal{F}'$, then $\mathcal{F}_j$ is not equidistant to $\mathcal{F}$ and $\mathcal{F}'$. It follows easily that when we add all the elements of S_2 to S_1 any face equidistant to $\mathcal{F}$ and $\mathcal{F}'$ is contained in the mid-level of $\mathcal{A}'$ with respect to $\mathcal{F}$ and $\mathcal{F}'$. See Figure 7.

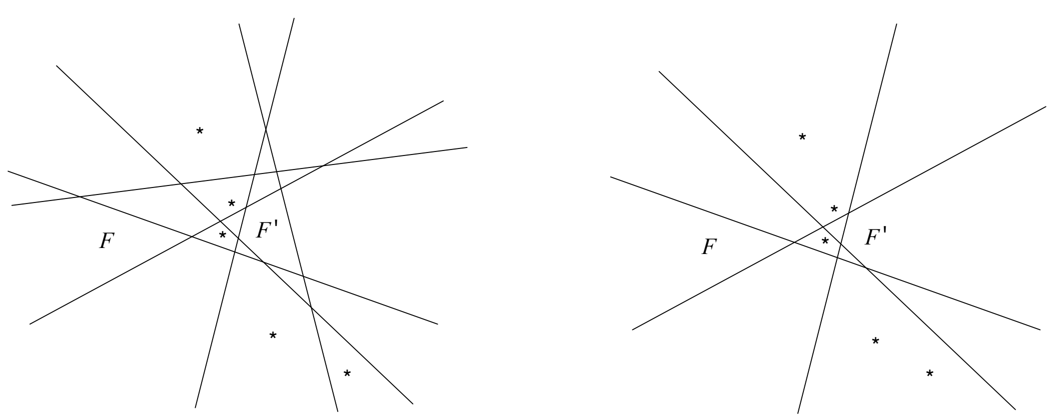

Figura 7: The bisector of two faces at evn distance.

As before, let $\mathcal{A}$ be an arrangement of n lines with its unbounded faces labelled $\mathcal{F}_0, \ldots, \mathcal{F}_{2n-1}$, and consider two unbounded faces $\mathcal{F}_i$ and $\mathcal{F}_j$ of $\mathcal{A}$ at even distance. By using the previous lemma, we can now define a curve $f_{i,j}$ contained in the mid-level of $\mathcal{F}$ and $\mathcal{F}'$ with respect to the arrangement $\mathcal{A}'$ defined by the lines of $\mathcal{A}$ that separate $\mathcal{F}_i$ and $\mathcal{F}_j$. It is easy to see that $f_{i,j}$ start and end in the unbounded faces $\mathcal{F}_k$ and $\mathcal{F}_{k+n}$ of $\mathcal{A}$, were $k = \frac{i+j}{2}$, addition taken *mod* $2n$.

Let $\mathcal{A}'$ be an arrangement with $3m$ lines, and let us assume as before that the unbounded faces of $\mathcal{A}$ are labeled $\mathcal{F}_1, \ldots, \mathcal{F}_{6m-1}$. Consider now the unbounded faces $\mathcal{F}_0$, $\mathcal{F}_{2m}$, and $\mathcal{F}_{4m}$. It is clear that $d(\mathcal{F}_0, \mathcal{F}_{2m}) = d(\mathcal{F}_{2m}, \mathcal{F}_{4m}) = d(\mathcal{F}_{4m}, \mathcal{F}_0) = 2m$.

Consider the curves $f_{0,2m}$ and $f_{2m,4m}$. It is easy to see that faces $\mathcal{F}_m$ and $\mathcal{F}_{4m}$ are in $Bis(\mathcal{F}_0, \mathcal{F}_{2m})$, and that $\mathcal{F}_{3m}$ and $\mathcal{F}_0$ are in $Bis(\mathcal{F}_{2m}, \mathcal{F}_{4m})$. It follows now that $f_{0,2m}$ start and end in faces $\mathcal{F}_m$ and $\mathcal{F}_{4m}$, and that $f_{2m,4m}$ starts in $\mathcal{F}_{3m}$ and ends in $\mathcal{F}_0$. Therefore $f_{0,2m}$ and $f_{2m,4m}$ intersect at a point p, see Figure 8. It could happen that p lies on the intersection of two lines in $\mathcal{A}$, this is the case in Figure 8. For our purposes, the worst case arises when p belongs to the interior of a face in $\mathcal{A}$, we analyze only this case. Since p belongs to $f_{0,2m}$ and $\mathcal{F}_{2m}$, it is equidistant to $\mathcal{F}_0$, $\mathcal{F}_{2m}$, and to $\mathcal{F}_{4m}$. Suppose that p is at distance r from $\mathcal{F}_0$. Since $\mathcal{F}_0$ and $\mathcal{F}_{3m}$ are opposite faces, then the sum of the distances of p to F_0 and to $\mathcal{F}_{3m}$ equals the number of lines of $\mathcal{A}'$, that is they add to 3m. It now follows that the distance from p to $\mathcal{F}_{3m}$ is $3m - r$. Similarly we can prove that the distance from p to $\mathcal{F}_m$ and to $\mathcal{F}_{5m}$ is $3m - r$, and thus p is also equidistant to $\mathcal{F}_m$, $\mathcal{F}_{3m}$, and $\mathcal{F}_{5m}$.

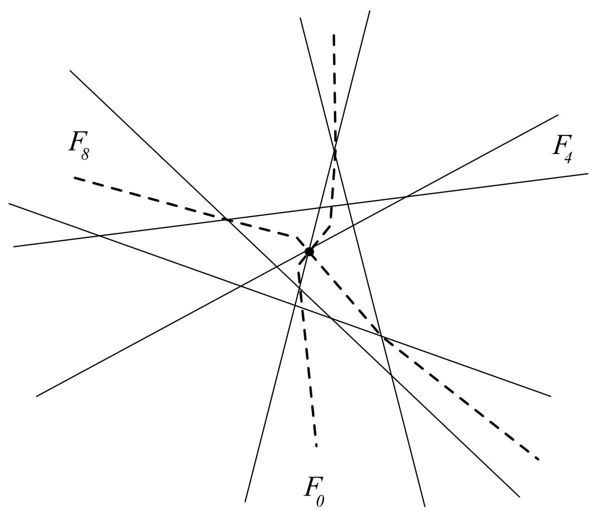

Figura 8: The curves $f_{0,4}$ and $f_{4,8}$ intersect at the point indicated with a small circle.

We now prove:

Lemma 2.3. *The distance of* p *to any face of* $\mathcal{A}'$ *is at most* $2m$.

Proof: Take a point q_i in $\mathcal{F}_i$, $i = 0, m, 2m, 3m, 4m, 5m$ such that they are the vertices of a convex hexagon $\mathcal{H}$ that contains in its interior all the intersection points of pairs of lines of $\mathcal{A}'$.

We now prove that any point in $\mathcal{H}$ is at distance at most $2m$ from p. Consider the triangle $\mathcal{T}$ with vertices p, q_0, and q_m. We now prove that exactly $2m$ lines of $\mathcal{A}'$ intersect the boundary of $\mathcal{T}$. Observe first that since the distance from $\mathcal{F}_0$ to $\mathcal{F}_m$ the line segment joining q_0 to q_m crosses exactly m lines of $\mathcal{A}$. Since the distance from p to $\mathcal{F}_0$ and to $\mathcal{F}_m$ are respectively r and $3m - r$, then the lines joining p to q_0, and p to q_m crossed respectively exactly r and $3m - r$ lines of $\mathcal{A}'$. Thus the edges of $\mathcal{T}$ are crossed exactly $m + r + (3m - r) = 4m$ times by the lines of $\mathcal{A}'$. Since each line that crosses the boundary of $\mathcal{T}$, crosses it twice, it follows that $2m$ lines of $\mathcal{A}'$ enter $\mathcal{T}$, and thus the distance from any point in $\mathcal{T}$ to p is at most $2m$. In a similar way we prove that any point in each of the six triangles formed by p and each of the edges of $\mathcal{H}$ is at distance at most $2m$ from p. Our result follows.

Theorem 2.1 follows directly from Lemma 2.3.

2.0.1. Two Modems

It is clear the if we place two modems with power $\lceil \frac{n}{2} \rceil$ in two unbounded opposite faces of an arrangement with n lines, they will cover the whole plane. The natural question now is: By placing them more carefully, can we improve on the $\lceil \frac{n}{2} \rceil$ previous bound? This is not possible. We now prove:

Theorem 2.4. *For every arrangement $\mathcal{A}$ of n lines on the plane, two $\lceil \frac{n}{2} \rceil$-modems are always sufficient and necessary to cover the plane.*

Proof: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ two k-modems placed on the plane that cover the plane. Suppose that they are placed at two faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{A}$, and that the line joining them is horizontal. Divide the lines of $\mathcal{A}$ into two subsets, S_1 containing the lines of $\mathcal{A}$ that separate $\mathcal{F}_1$ and $\mathcal{F}_2$, and S_2 the remaining lines of $\mathcal{A}$. Several cases arise, depending on the number of lines in $\mathcal{A}$, and the distance between $\mathcal{F}_1$ and $\mathcal{F}_2$. We analyze the case when $n = 2m$, and the distance between $\mathcal{F}_1$ and $\mathcal{F}_2$ is even. The other cases follow in a similar way.

Let $\mathcal{A}'$ be the arrangement induced by S_1, and suppose that the distance between $\mathcal{F}_1$ and $\mathcal{F}_2$ is $2k$. Then S_1 has $2k$ elements, and S_2 has $2r$ elements, $k + r = m$. Clearly $\mathcal{M}_1$ and $\mathcal{M}_2$ are in opposite unbounded faces of $\mathcal{A}'$. Thus there are two unbounded faces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{A}'$ at distance k from $\mathcal{F}_1$ and $\mathcal{F}_2$. It is easy to see that one of them, say $\mathcal{H}_1$ contains a point p_1 below all the lines in S_2, while $\mathcal{H}_2$ contains a point p_2 above all the elements in S_2. Since the lines in S_2 do not separate $\mathcal{M}_1$ from $\mathcal{M}_2$, for each $\ell_i \in S_2$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are above ℓ_i or below it. Then at least half of the elements of S_2 have both of $\mathcal{M}_1$ and $\mathcal{M}_2$ above them, or at least half of them have $\mathcal{M}_1$ and $\mathcal{M}_2$ below them. Thus the distance of p_1 or p_2 to both of $\mathcal{M}_1$ and $\mathcal{M}_2$ is at least $k + \frac{|S_2|}{2} \geq k + r = \lceil \frac{n}{2} \rceil$, see Figure 9.

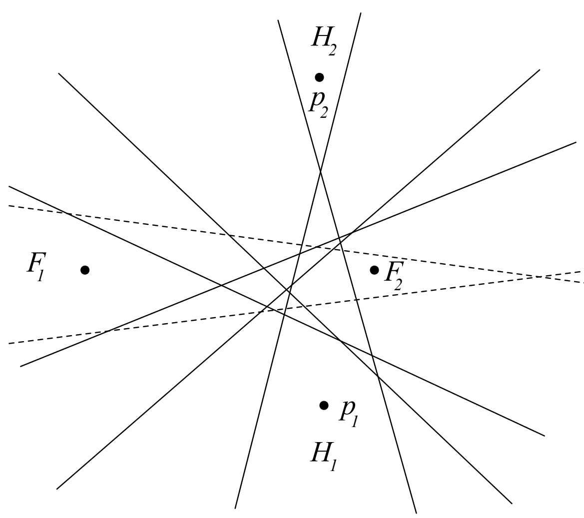

Figura 9: The elements of S_1 are the solid lines, the elements of S_2 are the dotted lines.

Using arguments similar to those used in the proof of Theorem 2.1, we can also prove:

Theorem 2.5. *Any arrangement of n lines, can be covered with four $\frac{5}{12}$-modems.*

The proof of this result will be given in a forthcoming paper. We believe that this bound is not tight, and we pose the following open problem:

Problem 2.6. Is it true that any arrangement $\mathcal{A}$ with n lines can always be covered with four $\frac{n}{3} + c$ modems, c a constant?

3. Covering Polygons

When $k = 0$, the probelm of covering polygons with k-modem is equivalent to the original Art Gallery Problem. However for $k \geq 1$, the k-modem illumination problem has turned out to be very difficult to solve. So far we only have significant results concerning *x-monotone* polygons, that is polygons such that the intersection of any any vertical line with them is empty or an interval. In [1] the following result is proved:

Theorem 3.1. *[1] Every monotone polygon with n vertices can be covered with at most $\lceil \frac{n}{2k} \rceil$ k-modems. There are There re monotone n-gons that require at least $\lceil \frac{n}{2k-2} \rceil$ k-modems to cover them.*

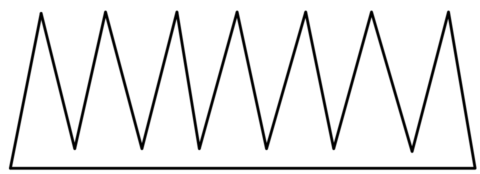

Figura 10: A monotone polygon that needs $\lceil \frac{n}{2k-2} \rceil$ k-modems to be covered.

The proof of the previous result is based on the following Lemmas which are given without proof. The interested reader is refered to [1]. Let P be a monotone polygon, and suppose that its vertices are labelled $p_1, \ldots, p_n$ from left to right. We have:

Lemma 3.2. *(The Splitting Lemma) Let P be a polygon with vertices $p_1, \ldots, p_n$, then for every $m < n$ there is a vertical line segment ℓ and two polygons R and L such that L has m vertices, R has $n-m+2$ vertices, and:*

- *Either ℓ is a chord of L and an edge of L, or viceversa*
- *p_m or p_{m+1} is an endpoint of ℓ*
- *Let L' be the polygon containing all the points of L on, or to the left of ℓ and R' the polygon containing all the points of R to the right of ℓ. Then $P = L' \cup R'$.*

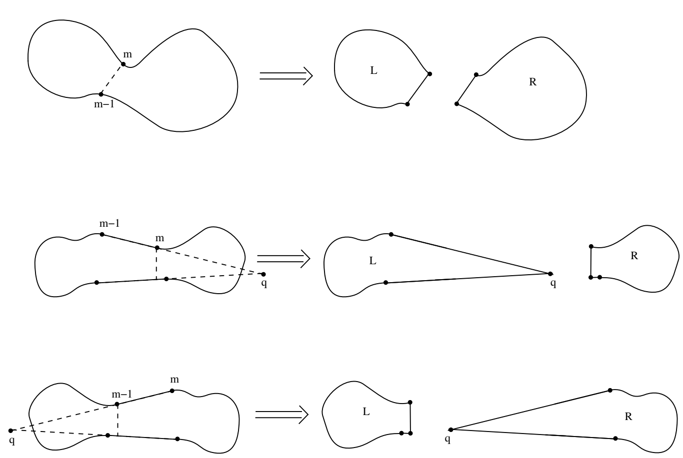

Figura 11: Illustrating the Splitting Lemma.

Lemma 3.3. *Any $k+2$ gon can be covered with a k-modem placed anywhere in the interior of P.*

Lemma 3.4. *Every $2k+2$-monotone polygon can be covered with a k-modem placed at either of its $k+1$-th or $k+2$-th vertex.*

The proof of Theorem 3.1 follows by applying the Splitting Lemma recursively to P to obtain $\lceil \frac{n}{2k} \rceil$ $2k+2$-monotone polygons, and then applying Lemma 3.4 to each of them.

4. Covering Polygons With Few Modems With High Power

In this section we study the problem of covering simple polygons using few modems with high power. In a recent paper Fulek, Holmsen, and Pach [5] proved the following result which will prove useful to us:

Theorem 4.1. *Let $\mathcal{F}$ be a family of n pairwise convex sets in $\mathbb{R}^d$, then there is a point p in the plane such that any ray emanating from p intersects at most $\frac{dn+1}{d+1}$ elements of $\mathcal{F}$.*

In the same paper, they proved that for any n, there is a set T_{3m} of $3m$ disjoint segments such that for any point p on the plane there is a ray emanating from p that intersects at least $2n-2$ elements of T_{3m}.

It can be shown that we can add $3m$ segments to the elements of T_{3m} to obtain a simple polygon P_{6m} with $6m$ vertices such that for every point p on the plane, there is a ray emanating from p intersects at least $4m-c$ edges of P_{6m}, that is to cover P_{6m} we need a $(4m-c)$-modem, c a constant. Using these results we can prove:

Theorem 4.2. *Any simple polygon P with n edges can always be covered with a modem of power $\lceil \frac{2n+1}{3} \rceil$, the bound is tight up to an additive constant.*

Proof: The sufficiency follows directly from Fulek, Holmsen, and Pach's Theorem 4.1. The proof that sometimes we need a $(\lceil \frac{2n+1}{3} \rceil - c)$-modem, follows from the fact that to cover P_{6m} we need a $(4m-c)$-modem.

4.0.2. Orthogonal Arrangements of Lines, Line Segments, and Polygons

In this section we study the problem of covering orthogonal arrangements of lines and orthogonal polygons, that is simple polygons such that its edges are all horizontal or vertical.

An orthogonal arrangement of lines is defined as in Section 2, except that $\mathcal{A}$ is now allowed to contain parallel lines, and all of them are horizontal and vertical. We prove first:

Theorem 4.3. *Let $\mathcal{F}$ be a family of disjoint segments, m horizontal, n vertical. Then there is a point p in the plane such that any ray emanating from p intersects at most $\lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$ elements of $\mathcal{F}$.*

Proof: It is easy to see that the worst case happens when no two elements of $\mathcal{F}$ are co-linear. Let ℓ_1 and ℓ_2 be a horizontal and a vertical line respectively, such that there are exactly $\lceil \frac{m}{2} \rceil$ horizontal elements of $\mathcal{F}$ above ℓ_1, and $\lceil \frac{m}{2} \rceil$ vertical elements of $\mathcal{F}$ to the left of ℓ_1. Let p be the intersection point of ℓ_1 with ℓ_2. It is straightforward to verify that any ray emanating from$p$ intersects at most $\lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$ elements of $\mathcal{F}$. The sufficiency of our result follows from a set of lines as shown in Figure 12.

In a similar way we can prove now:

Theorem 4.4. *Let $\mathcal{A}$ be an orthogonal arrangement of m horizontal and n vertical lines. Then there is a point p in the plane such that any ray emanating from p intersects at most $\lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$ elements of $\mathcal{A}$.*

Given a point $p = (a, b)$ on the plane let $\mathcal{C}_+^+(p) = \{q = (x, y) : a \leq x, b \leq y\}$. When p is the origin, $C(p)_+^+$ is the positive quadrant of the plane. Similarly we define $\mathcal{C}_+^-(p) = \{q = (x, y) : a \leq x, b \geq y\}$, $\mathcal{C}_-^-(p) = \{q = (x, y) : a \geq x, b \geq y\}$, and $\mathcal{C}_-^+(p) = \{q = (x, y) : a \geq x, b \leq y\}$ The following lemma will be used:

Lemma 4.5. *Let P be an orthogonal polygon and p be a point on the plane. Suppose that k vertices of P belong to the interior of $\mathcal{C}_+^+(p)$. Then any ray emanating from p contained in $\mathcal{C}_+^+(p)$ crosses at most k edges of P.*

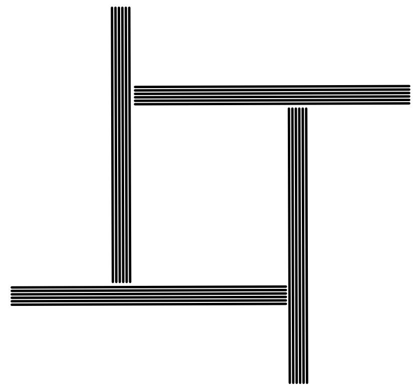

Figura 12:

Proof: Suppose that p is the origin, and let P be an orthogonal polygon such that P has k vertices in $\mathcal{C}_+^+(p)$. Let $\vec{p}$ be a ray contained in $\mathcal{C}_+^+(p)$ that emanates from p. If $\vec{p}$ intersects a horizontal edge e of P, charge this intersection to the right endvertex of e. If $\vec{p}$ intersects a vertical edge f of P, charge this intersection to the top endvertex of f. We can picture this as orienting the horizontal edges of P from left to right, and the vertical edges from bottom to top. It is now easy to see any ray $\vec{p}$ can charge at most one intersection to any vertex of P in $\mathcal{C}_+^+(p)$.

Clearly this lemma also holds for $\mathcal{C}_+^-(p)$, $\mathcal{C}_-^-(p)$, and $\mathcal{C}_-^+(p)$. We now prove

Theorem 4.6. *Let P be an orthogonal polygon with $2m$ edges. Then if m is even, P can always be covered with an $m-1$ modem located* in the interior *of P. The bound is tight. If m is odd, an m-modem is always sufficient. For m even, the bound is tight.*

Proof: Let P be an orthogonal polygon with $2m$ edges. Suppose that m is even. Let ℓ be a line that leaves $\frac{m}{2}$ horizontal edges of P in the interior of each of the half-planes it defines. Thus P has exactly m vertices above ℓ. Choose a point p on ℓ and in the interior of P. It is easy to see that since p belongs to the interior of P, each of $\mathcal{C}_+^+(p)$, $\mathcal{C}_+^-(p)$, $\mathcal{C}_-^-(p)$, and $\mathcal{C}_-^+(p)$ contains at most $m-1$ vertices in P. The upper bound now follows from the previous Lemma. The case when m is odd follows in a similar way.

Examples when an $(m-1)$-modem is required are shown in Figure 13.

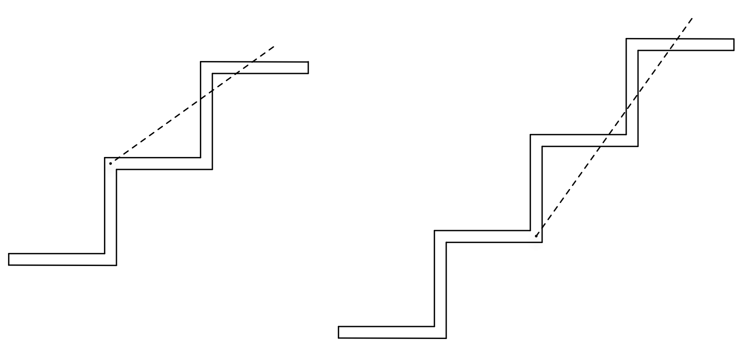

Figura 13:

When we allow our modems to be anywhere on the plane we can prove:

Theorem 4.7. *Any orthogonal polygon P with n edges can be covered with a $\lceil \frac{n}{3} \rceil$-modem.*

We only give a sketch of the proof. Suppose that P is contained in the unit square. Pick a point on the vertical band above the unit square, and high enough so that any ray emanating from p intersects at most one vertical edge of P. Let k be the maximum number of horizontal edges of P that any ray emanating from p intersects. If $k \leq \lceil \frac{n}{3} \rceil$ we are done as any such ray intersects at most one vertical edge of P. Suppose then that a ray $\vec{p}$ emanating from p intersects at least $k \geq \lceil \frac{n}{3} \rceil$ horizontal edges of P. Assume without loss of generality that $\vec{p}$ is vertical. Then the line ℓ containing $\vec{p}$ has at least $k \geq \frac{n}{3}$ vertices of P on each of its sides. Sweep a horizontal line h_ℓ from top to bottom until one of the two quadrants (above h_ℓ) defined by ℓ and h_ℓ contains exactly $\lceil \frac{n}{3} \rceil$ vertices.

It now follows easily that since $k \geq \lceil \frac{n}{3} \rceil$, each of the four quadrants defined by ℓ and h_ℓ contains at most $\lceil \frac{n}{3} \rceil$ vertices of P. Our result now follows from Lemma 4.5.

It is left as an open problem to determine the minimum k such that it is always possible to cover any orthogonal polygon with n edges with a k-modem. We conjecture that such k is within a constant factor of $\frac{n}{4}$.

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