MOTION PLANNING, TWO-DIRECTIONAL POINT REPRESENTATIONS, AND ORDERED SETS

by

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Abstract. Ordered sets are used as a computational model for motion planning problems. Every ordered set has a two-directional point representation using subdivisions. These subdivision points correspond to direction changes along the path of motion.

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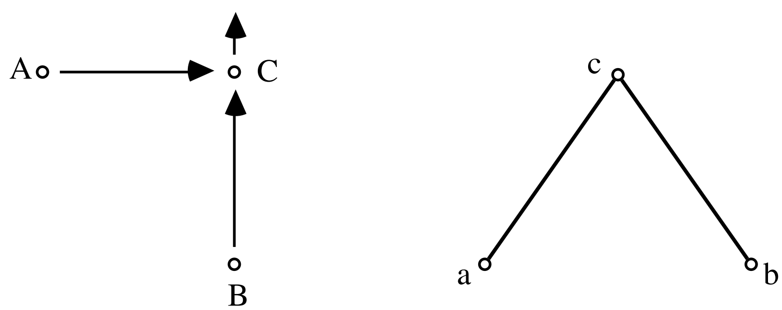
How may a robot arm be moved in order to grasp a delicate object from a crowded shelf without unwanted collisions?

How may a cluster of figures on a computer screen be shifted about to clear the screen without altering their integrity and without collisions?

These questions highlight instances of the recent and rapidly growing theme of 'motion planning'. Rival and Urrutia (1987) initiated the study of motion planning using a computational model based on the theory of ordered sets. Subsequently, Nowakowski, Rival and Urrutia (1987) proposed the problem to characterize the ordered sets here called 'two-directional orders'.



A two-directional point representation of the ordered set {a<b, a<c}.

An order diagram of the ordered set {a<c, b<c}.

Figure 1

For our purposes we cast the problem as follows. Given a finite collection of disjoint figures in the plane, is it possible to assign to each a single direction of motion so that this collection of figures may be separated, through an arbitrarily large distance, by translating each figure one at a time, along its assigned direction? In this model we have considered only convex figures in the plane. Indeed, given a collection of disjoint, convex figures, the separability problem always has a positive solution. Loosely speaking, at least one of the convex figures is on the 'outside' or 'boundary' of the collection, and therefore it may be removed.

To make the mathematical matter more definite, we shall here idealize each robot as a point (a circle of negligible radius) on the plane. Suppose that each point is assigned a single direction of motion not necessarily all the same. For points A and B we say that B *obstructs* A if there is a line joining a point of A to a point of B which follows the direction assigned to A. We write $A \sim B$. More generally, we write $A < B$ if there is a sequence $A = A_1 \sim A_2 \sim \ldots \sim A_k = B$. This relation $<$ is transitive. It is appropriate to call this binary relation $<$ a *blocking* relation. If the blocking relation has no directed cycles then it is antisymmetric too. In that case the blocking relation $<$ is a (strict) order on the set of these figures. If each of the points is assigmed the same direction, we call the relation *one-directional*. In that case, any maximal figure (with respect to $<$) is on the 'outside'.

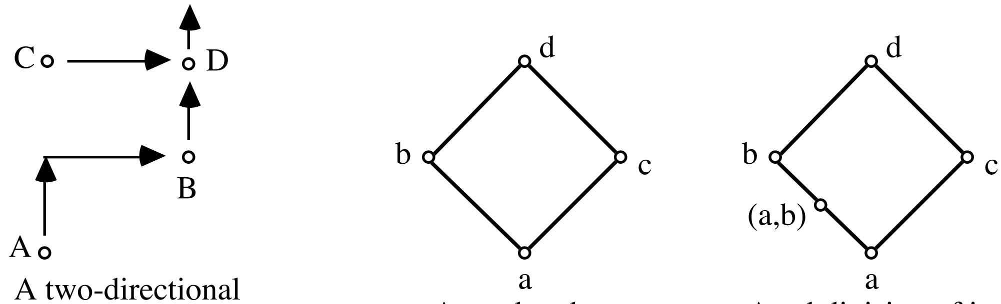
We say that a collection of points, each assigned one of m directions, is an *m-directional point representation* of an ordered set P, if its blocking relation is identical to the ordering of P.

Nowakowski, Rival and Urrutia (1987) considered ordered sets, each of whose points is assigned one of m directions, m a positive integer, and called these m-*directional point orders* (see Figure 1). Indeed, we may even imagine such point representations as models for an assembly line based on a many machine scheduling environment, in which the robots correspond to machines or machine parts.

Nowakowski, Rival and Urrutia showed *there are ordered sets with no* m-*directional point representation, for any positive integer* m, *yet every finite ordered set has a subdivision with such an* m-*directional point representation, for some* m. This subdivision consists precisely of the original ordered set with an extra element adjoined along some of the (covering) edges (with just the comparabilities induced, in each case, by just this edge).

We shall use throughout the customary *order diagram* of an ordered set in which the y-coordinate of a point b is larger than that of another point a if $a < b$ and an edge joins them just if b is an upper cover of a. (Say that b is an *upper cover of* a (b *covers* a or a is a *lower cover of* b or a is *covered by* b) if $a < b$ and if $a < c \leq b$ implies $b = c$.) Thus, an ordered set which contains an element with m lower covers requires at least m directions in its point representation — if it has one. We usually use upper case characters $A, B, C, \ldots$ to stand for the robots in the point representations and lower case characters $a, b, c, \ldots$ for the elements of ordered sets and the same symbol $<$ for the order relation in both contexts.

An alternate, perhaps more suggestive, interpretation of subdivision is this. Let b cover a and suppose a subdivision point (a,b) is placed along the corresponding covering edge. In a corresponding two-directional point representation a robot A may itself be assigned two directions, in succession, the first followed until a junction corresponding to the subdivision point (a,b) and the second followed from this junction to B (see Figure 2).



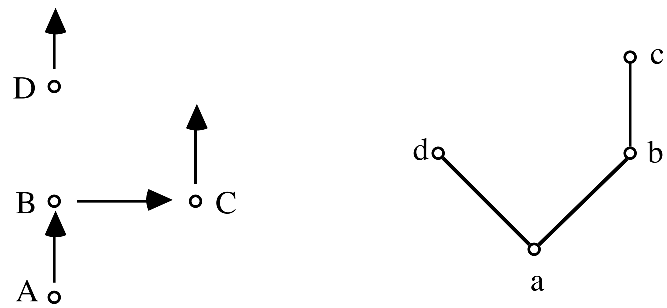
A two-directional point representation of a subdivision of $\{a<b<d, a<c<d\}$.

An ordered set $\{a<b<d, a<c<d\}$.

A subdivision of it.

Figure 2

Notice that, by transitivity, it may be that $A < B$ and $B < C$, that is, $A < C$, yet C is not 'visible' from A along either a horizontal eastward or a vertical upward path. At the same time, although D covers A it may be that B lies along the line of sight from A to D, apparently 'obstructing the visiblity' between them (see Figure 3).



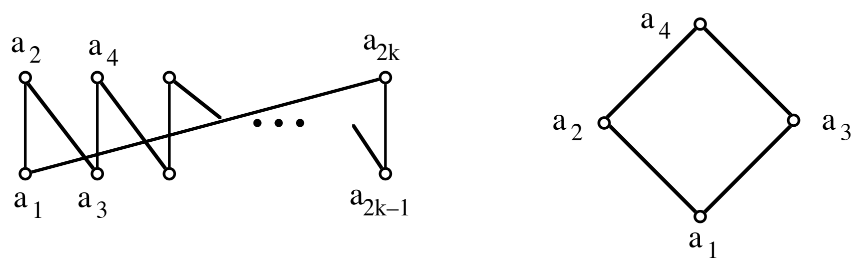
A two-directional point representation of $\{a<b<c, a<d\}$.

An order diagram of $\{a<b<c, a<d\}$.

Figure 3

From the viewpoint of motion planning we may suppose that once B begines to move along its intended direction of motion there is an unobstructed path from A to D. In the interest of continuity we shall insist, too, that all elements be assigned directions, including, in particular, the maximal elements, even though a maximal element is not constrained to precede any other.

Our leading problem is to characterize two-directional point orders among all orders. Here are our main results. The first highlights a class of ordered sets, each of whose members has a two-directional point representation. Call an ordered set a *tree* if its covering graph contains no cycle (a subset $a_1, a_2, \ldots, a_m$ of distinct points, $m \geq 4$, such that a_i covers a_{i+1} or a_{i+1} covers a_i for each $i = 1, 2, \ldots, m-1$ and a_1 covers a_m or a_m covers a_1). A *simple cycle* in an ordered set is a cycle $a_1, a_2, \ldots, a_{2k}$, $k \geq 2$, such that a_{2j} covers a_{2j-1} for each $j = 1, 2, \ldots, k$, and a_{2k} covers a_1. Moreover, we shall call a cycle a_1, a_2, a_3, a_4, in which $a_1 < a_2 < a_4$ and $a_1 < a_3 < a_4$ a simple cycle too (cf. Figure 4).



Simple cycles

Figure 4

THEOREM 1. *Every tree in which each element has at most two lower covers has a two-directional point representation, yet an ordered set which contains a simple cycle has no two-directional point representation at all.*

On the positive side we shall also show that any 'lexicographic sum' of ordered sets, with top and bottom, has a two-directional point representation, provided that both the index set and the blocks do too.

How many subdivision points along any covering edge ensure that an ordered set has a two-directional point representation? Or, in the language of motion planning, how many changes of direction for any robot guarantee that an order has a two-directional point representation?

THEOREM 2. *For any ordered set in which each element has at most two lower covers, at most one subdivision point along some of its covering edges ensures that it has a two-directional point representation.*

In some sense this result is best possible.

THEOREM 3. *There exist ordered sets in which each element has at most two lower covers such that almost half of its covering edges need be subdivided to ensure a two-directional point representation. Moreover, there are ordered sets in which each element has at most two lower covers with no two-directional point representation even if every covering edge is subdivided.*

Notice that while Theorem 2 ensures a two-directional point representation by subdividing some covering edges of P, according to Theorem 3 too many subdivisions may spoil the two-directional point representation.

We are still unable to characterize the ordered sets which have a two-directional point representation. Nevertheless, it seems to us that the solution to the bipartite case would shed light on the general problem.

TREES AND CYCLES

It is easy to see that an ordered set with a two-directional point representation also has one in which the two directions are perpendicular. We shall suppose throughout that these directions are northward (**n**) and eastward (**e**).

Our aim first of all is to show that no simple cycle has a two-directional point representation. Suppose that P is an ordered set with a two-directional point representation. Let a and b be distinct elements of P. If both a and b point northward and lie on the same vertical line in the representation of P then they must be comparable. For, if the y-coordinate of a is below the y-coordinate of b in this representation then, as a points northward, $a < b$; if the y-coordinate of b is below that of a then $b < a$. Now, let a, b be distinct lower covers of c in P. In the representation, c must be located along the 'line of sight' of a and of b. Thus, if a and b had the same direction, then each would be along the line of sight of the other and, according to our observation above, a and b would be comparable. Therefore, we may suppose that a points northward and b eastward, say, and that, therefore, c lies at the point of intersection of the northward and eastward lines from these points. It follows, of course, that every element in P has at most two distinct lower covers.

From these preliminary remarks it is an easy matter to deduce that no simple cycle $a_1, a_2, \ldots, a_{2k}$, $k \geq 3$ and k odd, has a two-directional representation. Suppose one did. As a_1, a_3 are lower covers of a_2 they have different directions **n**, **e**, respectively, say. Then, a_5 has direction **n**, a_7 **e**, and so on, alternately, which, of course, is impossible as k is odd.

We claim that no simple cycle at all has a two-directional point representation. The cases $a_1 < a_2 < a_4$ and $a_1 < a_3 < a_4$ as well as $a_1 < a_2$, a_4 and $a_3 < a_2$, a_4 can be checked directly to have none, as a simple longhand effort shows. For the remaining cases another remark is handy. Let $c_1, b_1, c_2, b_2, \ldots, c_m, b_m, c_{m+1}$, be a 'zigzag', that is, c_1 covers b_1 and c_i covers b_{i-1} and b_i, for $2 \leq i \leq m$, and c_{m+1} covers b_m, and consider a two-directional point representation of it. We may suppose that its minimal elements $b_1, b_2, \ldots, b_m$ alternate in direction **n**, **e**, ... As each b_i, $1 \leq k \leq m$, is covered by two of the c_j's, then both of them, namely c_{i-1} and c_i, lie along the line of sight of b_i. As c_{i-1} and c_i are noncomparable, neither can be along the line of sight of the other. It follows that in the representation, successive triples of the c_i's follow either an upward staircase pattern or a downward staircase pattern in which an upward staircase may meet a downward staircase with increasing subscript, yet a downward staircase continues only downward (see Figure 5).

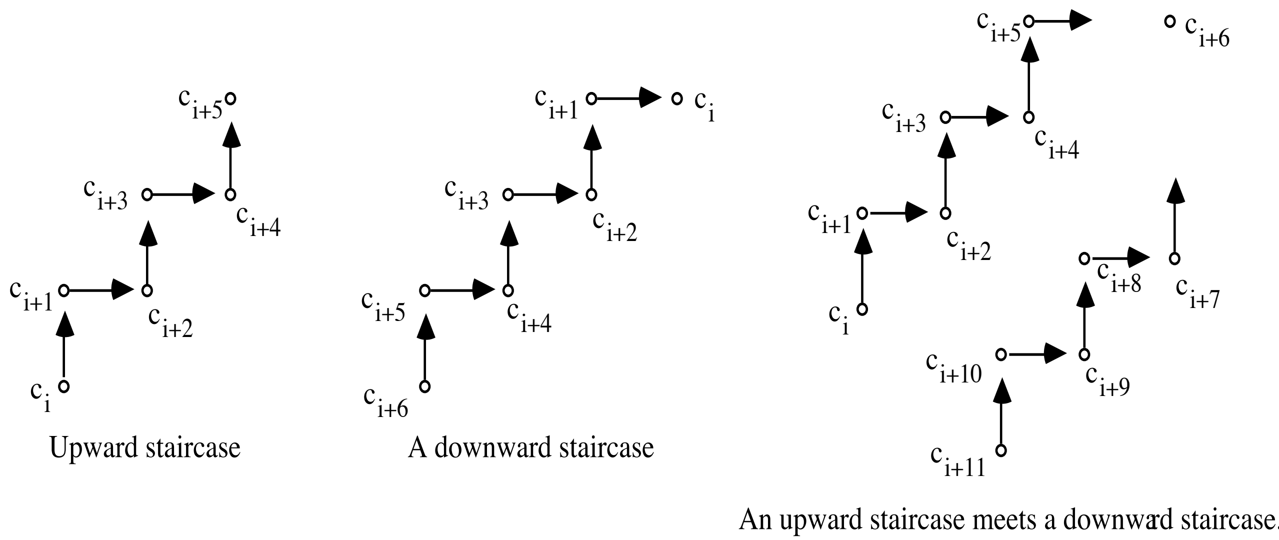


Figure 5

Let $a_1, a_2, \ldots, a_{2k}$, $k \geq 3$, be an arbitrary simple cycle, that is, a_{2j} covers a_{2j-1}, $j=1,2,\ldots,k$, and a_{2k} covers a_1. Suppose that it has a two-directional point representation. Then its maximal elements must follow the staircase pattern indicated above. Since the sequence $a_2, a_4, \ldots$ of maximal elements will repeat following the enumeration of the cycle, at least one portion must be a downward staircase, and, in that case, must continue as a downward staircase throughout — which is impossible. Thus, no simple cycle at all has a two-directional point representation.

We now show by induction on $|P|$ that any ordered set P which is a tree does have a two-directional point representation. Let a be an endpoint of the covering graph of P, that is, either a maximal element of P with precisely one lower cover or else a minimal element with precisely one upper cover. Suppose that a is maximal, that b is its unique lower cover and that a two-directional point representation of $P - \{a\}$ is given. We may assume that b has direction **n**. We shall locate a along the vertical from b above it. We may choose its y-coordinate less than any other point already on this vertical yet larger than b, and distinct from the y-coordinate of any other point. Assign a the direction **e**. This constitutes a two-directional point representation of P.

Suppose now that a is minimal with unique upper cover b and that $P - \{a\}$ has a two-directional point representation. There is no loss in generality to assume that b has direction **n**. By hypothesis b has at most one lower cover c, besides a. Suppose c has direction **e**. Then we may locate a on the vertical below b with y-coordinate distinct from the y-coordinate of any other point and above any point already on this vertical yet lower than b. We may assign a the direction **n** to obtain a two-directional point representation of P. Now, suppose that c has direction **n**, in which case c lies on the vertical through b beneath it. Before locating a we make a small change to the representation of $P - \{a\}$ by shifting the location of b just an 'epsilon' northward so that its y-coordinate is distinct from the y-coordinate of any other point. In this case we may locate a on the horizontal through b anywhere to the left of it and assign it the direction **e**. This gives a two-directional point representation of P.

Actually we can say somewhat more, for ordered sets constructed as a 'lexicographic sum'. For an ordered set P and a family $(Q_p \mid p \in P)$ of ordered sets, indexed by P itself, the *lexicographic sum* $\sum_p Q_p$ is the ordered set whose underlying set is the union of the Q_p's and in which $x < y$ if $x, y \in Q_p$, for some $p \in P$, and $x < y$ in Q_p or, if $x \in Q_p$, $y \in Q_r$ and $p < r$ in P.

PROPOSITION. *Let* $\sum_p Q_p$ *be a lexicographic sum of ordered sets. If* P *as well as each* Q_p *has a two-directional point representation, and if each* Q_p *has a top and a bottom, then* $\sum_p Q_p$ *itself has a two-directional point representation.*

Proof. Suppose a two-directional point representation of P is given.Let $p \in P$ with coordinates (x,y), let p be directed northward, and suppose that p', with coordinates (x',y'), is the first vertex on this vertical northward path from p. If Q_p is a chain then we may take a two-directional point representation of it in which each vertex is directed northward. Then if we contract the total vertical distance between the bottom vertex and the top vertex of Q_p to a total distance less than $y'-y$, we may insert this representation of Q_p into the vertical between p and p', replacing p by the bottom of Q_p and avoiding all y-coordinates already occupied by existing points.

Suppose that Q_p is not a chain. In this case we construct another two-directional point representation of P, by shifting each vertex r on the vertical along p by a small horizontal distance $\varepsilon > 0$ to the right less than the horizontal distance between p and any other vertex in its representation. We now contract the region occupied by the representation of Q_p into the ε by $y'-y$ rectangle from p to p', again replacing p by the bottom vertex of Q_p avoiding all y-coordinates already occupied.

In this way we may successively add the blocks to produce a two-directional point representation of the lexicographic sum itself.

It is not clear to us at this writing how we may naturally extend the class of ordered sets with a two-directional point representation. Lattices with at most two lower covers, even planar ones, need not have a two-directional point representation (see Figure 6).

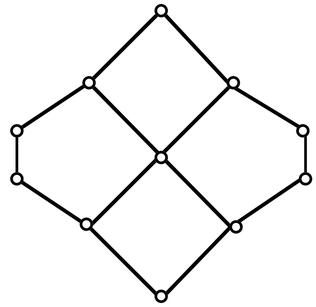


Figure 6

Elsewhere (cf. Czyzowicz, Pelc and Rival) we have studied ordered sets, and especially lattices, with a diagram using only two different slopes for its edges. For instance, the lattice diagram of Figure 6 uses five different slopes for its edges, although it is easy to draw another diagram of the same lattice using only two. On the other hand, there are ordered sets with no two-slope diagram (for nontrivial reasons) yet, which have a two-directional point representation (see Figure 7).

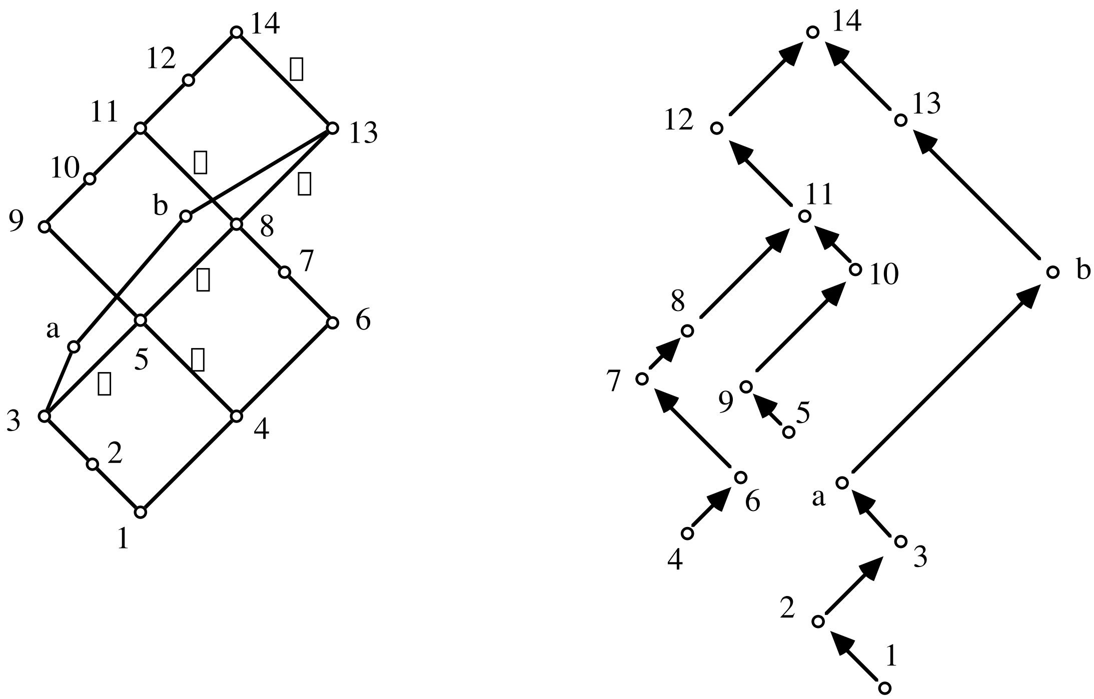


Figure 7

Still, there is an obvious connection between two-slope diagrams and point representations. If each vertex is allowed not just one of two directions, but both of the two directions, then it is easy to verify that there is a two-slope diagram. The converse, too, is obviously true.

SUBDIVISION

Let P be an ordered set in which each element has at most two lower covers. Even if P itself has no two-directional point representation, we shall show that there is an ordered set obtained from P by subdividing some edges of the diagram of P at most once which, in turn, has a two-directional point representation.

Before we do this, let us record a rather simple transparent construction which, however, proves less. We show that there is an ordered set P' constructed from P by adjoining at most two subdivision points along every covering edge which itself has a two-directional point representation.

An example of this construction is illustrated in Figure 8.

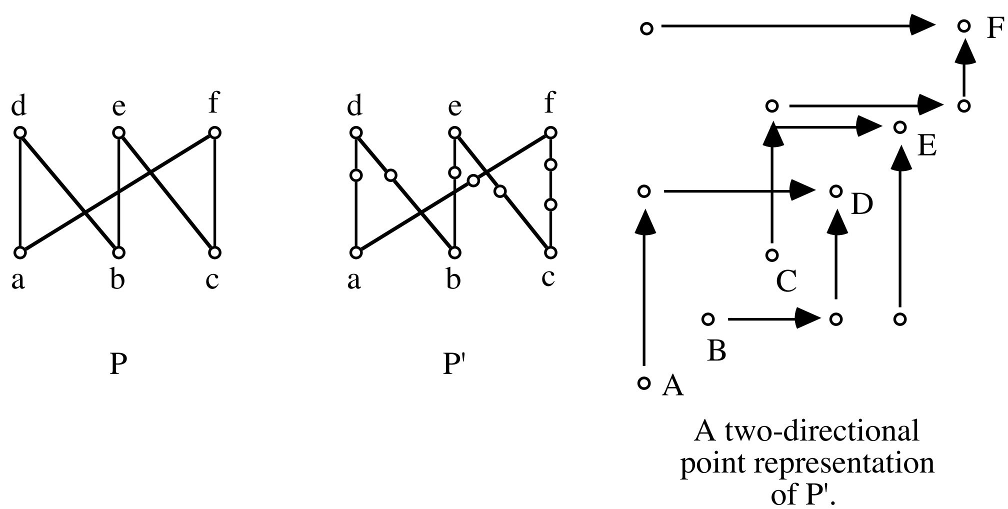


Figure 8

Although we have adjoined two subdivision points along just one edge of P and just one along the others, we may, of course, have introduced more along any other edge too with corresponding points in the representation using the same direction for them as used for the incoming edge. Thus any subdivision of an ordered set with an m-directional point representation also has an m-directional point representation.

Let L be a linear extension of P and arrange the elements of P as points at unit intervals along the $y = x$ line on the plane in the same increasing order as they occur in L. We proceed by induction on the *height* of an element in L (that is, the size of the longest chain in L from it to the bottom of L) to assign it successive directions changing at most twice in order to produce a two-directional point representation. Suppose that the elements of L labelled $A_1, A_2, \ldots, A_{m-1}$ are already directed. Suppose that A_n is an upper cover of A_m. As A_n has at most two lower covers in P either the eastward direction to A_n is available or else the northward direction to A_n is available. Suppose then that the eastward direction is available and is chosen from a single subdivision point on the (A_m, A_n) edge (cf. Figure 9). In fact, for any upper cover $A_{n'}$ of A_m for which this eastward direction is available we may choose a single subdivision point and direct A_m northward, as before, and the subdivision point eastward. Now, let C be an upper cover of A_m for which there already exists a point directed eastward toward it. In this case two subdivision points along the (A_m, B) edge suffice: the first located north of A_m at a point whose y-coordinate is distinct from the y-coordinate of any other point already constructed; the second located along the horizontal east from the first subdivision point and along the vertical below C.

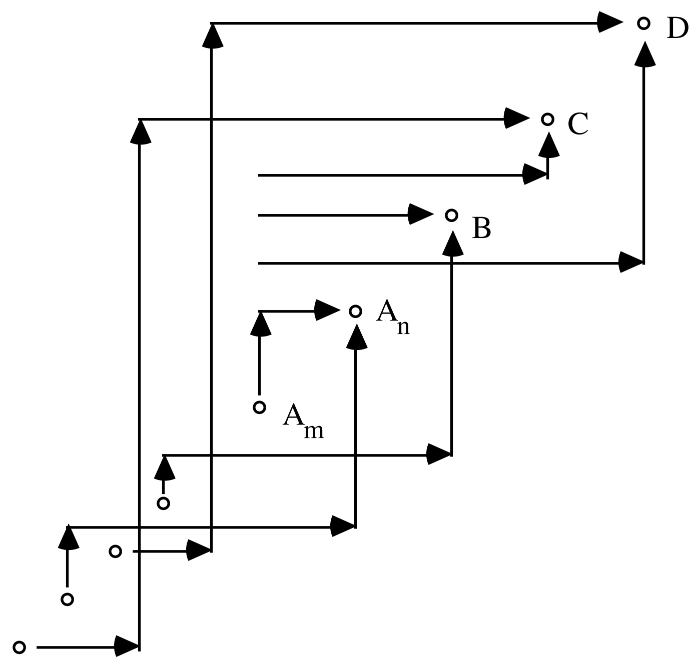


Figure 9

Then direct the first eastward and the second northward. The same construction can be carried out for any upper cover D of A_m whose incoming northward direction is available.

We turn now to the proof of Theorem 2. We first treat the special case that every chain in P has at most two elements, that is, P has 'height' at most two. Moreover, let us assume that P has a quite specific structure. Indeed, suppose that $P = P(G)$ is constructed from a graph G on the n vertices $v_1, v_2,\ldots, v_n$ with the minimal elements of P corresponding to these n vertices of G and the maximal elements of P corresponding precisely to those pairs $w_{ij} = (v_i, v_j)$ of vertices of G, joined by an edge in G. Then put $v_i < w_{ij}$ and $v_j < w_{ij}$. Evidently each element of $P(G)$ has at most two lower covers.

We shall now make P even more particular. Let $P = P(K_n)$, where K_n stands for the complete graph on n vertices, that is, every pair of vertices is joined by an edge. We shall show that there is an ordered set obtained from $P(K_n)$ by subdividing at most half of its edges which has a two-

directional point representation. To begin with select locations $p_1, p_2, \ldots, p_n$ for the vertices $v_1, v_2, \ldots, v_n$ on n horizontal lines with equations $y=y_1, y=y_2, \ldots, y=y_n$, say p_i has coordinates (x_i, y_i), $i = 1, 2, \ldots, n$. We locate the vertices w_{ij} beyond (that is, to the right of) the vertical line
$x = \max\{x_i \mid i = 1,2,\ldots,n\}$. For each w_{ij}, satisfying $i < j$, choose a location p_{ij} on $y=y_j$ with coordinates (x_{ij}, y_j) and define another location p_{ij}' on $y=y_i$ at (x_{ij}, y_i). We may suppose that all of these x-coordinates x_{ij} are distinct. Now, for each p_i assign it the horizontal direction to the right and, for each p_{ij} and p_{ij}' the vertical upward direction. The vertices of p_{ij}' correspond to subdivisions of the corresponding edges from v_i to w_{ij} (see Figure 10). In this way half of the edges of $P(K_n)$ are subdivided and this resulting subdivision has a two-directional point representation.

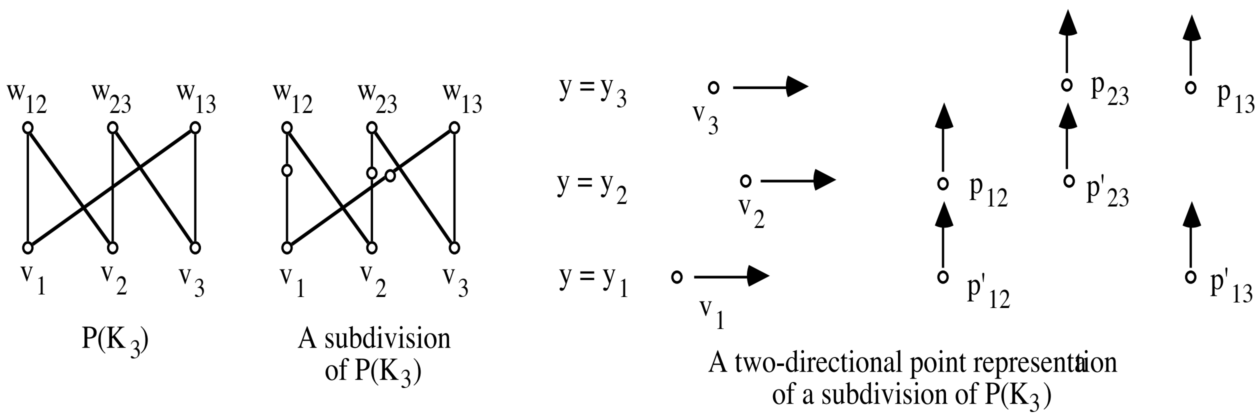


Figure 10

It is an easy consequence that, actually, for any graph G, the ordered set $P = P(G)$ also has a two-directional point representation. To see this, just erase the points p_{ij}, p_{ij}' from the representation of $P(K_n)$, n the number of vertices of G, whenever v_i, v_j are not joined by an edge in G.

We may now extend this idea to supply a two-directional point representation for any ordered set P in which each maximum chain has at most two elements. Indeed, just like the case for $P(K_n)$, subdividing at most half of the edges is enough. Locate the minimals of P, each on a different horizontal line. For each maximal element with two lower covers we proceed as for the representation of $P(G)$. In fact, if all the maximals of P have two lower covers then $P = P(G)$, where possibly G has some multiple edges (see Figure 11).

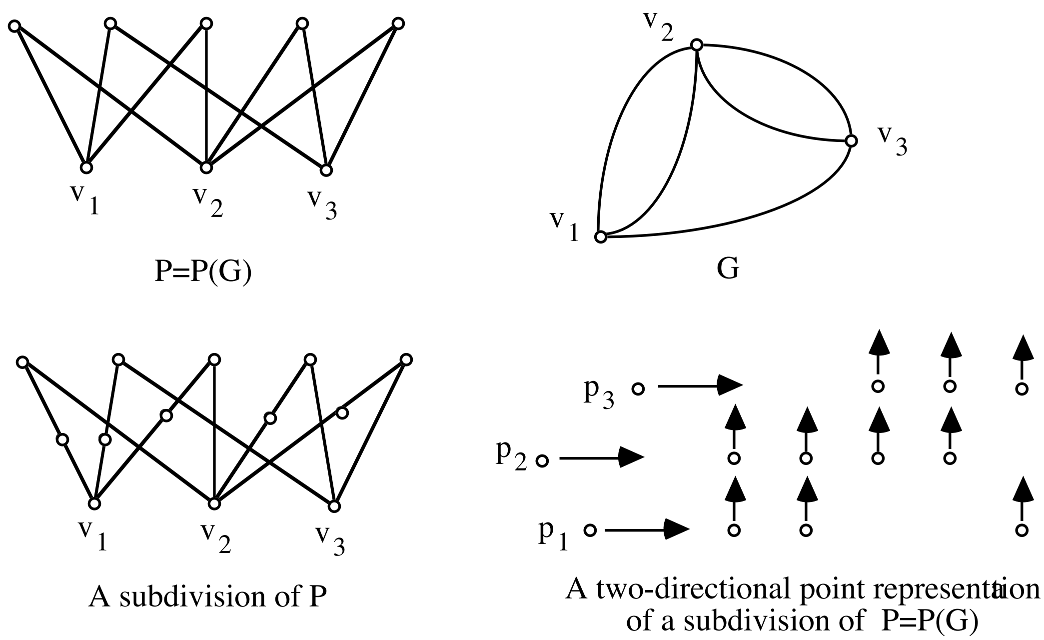


Figure 11

If, on the other hand, there are maximals with just one lower cover, then it suffices to locate these on the horizontal line corresponding to its unique lower cover and direct it upward (see Figure 12).

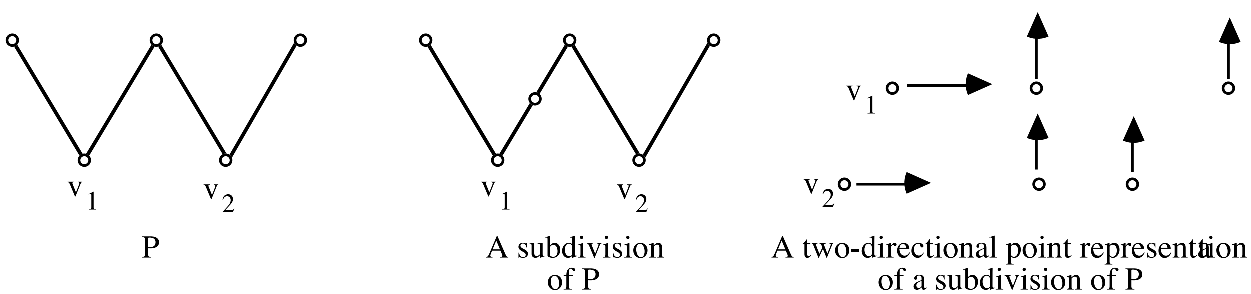


Figure 12

For this 'bipartite' case, we have consistently directed the minimals horizontally and the maximals, together with all subdivision points, vertically. Of course, we could have interchanged the two directions, with an appropriate change to all locations (a reflection along the diagonal $y = x$).

We are now ready to treat the general case. First, we partition P into 'levels':

$$L_1 = \min (P);$$

for $i > 1$, $\quad L_i = \min (P - \bigcup_{j<i} L_j)$,

where $\min (P)$ stands for the minimals of P. Notice that consecutive pairs L_i, L_{i+1} determine bipartite orders, each of which does have a two-directional point representation. In fact, as long as there are no covering relations between pairs of elements $x \in L_i$, $y \in L_j$, $j \geq i+2$, then we may successively locate positions for the elements of the levels, alternating directions for the levels. Thus, for L_1 □ L_2 all vertices associated with L_1 are directed horizontally, then the vertices of L_2, as well as subdivision points are directed vertically. At the next step in L_2 □ L_3 each vertex in L_3 is directed horizontally just as the subdivision points in L_2 □ L_3, and so on. Note that not all edges are subdivided; for instance, no edge associated with the lower cover of an element with only one lower cover is itself subdivided.

Let a two-directional point representation of a subdivision of the ordered set (L_1 □ L_2) □ (L_3 □ L_4) □ … be given. We suppose now that there are, however, covering edges joining elements in levels two or more apart. To this end let us suppose that x□L_i, i odd say, and y□L_j, $j \geq i+2$. Let p_x , p_y stand for the corresponding points in the representation of (L_1 □ L_2) □ (L_3 □ L_4) □ … . Then each coordinate of p_y is larger than the corresponding coordinate of p_x. By hypothesis, y can have precisely one other lower cover $z \neq x$ and z □ L_{j-1}, and, by construction, the covering edge y above z is not subdivided. Now, p_z is directed either horizontally or vertically. If horizontally, like p_x, then we insert a point p_{xy}, directed upward, at the intersection of the horizontal through p_x and the vertical through p_y (see Figure 13).

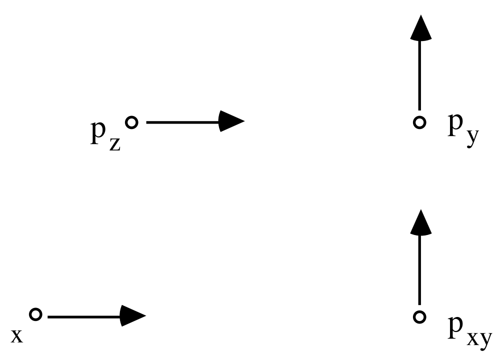


Figure 13

Now, if there is a point already located on the vertical between p_{xy} and p_y it cannot be comparable to p_y. As no point in the representation of the subdivision can be directed upward to it, we may shift it slightly to the right. This results in a representation of the subdivision again, along with the required comparability of $x < y$ using a single subdivision. Otherwise, p_z is directed upward. As z is not itself a subdivision point, p_y must be directed horizontally. Then move p_y slightly to the right, say a distance □> 0. Insert a point p_{zy}, directed horizontally, at the intersection of the vertical through p_z and the horizontal through p_y, that is, at the former location of p_y itself. Also insert a point p_{xy}, directed vertically at the intersection of the horizontal through p_x and the vertical through p_y, (now shifted a distance □> 0 horizontally). We may suppose that no other points lie on the segment between p_{xy} and p_y (else shift it horizontally by a small distance).

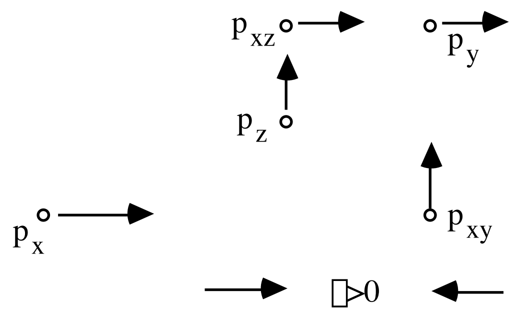


Figure 14

In this fashion we can produce a two-directional point representation of a subdivision of P. This completes the proof of Theorem 2.

We turn now to the proof of Theorem 3. We first prove the second part. We construct a family $(P_n \mid n \geq 7)$ of ordered sets, each member of which has no two-directional point representation. We will also prove that ifpof the diagram of P_n is subdivided, the ordered Q_n thus obtained, has no two directional point representation . Indeed, let $P_n = P(K_n)$.

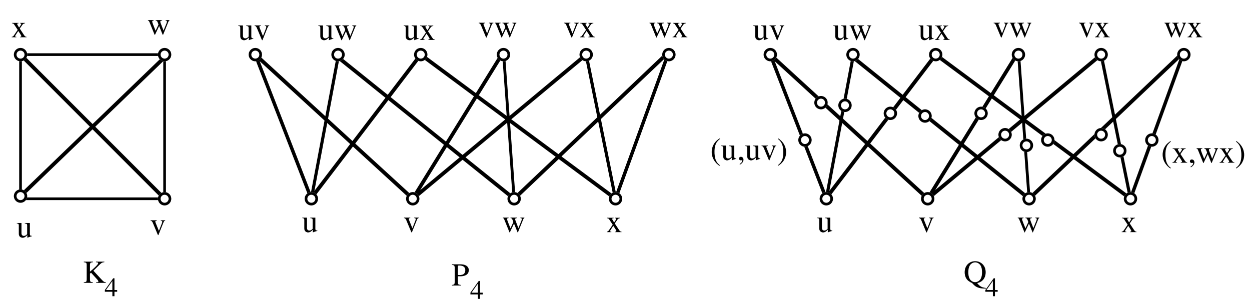
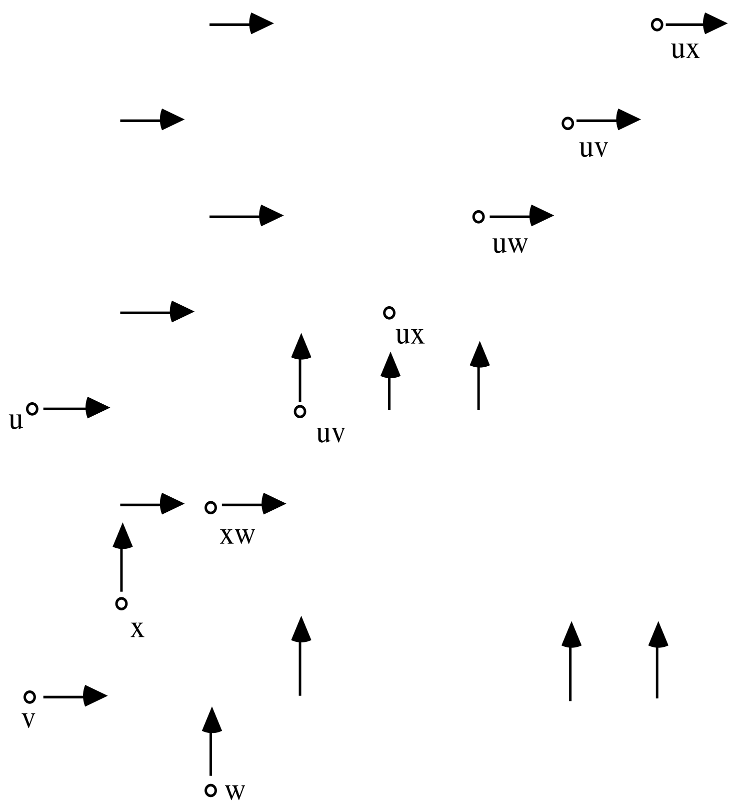


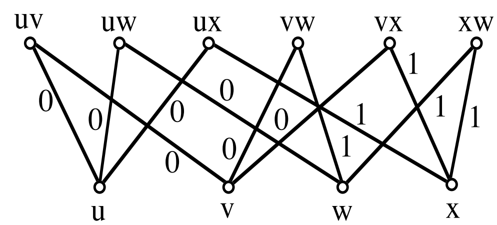
Figure 15

Of course, this bipartite order P_n, itself has no two-directional point representation, for any $n \geq 3$, as it certainly contains a simple cycle. We shall show that even adding at most one subdivision point along each edge of P_n, $n \geq 7$, cannot produce an ordered set with such a representation. Notice that if an ordered set has such a representation then it still has one with any number of subdivision points along any of its edges. So, for purposes of our argument, suppose, for contradictions, that every ordered set Q_n, $n \geq 7$, obtained from P_n by adding precisely one subdivision point (u, uv) along every covering edge u to uv does have a two-directional point representation.

Our aim is to construct a particular two-colouring of the edges of P_n based on the representation of Q_n. Let u be an arbitrary vertex of K_n. Note that, in the representation of Q_n, all, but at most one, of the upper covers of u, have direction different from that assigned to u. Colour the edge from u to uv 1 if (u, uv) is directed eastward, otherwise colour the edge 0. Notice that the two incident edges of each maximal vertex uv of P_n carry distinct colours.



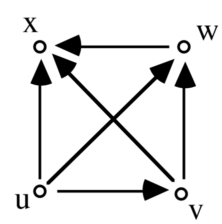
A two-directional point representation of Q_4



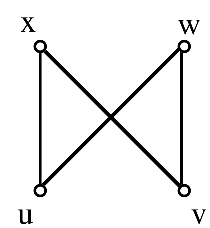
Edge-colouring of P_4

Figure 16

On the other hand, among the n incident edges of each minimal vertex u, all, but at most one, receive the same colour. Now, orient the edges of K_n according to this rule: u∅v if the edge u to uv in P_n has colour 1; v∅u if this edge u to uv in P_n has colour 0.



Orientation of K_4



Bipartite subgraph

Figure 17

Then, for any vertex u of K_n, either all, but at most one, of the edges are directed away from u or, all, but at most one, of the edges are directed into u. For each vertex u delete from K_n, $n \geq 4$, the minority edge, if it exists. Then, for every vertex u of the induced subgraph, either every edge is directed away from u or every edge is directed into u, that is, the induced subgraph must be bipartite. In summary, we have shown that the removal of at most n edges from K_n produces a bipartite graph. If $n \geq 7$ then one of the two parts of the bipartition contains at least four vertices whose six edges must have been removed, according to the construction. This is impossible if only n edges are removed in all, each one incident to a distinct vertex.

To prove the first part of Theorem 3, we will show that in any two directional representation of $P(K_n)$ there are at most n-1 vertices $w_{i,j}$ such that neither of the two edges v_i to $w_{i,j}$ or v_j to $w_{i,j}$ is subdivided. Let us consider any two directional representation of $P(K_n)$ in which the vertices v_1, $v_2,\ldots, v_n$ of $P(K_n)$ are represented by points $p_1, p_2,\ldots, p_n$ .Suppose that $p_1,\ldots, p_k$ move northward and $p_{k+1},\ldots, p_n$ move eastward. Let $L_1,\ldots, L_k$ be the vertical lines through $p_1,\ldots, p_k$ and $L_{k+1},\ldots, L_n$ the horizontal lines through $p_{k+1},\ldots, p_n$, respectively. In any such representation of $P(K_n)$ in which neither of the two edges v_i to $w_{i,j}$ or v_j to $w_{i,j}$ is subdivided, the directions of p_i and p_j are different. Suppose that p_i moves upwards and p_j moves eastward. Then it is easy to see that the point $p_{i,j}$ representing $w_{i,j}$ is the intersection point of L_i and L_j (see Figure 18). Moreover, in each vertical (horizontal) line L_i there is at most one point $w_{i,j}$ moving northward (eastward).It follows that at most n-1 different points $p_{i,j}$ representing distinct vertices $w_{i,j}$ of $P(K_n)$ can be placed in the intersection points of $L_1,\ldots,L_i$ with $L_{i+1},\ldots,L_n$ without blocking each other.

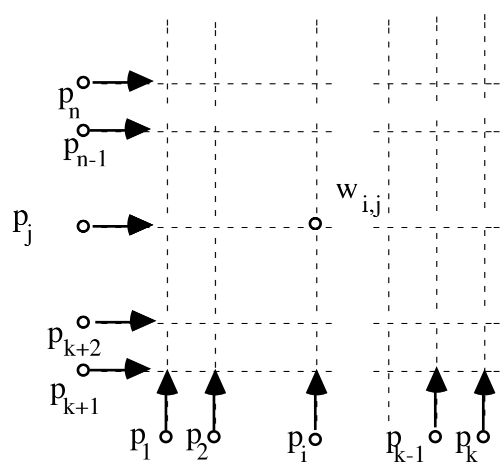


Figure 18.

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