SEPARATING CONVEX SETS ON THE PLANE

Jurek Czyzowicz[1], Eduardo Rivera-Campo[2], Jorge Urrutia[3] and Joseph Zaks[4]

1) Département d'Informatique, Université du Québec à Hull, Hull, Québec, Canada.

2) Departamento de Matemáticas, Universidad Autonóma Metropolitana-I, México D.F., México.

3) Department of Computer Science, University of Ottawa, Ottawa, Ontario, Canada.

4) Department of Mathematics, University of Haifa, Haifa, Israel.

1. Introduction.

Given a collection F of convex sets, an element $A \in F$ and a subcollection S of F; we say that a line L separates A from S if A is contained in one of the closed halfplanes defined by L, while every set in S is contained in the complementary closed halfplane.

In [4], H. Tverberg proves that for any positive integer k, there is a minimum integer N=N(k) such that in any family F of N disjoint convex plane sets, there is one that can be separated from a subfamily of F with at least k sets; he shows that N(k) is bounded from above by R(k) + k - 1, where R(k) is a Ramsey number. In this article we prove that N(k) is at most 12k.

We also show that for any collection F of n disjoint circles in R^2, there is a line L that separates a circle in F from a subcollection of F with at least $\lceil n/4 \rceil - 1$ circles. We produce configurations H_n and G_n, with n and 2n circles, respectively; such that no pair of circles in H_n can be simultaneously separated from any set with more than one circle of H_n ; and no circle in G_n can be separated from any subset of G_n with more than n circles.

In section 4 we present a set J_m with 3m line segments in R^2, such that no segment in J_m can be separated from a subset of J_m with more than m+1 elements. This disproves a conjecture by N. Alon, M. Katchalski and W.R. Pulleyblank presented in [1]. Finally, we show that if F is a set of n disjoint line segments in the plane such that they can be extended to be disjoint semilines, then there is a straight line L that separates one of the segments from a subset of F with at least $\lceil n/3 \rceil + 1$ elements.

2. Separating Convex Sets on the Plane.

In this section we deal with collections of disjoint, but otherwise arbitrary, convex sets on the plane. Our main result is the following.

Theorem 1. For any collection F of n disjoint convex sets on the plane, there is a line L that separates an element $A \in F$ from a subcollection of F with at least $\lceil n/12 \rceil$ sets . (**)

For the proof of theorem 1 we need two lemmas. The first lemma was proved implicitly in [2] and [5].

Lemma 1. For any family F of n disjoint convex sets on the plane there is a partitioning π of the plane using line segments or semilines $R_1,...,R_k$, with $k \leq 3n-6$, and such that every element in F lies on a different face of π and every element R_i of π lies on the boundary of exactly two faces of π containing elements of F; see figure 1.

For any line segment or semiline e, let us denote by $L(e)$ the line containing e. The next lemma is given without a proof. Let $C(n,k)$ denote the binomial coefficient.

Lemma 2. Let P and Q be two disjoint convex plane polygons. Then there is an edge e of P or Q such that $L(e)$ separates P from Q.

Proof of Theorem 1. Let F be a family of n disjoint convex sets and let π be as in lemma 1. For every element $S_i \in F$ let P_i be the face of π containing S_i .

Construct a bipartite graph $G(F,\pi)$ with one vertex $v(m)$ for every line segment R_m of π, $m=1,...,k \leq 3n-6$; and one vertex $s(i)$ for every set S_i in F. A vertex $v(m)$ is adjacent to a vertex $s(i)$ if the line $L(R_m)$ does not intersect the interior $int(S_i)$ of S_i.

Let us bound the number of edges in $G(F,\pi)$: consider any pair of elements S_i, S_j in F, and the polygonal faces P_i and P_j of π containing them. If P_i and P_j are disjoint, by Lemma 2, there is an edge R_m, say of P_i such that $L(R_m)$ separates P_i from P_j; then $L(R_m)$ does not intersect $Int(P_j)$ and $v(m)$ is adjacent to $s(j)$ in $G(F,\pi)$. When P_i and P_j share an edge R_m of π then $L(R_m)$ separates S_i from S_j; in particular $L(R_m)$ does not intersect $Int(P_l)$, where $l=\min\{i,j\}$, and $v(m)$ is adjacent to $s(l)$ in $G(F,\pi)$. None of these edges is counted more than once, therefore $G(F,\pi)$ contains at least $C(n, 2)$ edges.

By Lemma 1, any element R_m of π is in the boundary of two faces, say P_{m+} and P_{m-}, of π, containing elements of F denoted by S_{m+} and S_{m-}, respectively. Since π has k segments or semilines, $k \leq 3n-6$, then there is a vertex $v(m)$ of $G(F,\pi)$ with degree at least $C(n, 2)/k \geq C(n, 2)/3n-6 > n/6$. This implies that $L(R_m)$ does not intersect at least $\lceil n/6 \rceil$ elements of F. In the worst case, half of them lie on one side of $L(R_m)$ and the remaining on the opposite side. In any case, $L(R_m)$ separates either S_{m+} or S_{m-} from at least □n/12□ elements of F.

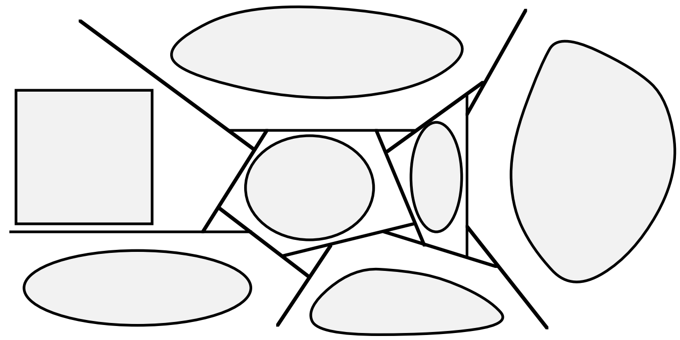

Figure 1

3. Separating Circles.

This section is devoted to the case where the convex sets are circles. In [1], N. Alon, M. Katchalski and W.R. Pulleyblank proved that there is a constant c>o such that for any family F with n disjoint congruent circles there is a line L that leaves at least $k/2 - c\sqrt{k}\sqrt{\log k}$ circles on each closed half semiplane defined by L. When the circles are allowed to have arbitrary radii the situation is entirely different.

We describe now a configuration H_n of n circles in which no pair C_i, C_j of circles in H_n can be simultaneously separated by one line L from any other pair C_k, C_l in H_n. Let $S_1 > S_2 > \ldots > S_n$ be n different slopes such that $0 \leq S_i \leq$ □, with □ small enough. Let H_n consist of n circles defined recursively as follows:

a) C_1 is any circle in R^2.

b) C_{i+1} is a circle tangent to C_i such that the slope of the line that separates them is S_i.

c) C_{i+1} is large enough such that any line L separating C_j from C_{i+1}, $1 \leq j < i+1$ has slope s(L) contained in the interval (S_i-□, S_i+□) , □>0, □ much smaller than □. Observe that s(L) is contained in the interval (-□, □+□) since $0 \leq S_i \leq$ □.

Moreover, if □ is small enough, C_{i+1} can be chosen such that:

d) Any line separating C_j from C_i, $1 \leq j < i$, intersects C_{i+1}.

It follows that there are no different pairs of circles $\{C_i, C_j\}$ and $\{C_k, C_l\}$ in H_n, such that there is a line separating $\{C_i, C_j\}$ from $\{C_k, C_l\}$. For let us assume that i is the smallest of i, j, k and l and that $k < l$. It now follows from (d) that any line separating C_i from C_k must intersect C_l . Notice that in H_n, C_i can be separated from $C_1,\ldots,C_{i-1}$, $i=1,\ldots,k$, and that C_i can not be separated from any pair C_k, C_l, $i < k < l$.

For any family of disjoint plane circles we have the following theorem.

Theorem 2. In any family F of n disjoint circles, there is one that can be separated from a subfamily of F with at least $\lfloor n/4 \rfloor - 1$ circles.

The following lemma will be used in the proof; the reader may wish to verify it.

Lemma 3. Let $C_1',\ldots. C_m'$ be m disjoint circles not containing the origin. Assume all of them intersect the x and y-axes and all of their centers are in the same quadrant, say the positive quadrant. If they intersect the axes in increasing order $C_1', C_2',\ldots, C_m'$, then any line separating C_m' from C_{m-1}' also separates C_m' from each Cj', with $1 \leq j < m$.

Proof of theorem 2. Start by sweeping a line L_1, from left to right and parallel to the y-axis, until one circle of F, say C_1, is left to the left of L_1 . Then sweep a line L_2 from bottom to top and parallel to the x-axis, until one circle, say C_2, is left below L_2.

If there are at least $n_1 \geq \lfloor n/4 \rfloor - 1$ circles to the right of L_1 or $n_2 \geq \lfloor n/4 \rfloor - 1$ circles above L_2, the result holds. Suppose then that n_1 and n_2 are both smaller than $\lfloor n/4 \rfloor - 1$. Then there is a subset H of F with $n - (n_1+n_2+2)$ circles that intersect both of L_1 and L_2 , and at most one of them, say C_3, contains the intersection point of L_1 and L_2.

Consider L_1 and L_2 as the coordinate axes, and divide $H\backslash\{C_3\}$ into four subsets as follows: each one of the four quadrants q_i of the plane defines a subset S_i of $H\backslash\{C_3\}$ consisting of all of the elements of $H\backslash\{C_3\}$ with center in q_i, $i=1,..,4$. Suppose, without loss of generality, that the union of the subsets $S_1 = \{C_1',\ldots,C_j'\}$ and $S_2 = \{C_1'',\ldots, C_k''\}$, corresponding to the first and second quadrants contain at least half of the elements of $H\backslash\{C_3\}$. Assume that the elements of S_1 and the elements in S_2 intersect the y-axis in increasing order $C_1', C_2',\ldots, C_j'$ and $C_1'', C_2'',\ldots, C_k''$, respectively. Let M_1 be any line separating C_j' from C_{j-1}' and M_2 be any line separating C_k'' from C_{k-1}''. By observation 1, M_1 separates C_j' from $S_1\backslash\{C_j'\}$ and M_2 separates C_k'' from

$S_1\setminus\{C_k''\}$. It is easy to verify that either M_1 separates C_j' from $(S_1\setminus\{C_j'\})\cup(S_2\setminus\{C_k''\})$ or M_2 separates C_k'' from $(S_1\setminus\{C_j'\})\cup(S_2\setminus\{C_k''\})$ and that $|(S_1\setminus\{C_j'\})\cup(S_2\setminus\{C_k''\})| \geq \lfloor n/4 \rfloor - 1$.

We now construct a family G_n of 2n circles in which no circle in it can be separated from more that n circles in G_n. To construct the family G_n let us take a copy $H_n' = \{C_1', C_2', \ldots, C_n'\}$ of the configuration H_n as follows: reflect H_n along the x-axis and translate it in the north-west direction until all the lines separating C_i from C_j intersect only C_n' in H_n' and all lines separating C_i' from C_j' intersect only C_n in H_n, see figure 2.

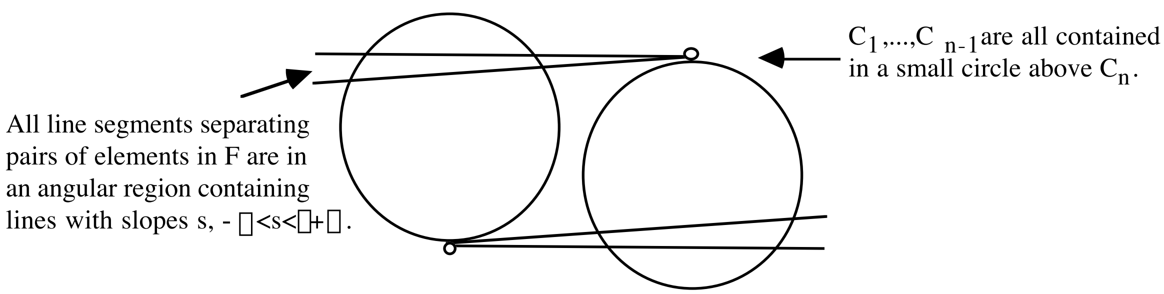

Figure 2

Any line separating two elements C_i, C_j in H_n leaves at most $C_1,\ldots,C_i$ on one side and $C_1',\ldots,C_{n-1}'$ on the other; similarly for any line separating two elements in H_n'. Then G_n is a configuration with 2n circles and none of them can be separated from any set of circles in G_n with more than n circles.

4. Separating Line Segments.

In [1], the following conjecture is presented: for any collection F of n disjoint line segments on the plane, there is an element S of F that can be separated from close to n/2 elements of F. In this section we disprove the conjecture by producing a family J_m of 3m line segments such that no element of J_m can be separated from more than m+1 elements of J_m.

To describe J_m we use a configuration due to K.P. Villanger, see [4]. He constructs a family T of m line segments $L_1, L_2, \ldots, L_m$ with the property that for each $k = 3,\ldots,m$; L_k

intersects the convex closure of $L_i \cup L_j$, $1 \le i < j < k$ and therefore L_k cannot be separated by a line from $\{L_i, L_j\}$, see figure 3.

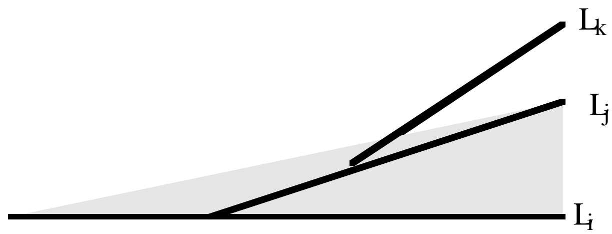

Figure 3

His construction may be reproduced in such a way that $L_1, L_2, \ldots, L_m$ have slopes $0=S(L_1)<S(L_2)<\ldots<S(L_m)=\alpha<\pi/2$, respectively; and such that for $i=1,2\ldots,m$, the left endpoint of L_{i+1} lies in an interior point of L_i within distance ε of the left endpoint of L_1.

Our example is a set J_m of $3m$ line segments consisting of three copies $T_0=\{L_{0,1},\ldots, L_{0,k}\}$, $T_1=\{L_{1,1},\ldots, L_{1,k}\}$ and $T_2=\{L_{2,1},\ldots, L_{2,k}\}$ of T placed around a triangle Q with vertices v_0, v_1, v_2. The values of α and ε are chosen in such a way that any element of T_i, when extended to be a whole line, intersects all the elements of T_{i+1}; addition taken mod 2.

Theorem 3. There is no element in J_m that can be separated from more than $m+1$ elements in J_m.

Proof. No element of T_i can be simultaneously separated by a single line from two elements of J_m, one in T_{i+1} and the other in T_{i+2}; addition taken mod 2. The result follows from the properties of T.

Let us consider the case where the segments in F can be extended to semilines so that they remain pairwise disjoint.

Theorem 4. Let $F=\{L_1,\ldots, L_n\}$ be a family of n disjoint line segments, $n\ge 4$. If they can be extended to form a collection of disjoint semilines, then there is a line L that separates an element L_i of F from a subset of F with at least $\lfloor n/3 \rfloor+1$ elements.

Proof. If there is an element L_i of F that can be extended to a whole line without intersecting any other element of F, then L_i can be separated from a subfamily of F with at least $\lfloor (n-1)/2 \rfloor$ elements of F. Suppose then that the line containing each L_i intersects at least another element L_i of F. Extend the elements of F as much as possible until a family $F' = \{L'_1, \ldots, L'_n\}$ of semilines is obtained such that:

1) The end point of every element of F' lies on an interior point of another element of F'.
2) No two elements of F' cross each other.

We say that L'_i hits L'_j if the end point of L'_i lies on L'_j. It is easy to see that in F' there is a cyclic sequence of elements, say $L'_1, \ldots, L'_j$, $j \leq n$ such that L'_{i+1} hits L'_i, $i=1, \ldots, j-1$, and L'_1 hits L'_j.

For the case when $j=n$ we can easily show that there is an element of F separable from a set with at least $\lceil n/2 \rceil$ elements of F; in the remainder of this section we will assume that $j<n$.

For every $i=2,...,j$ let S_i be the subset of F' consisting of L'_i together with all the elements of F' contained in the region bounded by L'_i and L'_{i-1}, and let S_1 be the subset of L' consisting of L'_1 and all elements of F' contained in the region bounded by L'_1 and L'_j. Let i be the smallest index such that the line L containing L'_1 intersects L'_i. Then it is easy to see that the set $A = S_2 \cup ... \cup S_{i-1}$ is separable from L_1. It is also easy to see that $B = S_i$ is separable from L'_{i-1} and that $C = S_{i+1} \cup ... \cup S_j \cup S_1$ may be separated from L'_i.

However, since $A \cup B \cup C = F'$, at least one of them has $\lceil n/3 \rceil$ elements; moreover if not all their cardinalities are the same, then at least one of them has $\lceil n/3 \rceil + 1$ elements and the result is proved. Assume then that A, B and C have the same cardinality. Since $j<n$, then at least one of the sets S_i, without loss of generality say S_1, contains more than one element $L'_a \in S_1$, $L'_a \neq L'_1$. It is now easy to see that L_a is separable from $A \cup \{L'_1\}$.

The segments in the example J_m may be extended to semilines in such a way that they remain pairwise disjoint. This shows that the bound in theorem 4 is tight.

4. Triangles and Rectangles

Similar results to the ones presented here for families of rectangles, triangles, etc. can also be obtained. We list some results that are easy to obtain using sweeping line arguments. No proofs will br given.

Theorem 4. In any family of n isothetic rectangles, it is always possible to separate one rectangle from $\lfloor 2n/3 \rfloor + 1$. Moreover, in this case we can always separate $\lfloor n/4 \rfloor$ rectangles from $\lfloor n/4 \rfloor$. These bounds are tight.

Theorem 5. Given any family of n disjoint homotetic triangles, there is one that can be separated from at least 3n/5±c triangles. There are some families with 3m triangles in which we cannot separate any triangle from more than 2m triangles.

5. Conclusions

We believe that the lower bound of $\lfloor n/12 \rfloor$ sets given in theorem 1 is far from optimal; the best upper bound we know is $\lfloor n/3 \rfloor + 1$, given by the example J_n in the previous section. For the case of circles we think that the $\lfloor n/4 \rfloor - 1$ lower bound given in theorem 2 should be improved to something close to n/2. We believe that in any family F of n disjoint line segments there is one that can be separated from considerably more than $\lfloor (n-1)/4 \rfloor$; perhaps from close to n/3 segments. The lower bound in Theorem 5 for homothetic triangles is not tight, we believe that the correct lower bound is close to 2n/3.

(**) Theorem 1 was independently proved by K. Hope and M. Katchalsky [3]

References

[1] N. Alon, M. Katchalski, and W.R. Pulleyblank. "Cutting disjoint disks by straight lines", *Discrete Comput. Geom.* **4** (1989), 239-243.

[2] H. Edelsbrunner, A.D. Robin and X. Shen."Covering convex sets with non-overlapping polygons", Report no.UIUCDCS-R-87-1364. Aug. 1987.

[3] K. Hope and M. Katchalski. "Separating plane convex sets", to appear in *Math Scand.*

[4] H. Tverberg. "A separation property of plane convex sets", *Math. Scand.* **45** (1979), 255-260.

[5] J. Urrutia and J . Zaks, "Illuminating convex sets", Technical Report TR-89-31. Department of Computer Science, University of Ottawa.