Two-Convex Polygons*

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Abstract

We introduce a notion of k-convexity and explore some properties of polygons that have this property. In particular, 2-convex polygons can be recognized in $O(n \log n)$ time, and k-convex polygons can be triangulated in $O(kn)$ time.

1 Introduction

The notion of convexity is central in geometry. As such, it has been generalized in many ways and for different reasons. In this note, we consider a simple and intuitive generalization, which to the best of our knowledge has not been worked on. It leads to an appealing class of polygons in the plane, with interesting structural and algorithmic properties.

A set in $\mathbb{R}^d$ is convex if its intersection with every straight line is connected or empty. This definition may be relaxed to directional convexity or *D-convexity* [9, 14], by considering only lines parallel to one out of a (possibly infinite) set D of vectors. A special case is *ortho-convexity* [16], where only horizontal and vertical lines are allowed. For any fixed D, the family of D-convex sets is closed under intersection, and thus can be treated in a systematic way using the notion of *semi-convex* spaces [17], which is sometimes appropriate for investigating visibility issues. The *D-convex hull* of a set M is the intersection of all D-convex sets that contain M. If D is a finite set, this definition of a convex hull may lead to an undesirably sparse structure—an effect which can be remedied by using a stronger, functional (rather than set-theoretic) concept of D-convexity [14].

***k*-Convex Sets** We consider a different generalization of convexity. A set M in $\mathbb{R}^d$ is called *k-convex* if there exists no straight line that intersects M in more than k connected components. Note that 1-convexity refers to convexity in its standard meaning[1]. To reformulate in terms of visibility, call two points $x, y \in M$ to be *k-visible* if $\overline{xy} \cap M$ consists of at most k components. Thus, a set is k-convex if and only if any two of its points are mutually k-visible. Applications of this concept may stem from placement problems for modems that have the capacity of sending through a fixed number of walls. Unlike directional convexity, k-convexity fails to show the intersection property: The intersection of k-convex sets is not k-convex, in general (k fixed), cf. Figure 1. For $k \geq 2$, a k-convex set M may be disconnected, or if being connected, its boundary may be disconnected. In this note, we will restrict attention to simply connected sets in two dimension, namely, simple polygons in the plane.

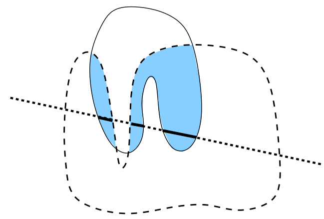

Figure 1: Intersecting two 2-convex sets

There are two notions of planar convexity that appear to be close to ours. One is *k-point convexity* [18, 4] which requires that, for any k points in a set M in $\mathbb{R}^2$, at least one of the line segments they span is contained in M. (Thus 2-point convex sets are precisely the convex sets.) The other is *k-link convexity* [13], being fulfilled for a given polygon P if, for any two points in P, the geodesic path connecting them inside P consists of at most k edges. (The 1-link convex polygons are just the convex polygons.) While there is a relation between k-convexity and the former concept, the latter concept is totally unrelated.

In the following we study basic properties of k-convex polygons (Section 2), give a characterization of 2-convex polygons (Section 3), and present efficient algorithms for recognizing (Section 4) and triangulating (Section 5) such polygons. Finally, Section 6 offers a discussion of our results.

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[1]We face notational ambiguity. The term 'k-convex' has, maybe not surprisingly, been used in different settings, namely, for functions [15], for graphs [3], and for discrete point sets [11]. Also, the concept of k-point convexity [18] has later been called k-convexity in [4].

2 Two-Convex Polygons

From now on, all geometric objects we will consider are closed sets in the Euclidean plane. Let P be a simple polygon, and denote with n the number of edges of P. Two line segments e and e' are said to *cross* if $e \cap e'$ is a point in the relative interior of both e and e'. Clearly, a polygon P is k-convex if every line segment with endpoints in P crosses at most $2(k-1)$ edges of P. The *stabbing number* [8] of a set of (interior-disjoint) line segments is the largest number of crossings attainable with a straight line. A polygon is k-convex if and only if its stabbing number is at most $2k$. Thus, all our observations on k-convexity could be reformulated in terms of stabbing numbers.

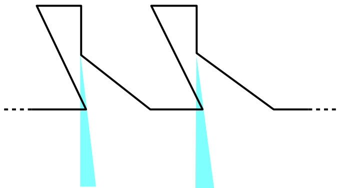

Figure 2: Quadratic 2-kernel construction

To see that 2-convexity is already significantly more complex than standard convexity, consider the *k-kernel* of a polygon P, i.e., the subset $M_k \subset P$ such that the entire polygon P is k-visible from each point $x \in M_k$. Note that P is k-convex if and only if $P = M_k$. Whereas M_1 is known to be a convex set which is computable in $O(n)$ time [12], M_2 may have an $\Omega(n^2)$ description; see Figure 2. The shaded areas emanating from the spikes are not part of M_2, and arranging such spikes along the boundary of a rectangle leads to a grid-like structure that partitions the 2-kernel into a quadratic number of components.

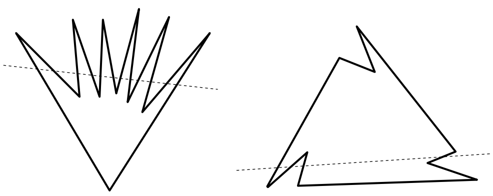

Figure 3: Star-shaped versus 2-convex

There is also no immediate relation to *star-shaped* polygons, i.e., polygons P with $M_1 \neq \emptyset$. Figure 3 shows, on the lefthand side, a polygon which is star-shaped but only $\frac{n}{2}$-convex. On the righthand side, we see a polygon which is 2-convex but not star-shaped. Visually, 2-convexity seems to be closer to convexity than is star-shapedness.

While 2-convexity clearly restricts the winding number [19] of a polygon, its link distance [13] is unaffected and may well be $\Theta(n)$. Conversely, a polygon which is 2-link convex (such that any two of its points are at link distance 2 or less) may fail to be k-convex for sublinear k. The star-shaped polygon in Figure 3 (left) is an example.

There is, interestingly, a relation to *k-point* convexity as defined in [18]. Every k-point convex polygon P is $(k-1)$-convex. To verify this, assume the contrary, which implies the existence of a straight line L which intersects P in at least k components. Select a point in each component. Now, by the assumed k-point convexity of P, at least one pair among the selected points yields a line segment, S, which entirely lies in P. As, clearly, $S \subset L$, the corresponding two components cannot be different—a contradiction. No implication exists in the other direction, however. For example, the 2-convex polygon in Figure 3 (right) fails to be 3-point convex. The class of k-convex polygons also differs from the class of k-guardable polygons defined in [1]. A linear number of (point) guards may be needed already to watch a 2-convex polygon. For further details see the full version of this paper.

3 Characterization

The definition of a k-convex polygon does not translate into an algorithm for recognizing such polygons. We give a characterization of k-convex polygons that allows a decision in time $O(n \log n)$.

Let P be the polygon under consideration, and denote with ∂P its boundary. A line L is called a *j-stabber* of P if L crosses ∂P at least j times. Note that a j-stabber may totally contain edges of P; these are *not* considered to contribute to the count. An *inflection edge* of P is an edge between a convex and a reflex vertex of P. Finally, an *inner tangent* of P is a line segment $T \subset P$ such that T contains two non-adjacent reflex vertices of P in its relative interior.

Lemma 1 *A simple polygon P is 2-convex if and only if P has no inner tangent, and no 3-stabber that contains an inflection edge.*

Proof. Omitted. □

4 Recognition

Let us assume that the given polygon P is nonconvex, as things are trivial, otherwise. The recognition algorithm is based on Lemma 1. It looks for inner tangents and 3-stabbers at inflection edges, trying to witness that P is not 2-convex. To this end, ray shooting [5] is performed for each reflex vertex of P, in the directions of its two incident edges. If ∂P is intersected more than once in a fixed direction, then a 6-stabber exists and we report that P is not 2-convex and stop. This covers the necessary check at each inflection edge.

The algorithm continues if all these directions yield a *unique* intersection point with ∂P. Let us assume, for the remainder of this section, that P is of this form. We store the points of intersection, and use them to check for inner tangents. Define, for each reflex vertex v of P, its *critical range* $C(v)$ as the set of all points $x \in \partial P$ such that $\overline{xv}$ can be prolonged to a line segment being tangent to P at v. Note that such a segment need not lie entirely in P. However, $C(v)$ consists of exactly two connected intervals on ∂P, whose endpoints are among the stored points obtained from ray shooting. See Figure 4, where $C(v)$ is drawn with bold lines.

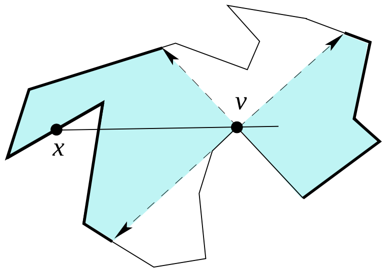

Figure 4: Critical range for vertex v

Observation 1 *P admits an inner tangent if and only if P has two reflex vertices v and v' such that $v \in C(v')$ and $v' \in C(v)$.*

Proof. Omitted. □

The strategy for detecting inner tangents is now clear. We first augment each reflex vertex with the two intervals of polygon vertices that lie inside its critical range. Then, we scan around ∂P with a point x and maintain, in some search tree, the set $R(x)$ of reflex vertices whose critical ranges contain x. That is, after initialization for a fixed position of x, we update $R(x)$ whenever x scans over some hitting point from ray shooting. Moreover, when x reaches some reflex vertex v of P, we search the tree with the four vertices u_i, u_j, u_k, u_ℓ that delimit $C(v)$, to check for $R(x) \cap [u_i, u_j] = \emptyset$ and $R(x) \cap [u_k, u_\ell] = \emptyset$.

The number of events where $R(x)$ undergoes some change or is searched is $O(n)$. This gives $O(n \log n)$ time, as does the total time spent for the $O(n)$ initial ray shooting queries; see [5]. The space requirement remains in $O(n)$.

Theorem 2 *For a simple polygon with n vertices, 2-convexity can be decided in $O(n \log n)$ time and $O(n)$ space.*

5 Triangulation

Triangulating a polygon in $O(n)$ time with a reasonably simple algorithm is still outstanding, except for special classes of polygons described, e.g., in [7, 1]. We will show below that 2-convex polygons add to this list. A simple ear-cutting-type triangulation method can be used, based on the following property.

Observation 2 *If P is a 2-convex polygon then, for each reflex vertex v of P, its critical range $C(v)$ is visible from v.*

Proof. Let P be 2-convex. Consult Figure 4 again, and consider any point $x \in C(v)$. The line segment $\overline{xv}$ does not cross ∂P because, otherwise, $\overline{xv}$ could be prolonged and slightly translated to yield a 6-stabber of P. □

Algorithm CUT-TO-PIECES
  $v_0 \leftarrow$ reflex vertex of P
  $v \leftarrow v_0$
  repeat
    Triangulate from v to $C(v)$
    $v \leftarrow$ next reflex vertex along ∂P
  until $v = v_0$

Triangulating from a given vertex refers to the yet untriangulated part of the polygon P. Figure 5 illustrates the effect of visiting the reflex vertices of P in clockwise order. After the loop, each subpolygon Q left untriangulated has a special property: Each vertex w of Q sees *all* vertices of Q in its internal angle (not just those in its critical range if w is reflex). Assuming the contrary implies that w is endpoint of some line segment tangent to the original polygon P at a reflex vertex, say v. But then we have $w \in C(v)$, and Q would have been split with the edge $\overline{vw}$ by Algorithm CUT-TO-PIECES. Observe that this argumentation does *not* imply that left-over polygons are star-shaped. Still, we can easily complete the triangulation for P by adding diagonals for such polygons.

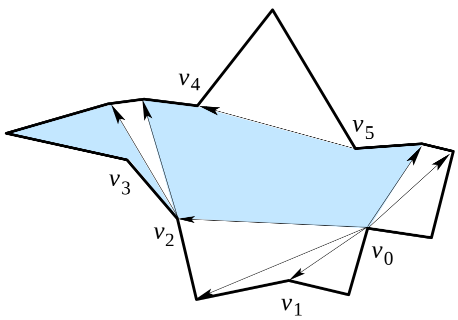

Figure 5: Ear cutting leaves two subpolygons

An alternative fast triangulation method for general k-convex polygons results from the fact that we can sort the vertices of a k-convex polygon P in any given direction (say, x-direction) in $O(kn)$ time: Simply scan around ∂P and use insertion sort, starting each time from the place where the x-value of the previous vertex has been inserted. Then any fixed value x_j, once being inserted, takes part in later comparisons at most $2k-1$ times because, otherwise, the vertical line $x = x_j$ would intersect P in more than k components. Having x-sorted P's vertices, a simplified plane sweep method can be used to build a vertical trapezoidation [6, 10] (and then a triangulation) of P. Only trivial data structures are needed, as the scenario on the sweep line is of complexity $O(k)$, by the k-convexity of P. Thus, each vertex of P can be processed in $O(k)$ time during the sweep. An $O(kn)$ time triangulation algorithm results.

Theorem 3 *Any k-convex polygon can be triangulated in $O(kn)$ time and $O(n)$ space.*

6 Discussion

Among the open algorithmic problems raised by this paper is the recognition of 2-convex polygons in linear time. In general, for $k \geq 3$, recognizing k-convexity of a polygon in subquadratic time is open. Also, no computational discussion of k-point convexity apparently exists.

As a combinatorial question, is it always possible to build, on top of a given planar point set, a 2-convex decomposition with a sublinear number of polygons? The problem of constructing a polygonization (a polygonal cycle through the points) which has k-convexity as low as possible seems to be hard. Is there a relation to the reflexivity [2] of point sets? How fast can we decide whether a point set admits a 2-convex polygonization?

Let us finally show that there are point sets where the best polygonization is at least $\Omega(\sqrt{n})$-convex. To this end, let S be the n points of a $\sqrt{n} \times \sqrt{n}$ grid, slightly perturbed to be in general position. Let L be a set of $\sqrt{n} - 1$ horizontal and $\sqrt{n} - 1$ vertical lines which can be drawn between the different rows and columns to separate the grid points. Then any edge of an arbitrary polygonization P of S intersects at least one element of L. Assign each edge of P to one of the elements in L it intersects. This way on average each line in L gets assigned $\frac{n}{2\sqrt{n}-2}$ edges of P. Thus, by the pigeon-hole principle, there is at least one line in L which intersects $\Omega(\sqrt{n})$ edges of P, that is, P is at least $\Omega(\sqrt{n})$-convex. We close with the question whether we can always find a polygonization which is $o(n)$-convex.

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