Modem Illumination of Monotone Polygons

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Abstract

We study a generalization of the classical problem of illumination of polygons. Instead of modeling a light source we model a wireless device whose radio signal can penetrate a given number k of walls. We call these objects k-modems and study the minimum number of k-modems necessary to illuminate monotone and monotone orthogonal polygons. We show that every monotone polygon on n vertices can be illuminated with $\left\lceil \frac{n}{2k} \right\rceil$ k-modems and exhibit examples of monotone polygons requiring $\left\lceil \frac{n}{2k+2} \right\rceil$ k-modems. For monotone orthogonal polygons, we show that every such polygon on n vertices can be illuminated with $\left\lceil \frac{n}{2k+4} \right\rceil$ k-modems and give examples which require $\left\lceil \frac{n}{2k+4} \right\rceil$ k-modems for k even and $\left\lceil \frac{n}{2k+6} \right\rceil$ for k odd.

1 Introduction

New technologies inspire new research problems, and wireless networking is a clear example of this. One such new problem is what we call here *modem illumination* of polygonal regions. Our problem arises in the following setting. It is well known that while trying to connect a laptop to a wireless modem, there are two factors that have to be considered, the *distance* to the wireless modem and, perhaps more important in most buildings, the *number of walls* separating our laptop from the wireless modem. (From now on, the term *modem* will be used to refer to a wireless modem.) We call a modem *a k-modem* if it is strong enough to transmit a stable signal through at most k walls along a straight line. Thus we say that a point p in a polygon P is covered by a k-modem m in P if the line segment joining p to m *crosses* at most k walls (edges) of P.

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We point out that we allow a modem to be located at a point q *on an edge* e of P; In this case, if p is an interior point of P, the line segment connecting p to q may cross an odd number of edges of P. This follows from the fact, that the line segment connecting p to q does not *cross* the edge e of P containing q. In this paper we consider the following problem:

Modem Illumination Problem: Let P be an art gallery modeled by a polygon P with n vertices. How many k-modems located at points of P are always sufficient, and sometimes necessary, to cover all points in P?

For $k = 0$ our problem corresponds to Chvátal's Art Gallery Theorem [2] which states that $\left\lfloor \frac{n}{3} \right\rfloor$ watchmen are always sufficient and sometimes necessary to guard an art gallery with n walls. Many generalizations of the original Art Gallery problem have been studied, see [4] for a comprehensive survey.

Illumination of polygons with wireless devices has been studied recently in [3, 1] in a slightly different context, the so-called sculpture garden problem. There, each device only broadcasts a signal within a given angle of the polygon and has unbounded range. The task is to describe the polygon (distinguish it from the exterior) by a combination of the devices, meaning that for each point p in the interior of the polygon no point outside the polygon receives signals from the same devices as p. See also [5] for related problems on wireless guarding.

In this paper we provide lower and upper bounds for the Modem Illumination Problem for monotone polygons and monotone orthogonal polygons, which perfectly model most real life buildings.

For technical reasons we make the following assumptions: for a non-orthogonal monotone polygon, we assume that no two of its edges are parallel; when we speak of a polygon we refer to both the boundary and the interior of the polygon.

2 Modem Illumination of Monotone and Monotone Orthogonal Polygons

2.1 Monotone Polygons

The next lemma provides our main tool for proving upper bounds on the number of modems required to illuminate monotone polygons. It allows us to *split*

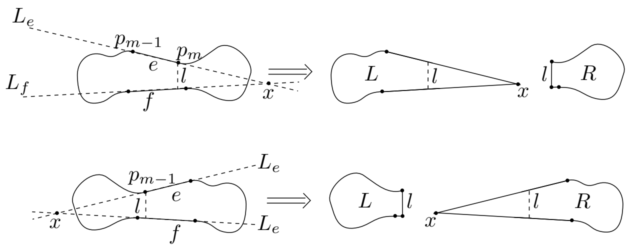

Figure 1: Illustrating the proof of Lemma 1.

monotone polygons into smaller ones, in such a way that we can illuminate these sub-polygons independently of each other.

Lemma 1 *(Splitting Lemma) Let P be a monotone polygon with vertices $p_1, p_2, \ldots, p_n$, ordered from left to right. For every positive integer $m < n$, there exist a vertical line segment l and two monotone polygons L and R, such that:*

- *L has m vertices and R has $n - m + 2$ vertices.*
- *Either l is chord of L and an edge of R, or l is an edge of L and a chord of R.*
- *p_m or p_{m+1} is an end point of l.*
- *Denote as L' the subset of L to the left of l, and denote as R' the subset of R to the right of l; then $P = L' \cup l \cup R'$.*

Proof. Without loss of generality we assume that p_{m-1} lies on the upper polygonal chain of P. Let f be the edge of P directly below p_{m-1} and let e be the edge of P having p_{m-1} as its left endpoint. Also let $e_l = p_{m-1}$ and e_r be the left and right endpoints of e, respectively. Likewise let f_l and f_r denote the left and right endpoints of f, respectively.

We extend both e and f to straight lines, so that L_e is the straight line containing e and L_f is the straight line containing f.

Since we are assuming non-parallel edges, L_e and L_f intersect at a point x. There are two cases (see Figure 1):

1. x is to the left of p_{m-1}.
   Draw a vertical line through p_{m-1} and let l be its intersection with P. Let P^- be the subset of P to the left of l and set $L = P^- \cup l$. We define R as the polygon enclosed by:
   - the upper polygonal chain of P from e_r to p_n,
   - the lower polygonal chain of P from f_r to p_n,
   - the line segment from x to e_r and the line segment from x to f_r.
2. x is to the right of p_{m-1}.
   Draw a vertical line through p_m and let l be its intersection with P. Let P^+ be the subset of P to the right of l and set $R = P^+ \cup l$. We define L as the polygon enclosed by:
   - the upper polygonal chain of P from p_1 to p_{m-1},
   - the lower polygonal chain of P from p_1 to f_l,
   - the line segment from $e_l = p_{m-1}$ to x and the line segment from f_l to x.

Note that in both cases the three stated properties between L, R, l and P hold. □

Before stating our main theorem for monotone polygons, we remark two useful lemmas.

Lemma 2 *Every $(k+2)$-gon can be illuminated with a k-modem placed anywhere in the interior or on the boundary of the polygon.*

Proof. A $(k+2)$-gon P contains $k+2$ edges. Note that any line segment joining points of the boundary of P intersects at most k edges of P in its interior. Therefore a k-modem placed anywhere in the interior or on the boundary of P illuminates the whole polygon. □

Lemma 3 *Every $(2k+2)$-gon can be illuminated with a k-modem placed either at its $(k+2)$-th or $(k+1)$-th vertex.*

Proof. We apply Lemma 1 to P and obtain a line segment l and two monotone polygons L and R of $k+2$ vertices each; all of them satisfying the properties given by Lemma 1. We place a k-modem at an endpoint of l. By Lemma 2 this k-modem illuminates both L and R. Since in particular it also illuminates L' and R', by Lemma 1 all of P is illuminated. □

We remark that for the particular cases of $k = 1, 2, 3$, Lemma 3 can be strengthened as follows (for lack of space, we omit the proofs):

Lemma 4 *For every monotone heptagon there exists a point inside the polygon, between its second and sixth vertex, where a 1-modem can be placed to illuminate the whole polygon.*

Proof. Omitted. □

Lemma 5 *Every monotone 8-gon can be illuminated with one 2-modem, placed either at its fourth or fifth vertex.*

Proof. Omitted. □

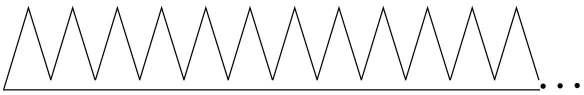

Figure 2: A monotone n-gon requiring $\lceil \frac{n}{2k+2} \rceil$ k-modems.

Lemma 6 *Every monotone 9-gon can be illuminated with one 3-modem placed at its fifth vertex.*

Proof. Omitted. □

We are now ready to state our main theorem.

Theorem 7 *Every monotone n-gon can be illuminated with $\lceil \frac{n}{2k} \rceil$ k-modems, and there exist monotone n-gons that require at least $\lceil \frac{n}{2k+2} \rceil$ k-modems to be illuminated.*

Proof. An example achieving the lower bound is given in Figure 2. So it remains to prove the upper bound. Using Lemma 1 recursively, we split the n-gon into $m = \lceil \frac{n}{2k} \rceil$ $(2k+2)$-gons as follows: apply Lemma 1 to P and obtain a line segment l_1, a monotone polygon L_1 of $2k+2$ vertices and a monotone polygon R_1 of $n-2k$ vertices; all satisfying the properties of Lemma 1. Apply now Lemma 1 to R_1 to obtain a line segment l_2, a monotone polygon L_2 of $2k+2$ vertices and monotone polygon R_2 of $n-4k$ vertices.

Continue this process and obtain the $\lceil \frac{n}{2k} \rceil$ monotone $(2k+2)$-gons $L_1, L_2, \ldots, L_m$ and the line segments $l_1, \ldots, l_{m-1}$; all satisfying the properties of Lemma 1.

For each L_i $(1 < i < m)$, let Q_i be the subset of L_i to the left of l_i and to the right of l_{i-1}. For L_1 and L_m, let Q_1 be the subset of L_1 to the left of l_1 and Q_m the subset of L_m to the right of l_{m-1}.

By Lemma 3, each L_i can be illuminated with a k-modem placed in Q_i. Note that since $P = (\bigcup Q_i) \cup (\bigcup l_i)$, this $\lceil \frac{n}{2k} \rceil$ modems of power k illuminate all of P. □

The proof of the upper bound in Theorem 7 uses Lemma 3, however for $k = 1, 2, 3$ the corresponding strengthened version of Lemma 3 can be used instead to obtain the following (better) upper bound:

Theorem 8 *For $k = 1, 2, 3$, any monotone n-gon can be illuminated using $\left\lceil \frac{n}{k+4} \right\rceil$ k-modems.*

Proof. Omitted. □

We remark that for the particular case of 1-modems, Theorem 8 gives an upper bound of $\lceil \frac{n}{5} \rceil$ 1-modems, and that there exist monotone n-gons (see Figure 3) that achieve this as a lower bound.

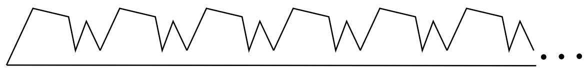

Figure 3: A monotone n-gon requiring $\lceil \frac{n}{5} \rceil$ 1-modems.

2.2 Monotone Orthogonal Polygons

In this section we give lower and upper bounds on the number of k-modems needed to illuminate orthogonal and monotone orthogonal polygons. Recall that for the main motivation of our research, namely to place modems inside buildings in order to cover the interior of the building with wireless reception, often orthogonal polygons are a quite realistic scenario.

Proposition 9 *Every orthogonal at most $(k+4)$-gon P is illuminated by a k-modem placed anywhere in the interior or on the boundary of P.*

Proof. Any line segment l with endpoints inside or on the boundary of the polygon P cannot properly (i.e., in the interior of l) intersect any of the leftmost vertical, topmost horizontal, rightmost vertical or bottom-most horizontal edge of P. Therefore, being at most $k+4$ edges in total, at most $k+4-4 = k$ edges are intersected by l. Thus, a k-modem placed anywhere inside or on the boundary of P illuminates the whole polygon. □

Proposition 10 *For any x-monotone orthogonal $(k+5)$-gon there is a point on its leftmost (or rightmost) edge where a k-modem can be placed to illuminate the polygon.*

Proof. If (at least) one of the two horizontal edges adjacent to the leftmost vertical edge, say e, is not the topmost or bottom-most horizontal edge, respectively, then placing the modem at the common vertex of e and the leftmost vertical edge illuminates the polygon. This follows from the proof of Proposition 9 and the fact that e does not block the rays of the modem.

Otherwise consider the two horizontal edges adjacent to the rightmost vertical edge. At least one of them is not the topmost or bottom-most horizontal edge. Let e' be this edge and w.l.o.g. assume that the interior of the polygon lies below e'. The horizontal line supporting e' intersects the leftmost vertical edge in its interior. Placing the modem on the leftmost vertical edge and below this intersection point illuminates the whole polygon by arguments similar to the previous case. Note that this is where we need the x-monotonicity of the polygon to guarantee that e' cannot block the rays of the modem. □

Using the previous observations we now can prove the following.

Proposition 11 *Every x-monotone orthogonal $(2k+6)$-gon can be illuminated with a k-modem.*

Proof. If k is even, we split the polygon vertically into two $(k+4)$-gons and place a k-modem in their common intersection; Proposition 9 ensures that the whole polygon is illuminated.

For odd k, split the polygon into one $(k+3)$-gon and one $(k+5)$-gon. By Proposition 10, there exists a point in their common intersection where a k-modem can be placed to illuminate the $(k+5)$-gon; Proposition 9 ensures that the $(k+3)$-gon is also illuminated. □

Our main result for monotone orthogonal polygons is thus:

Theorem 12 *Every x-monotone orthogonal polygon on n vertices can be illuminated with $\lceil \frac{n-2}{2k+4} \rceil$ k-modems.*

Proof. Split the n-gon into $(2k+6)$-gons and apply Proposition 11. □

For the case when k is even, the bound of Theorem 12 is tight, as shown by Figure 4 (right), where a monotone orthogonal n-gon requiring $\lceil \frac{n-2}{2k+4} \rceil$ k-modems to be illuminated is shown. For odd k, the same example gives a lower bound of $\lceil \frac{n-2}{2k+6} \rceil$.

However, for 1-modems, an example of a monotone orthogonal n-gon requiring $\lceil \frac{n-2}{6} \rceil$ 1-modems to be illuminated is shown in Figure 4 (left). Thus in this case the bound is also tight.

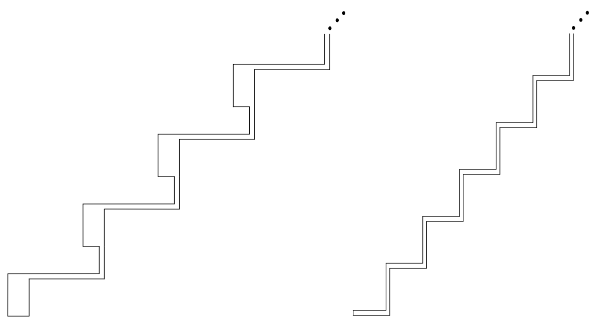

Figure 4: Lower bound constructions for monotone orthogonal polygons.

3 Conclusions

Inspired by current wireless networks, we studied a new variant of the classic polygon-illumination problem. To model the way wireless devices communicate within a building, we now allow light to cross a variable number of walls.

Using as a main tool the Splitting Lemma that allows us to divide a polygon into simpler, overlapping polygons, we give an upper bound on the number of k-modems needed to illuminate any given monotone polygon. We also presented a family of monotone polygons that need at least a number of k-modems close to that of our upper bound.

Moreover we studied the particular case when the monotone polygons are orthogonal, where we have been able to give, in most cases, tight bounds.

The natural open problem remaining is to close the gap between lower and upper bounds for both types of polygons, monotone and monotone orthogonal.

The modem illumination problem for general polygons, has proved to be rather challenging. We believe that obtaining tight bounds for this case is a non-trivial problem. At the moment the best lower bounds we have, are those we have for monotone polygons, and no significant upper bounds are known to us. Noteworthy, we might not be surprised if the lower bounds for general polygons happen to be realized by monotone polygons also; note that this happens for the $\lfloor n/3 \rfloor$ lower bound for the classical polygon illumination problem.

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