On measuring areas of polygons

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(Abstract)

The measurement of areas and volumes of sets is a fundamental problem in mathematics and Computational Geometry. It is generally accepted that one of the motivations that fueled the development of geometry in early civilizations was the need to measure land for taxation purposes [1].

It is straightforward to see that calculating areas of polygons in the plane, and that volumes of polyherdra in R^3 can be done in linear time. In this paper we study the following problem: Suppose that we split a simple polygon P into two subpolygons Q_1 and Q_2 by cutting it along a line segment joining two mutually visible points p and q on its boundary, see Figure 1.

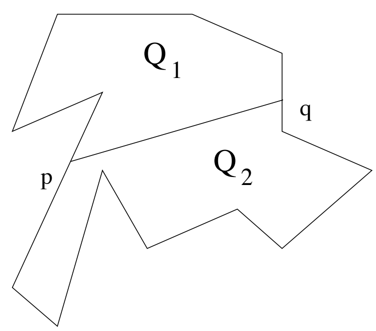

Figure 1: Cutting P along the line segment joining p to q.

How quickly can we measure the areas of Q and R? It is easy to see that this problem can be solved optimally in linear time. Our main objective here, is to study how to speed up the measurement of the areas of Q_1 and Q_2 by using some preprocessing.

We prove that after a *linear* amount of preprocessing the problems listed below can be solved as follows:

1. Measuring the areas of the sections obtained by cutting a convex polygon along a line segment joining two vertices.
   This can be done in constant time.

2. Measuring the areas of the sections into which an arbitrary line splits a convex polygon.
   Solvable in $O(\ln n)$ time.

3. Given a convex polygon of area A and an arbitrary direction d, finding a line parallel to d that divides our polygon into sections of size $\frac{A}{k}$ and $\frac{A(k-1)}{k}$, $k > 1$. Solvable in $O(\ln n)$ time.

It is worth noting that the preprocessing required to achieve these complexities is simple, and thus the solutions obtained are useful in practice. The preprocessing consists of measuring the areas of some triangles, and a few binary searches.

We also show that using $O(n \ln n)$ time preprocessing and space we can measure in $O(\ln n)$ time the areas of the sections resulting when we cut a simple polygon along a line segment joining two mutually visible points on its boundary. Our preprocessing, although a more complicated than the one used for convex polygons, remains practical.

Some generalizations of our problem, as well as open problems in dimension 3 will be discussed.

References

[1] F. P. Preparata and M. I. Shamos. *Computational Geometry: An introduction.* Springer-Verlag, New York 1985.