Bichromatic Discrepancy via Convex Partitions

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Abstract

Let R be a set of red points and B a set of blue points on the plane. In this paper we introduce a new concept for measuring how mixed the elements of $S = R \cup B$ are. The discrepancy of a set $X \subset S$ is $||X \cap R| - |X \cap B||$. We say that a partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of S is convex if the convex hulls of its members are pairwise disjoint. The discrepancy of a convex partition of S is the minimum discrepancy of the sets S_i. The discrepancy of S is the discrepancy of the convex partition of S with maximum discrepancy. We study the problem of computing the discrepancy of a bichromatic point set. We divide the study in general convex partitions for both general set of points and points in convex position, and also when the partition is given by a line. In this case we prove that this problem is 3SUM-hard.

1 Introduction

In this paper we study a parameter that measures how *mixed* two point sets are. In the rest of this paper $S = R \cup B$ will always be a set of points on the plane in general position whose elements are colored either red (the elements of R), or blue (the elements of B). We will also assume that R and B are non-empty and have r and b elements respectively. Intuitively speaking R and B are *well mixed* if for any convex region C of the plane the proportion of red elements of S is aproximately $\frac{r}{b+r}$, e.g. if $r = 2b$, we would expect C to contain twice as many red points as blue. It is clear that we must be careful on how we define well mixed sets of points, as since S is in general position, we can always find many convex sets containing only two points of S with the same color. In this paper we introduce a parameter that seems like a good candidate to measure how well mixed a bicolored point set is, we call this parameter the *discrepancy* of S.

For any set of points P on the plane, let $CH(P)$ denote the convex hull of P. Let $S = R \cup B$ be a bicolored point set, and $X \subset S$. The discrepancy of X is defined as $\nabla(X) = ||X \cap R| - |X \cap B||$. We say that a partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of S is a *convex partition* if $CH(S_i) \cap CH(S_j) = \emptyset$ for all $1 \leq i < j \leq k$. The *discrepancy of* S *with respect to* Π is defined as $d(S, \Pi) = \min_{i=1..k} \nabla(S_i)$. The *discrepancy of* S, $d(S)$, is defined as the largest $d(S, \Pi)$ over all the convex partitions Π of S.

Notice that if the discrepancy of a set is large, then there is a convex partition Π of S in which each element of Π has large discrepancy. If $d(S) = 1$ *any convex partitioning* of S has at least one element with discrepancy one. For example if $S = R \cup B$ is separable, that is there is a line ℓ that leaves all the elements of R on one of the half-planes it determines, and all the elements of B on the other, then the discrepancy of S is at least the minimum of r and b.

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If we restrict ourselves to convex partitions of S with exactly k elements, we obtain the k-discrepancy of S, which will be denoted as $d_k(S)$. When $k = 1$ then the partition Π has only one element, and thus $d_1(S) = \nabla(S) = |r - b|$. If $k = 2$ then we have what we call *linear discrepancy*, that is the discrepancy obtained by partitions of S induced by lines that split S into two subsets.

Our concept of discrepancy has applications in Data Analysis and Clustering in sets of data of two classes, say red and blue. We can state that a red-blue dataset is not good for clustering when its discrepancy is low. Hence our concept can be used as a priori tester to a dataset for clustering. The extreme case is when $d(S) = 1$, in this case we say that S is *locally balanced*. Many of the results of this paper focus on the hardness of deciding if a bicolored point set is locally balanced or not.

The discrepancy between two objects is a measure of how different the objects are. In [1] they study a concept of discrepancy for hypergraps, they study the problem of assigning weight $+1$ or -1 to the vertices of a given hypergraph in such a way the maximum weight of a edge (i.e. the absolute value of the sum of the weights of its vertices) is minimized. Geometric Discrepancy Theory [4] studies how uniform nonrandom structures can be. For example, how to color n points in the plane to minimize the difference between the number of red points and the number of blue ones within any disk. In [2, 6, 7] the concept of bichromatic discrepancy is considered by computing the object (e.g. box, triangle, strip, convex polygon, etc.) of maximum absolute difference between red and blue points inside it. In this paper we introduce a new parameter to measure the discrepancy of bicolored point sets.

The outline of this work is as follows. In section 2 we provide combinatoric results for measuring discrepancy for general convex partitions. We consider two cases, when the points are in convex position and when they are not. In section 3 we give combinatoric and hardness proofs on measuring discrepancy by using partitions by a line.

2 General Convex partitions

In this section we explore sets of red and blue points with discrepancy equal to one. A point set P is in *convex position* if the elements of P are the vertices of a convex polygon. We consider two cases, points in convex position and points in general position. We start with the following lemma which is straightforward to prove.

Lemma 2.1. *Let $S = R \cup B$ then, $d(S) \geq 1$. Moreover if $d(S) = 1$ then $|r - b| \leq 1$.*

2.1 Point Sets in Convex Position

Definition 2.2. Suppose that the elements of S are in *convex position* and that $|r - b| \leq 1$. We say that they form an *alternating convex chain* if we can label them $p_1, p_2, \ldots, p_{r+b}$ in the counterclockwise order around $CH(S)$ such that, for every $1 \leq i \leq r + b - 1$, p_i and p_{i+1} have different color.

Lemma 2.3. *If S is in convex position then $d(S) = 1$ if and only if S is an alternating convex chain.*

Proof. Suppose that $d(S) = 1$ and that S is not an alternating convex chain. By Lemma 2.1 $|r - b| \leq 1$. If $r = b$ and for some i we have that p_i and p_{i+1} have the same color, the partition $\Pi = \{S_1, S - S_1\}$ where $S_1 = \{p_i, p_{i+1}\}$ has discrepancy 2. If $r = b + 1$ and there is an i such that p_i and p_{i+1} are red points, then if $S_1 = \{p_i, p_{i+1}\}$, $\nabla(S_1) = 2$, $\nabla(S - S_1) = 3$, and the partition $\Pi = \{S_1, S - S_1\}$ is such that $d(S) \geq d(S, \Pi) = 2$. It follows by contradiction that S is an alternating convex chain.

Suppose now that S is an alternating convex chain. It is easy to see that in any convex partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of S, there is at least one S_i $(1 \leq i \leq k)$ such that S_i is an alternating convex chain, and thus has discrepancy less than or equal to one. Thus $\nabla(S_i) \leq 1$, and that $d(S, \Pi) \leq 1$ for all Π. Therefore $d(S) = 1$. □

To make some of our proofs easier, for any bicolored point set X with r' red points, and b' blue points, let $\nabla'(X) = r' - b'$. Observe that $\nabla(X) = |\nabla'(X)|$. Next we prove:

Theorem 2.4. *If S in convex position then $d(S) = \max_{k=1,2,3} d_k(S)$.*

Proof. Let $d = d(S)$, observe that in particular $0 \le \nabla(S) \le d$. Assume w.l.o.g. that $0 \le \nabla'(S) \le d$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a convex partition of S with minimum cardinality such that $d(S, \Pi) = d$. By definition, $\nabla(S_i) \ge d$ $(1 \le i \le k)$. Suppose that $k > 3$. Then S has at least two elements, say S_1 and S_2, such that both of them contain only consecutive elements of S along the boundary of $CH(S)$.

If any of S_1 or S_2, say S_1, is such that $\nabla'(S_1) \le -d$ then $\nabla'(S - S_1) = \nabla'(S) - \nabla'(S_1) \ge 0 + d = d$, and thus $\nabla(S - S_1) = |\nabla'(S - S_1)| \ge d$. This is a contradiction because the convex partition $\Pi' = \{S_1, S - S_1\}$ has cardinality 2 and $d(S, \Pi') \ge d$. Suppose then that $\nabla'(S_1) \ge d$ and $\nabla'(S_2) \ge d$. Observe that $\nabla'(S - S_1 - S_2) = \nabla'(S) - \nabla'(S_1) - \nabla'(S_2) \le d - d - d = -d$, and thus $\nabla(S - S_1 - S_2) = |\nabla'(S - S_1 - S_2)| \ge d$. This is a contradiction because $\Pi'' = \{S_1, S_2, S - (S_1 \cup S_2)\}$ has cardinality 3 and $d(S, \Pi'') \ge d$. □

This implies:

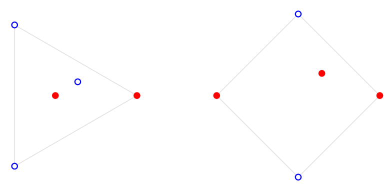
Theorem 2.5. *The discrepancy of a bicolored point set in convex position can be computed in polynomial time.*

2.2 Point Sets in General Position

In this section we deal with bicolored point sets in general position. The following lemmas are straightforward.

Proposition 2.6. *If S has at most four elements and $d(S) = 1$ then S is an alternating convex chain.*

Proposition 2.7. *There are two combinatorially different point configurations of a bichromatic point set S with five points such that $d(S) = 1$ (see Fig. 1).*



(a) (b)

Figure 1: Two different point configurations with five points with $d(S) = 1$.

We now show configurations of points in general position that are locally balanced.

Proposition 2.8. *For all $n \ge 4$ there are bichromatic point sets, not in convex position, with $d(S) = 1$.*

Proof. The proof is based on the following constructions. For $n = 2m + 1$ let Q be a regular convex polygon with $2m$ vertices such that their colors alternate blue and red along the boundary of Q. Let S be the set of vertices of Q plus a red point p close to the center of Q (see Fig. 2 (a)). Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be any convex partition of S. If $k = 1$ then $d(S, \Pi) = 1$. Suppose that $k > 1$, then there is some $S_i \in \Pi$ $(1 \le i \le k)$ such that $r \notin S_i$ and S_i contains a set of consecutive vertices of Q. Then $\nabla(S_i) \le 1$ and therefore $d(S, \Pi) \le 1$.

If $n = 2m + 2$, place two points p and q in the interior of Q such that p and q are close enough to the middle of an edge e of Q, and the line joining then is almost parallel to e. It is easy to see now that $d_2(S) = 1$ and that $d(S, \Pi) \le 1$ for all the convex partitions Π of S (see Fig. 2(b)). □

Constructing point sets with large discrepancy is straightforward. As we already mentioned in the introduction of this paper, if R and B are linearly separable then $d(S) \ge min\{r, b\}$.

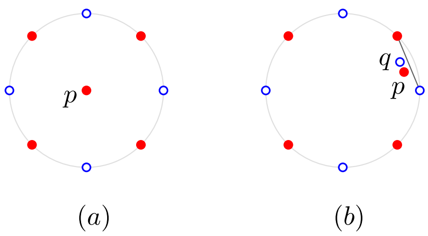


Figure 2: Point sets with n points and $d(S) = 1$. (a) n is odd. (b) n is even.

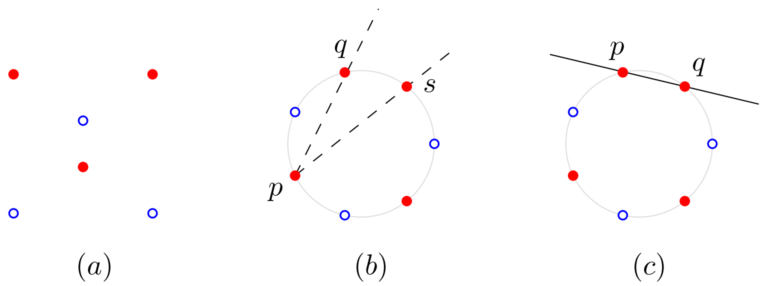


Figure 3: (a) There are no three consecutive points of the same color in the projection on any line and $d_2(S) = 2$. (b) The red points q and s are consecutive in the angular sorting of $S - \{p\}$ with respect to p and $d_2(S) = 1$. (c) The number of red and blue points in the half-plane above the line through p and q is zero and $d_2(S) = 1$ because S is an alternating convex chain.

3 Partitions with a line

The problem of deciding the discrepancy of point sets in general position seems to be non-trivial. At this point, we are unable even to characterize point sets with discrepancy one. In this section we characterize sets with linear discrepancy one and show how to decide if the linear discrepancy of a bicolored point set S is equal to d. We introduce the following notation.

Let $\Pi_{\ell+}$ and $\Pi_{\ell-}$ be the open half-planes bounded below and above respectively by a non vertical line ℓ. Let $S_{\ell+} = S \cap \Pi_{\ell+}$, $S_{\ell-} = S \cap \Pi_{\ell-}$ and $\Pi_\ell = \{S_{\ell+}, S_{\ell-}\}$. The *linear discrepancy of* S is $d_2(S) = \max_\ell d(S, \Pi_\ell)$ where the lines ℓ contain no point in S.

Proposition 3.1. *Let $S = R \cup B$ such that $r = b$ and $d_2(S) = 1$. Then the following properties hold:*

1. *The convex hull of S is an alternating chain.*
2. *When projected on any line, the points of S form a sequence such that no three consecutive points have the same color.*
3. *For every $p \in S$ on the convex hull of S, the angular ordering of the elements of $S - \{p\}$ with respect to p is a sequence with alternating colors.*
4. *For every line ℓ passing through two points of the same color, say red, the number of red points in each of $S_{\ell+}$ and $S_{\ell-}$ is exactly one less than the number of blue points in $S_{\ell+}$ and $S_{\ell-}$ respectively.*

Property 2 in Proposition 3.1 is not sufficient to guarantee that $d(S) = 1$, e.g. see Fig. 3 (a). If $r \neq b$ properties 3 and 4 are not necessarily true, see Fig. 3 (b) and (c). We now show that if $r = b$, Property 4 is sufficient.

The next result, proven in [5] will be useful:

Theorem 3.2. *Let P and Q be two disjoint convex polygons on the plane. Then there is at least one edge e of P or Q such that the line ℓ_e containing e separates the interior of P from the interior of Q.*

Lemma 3.3. *If $r = b$ then the following two conditions are equivalent: (a) $d_2(S) = 1$, (b) for every line ℓ passing through two points of S with the same color $\nabla(S_{\ell^+}) = \nabla(S_{\ell^-}) = 1$.*

Proof. It is easy to prove that (a) implies (b). We show here that (b) implies (a). Suppose that $d_2(S) = d \geq 2$. We now show that there exists a line ℓ containing two points of the same color of S such that $\{\nabla(S_\ell^-), \nabla(S_\ell^+)\} = \{d, d-2\}$.

Let ℓ_0 be a line containing no elements of S such that $d_2(S) = d(S, \Pi_{\ell_0}) = d$. Assume w.l.o.g. that ℓ_0 is horizontal. Since $r = b$ we have that $d_2(S) = d(S, \Pi_{\ell_0}) = \nabla(S_{\ell_0}^+) = \nabla(S_{\ell_0}^-) = d$ and $\nabla'(S_{\ell_0}^+) = -\nabla'(S_{\ell_0}^-)$. Assume w.l.o.g. that $\nabla'(S_{\ell_0}^+) > 0$ (i.e. $S_{\ell_0}^+$ has more red points than blue, and $S_{\ell_0}^-$ has more blue points than red).

Let P and Q be the polygons induced by the convex hulls of $S_{\ell_0}^+$ and $S_{\ell_0}^-$ respectively. Let p be a vertex of P such that there is a line ℓ' passing trough p that separates P from Q. Then p must be a red point, for otherwise by translating ℓ' up by a small distance, we obtain a partitioning Π' of S with discrepancy $d+1$. Similarly any point q in Q such that there is a line trough q that separates P from Q must be blue.

By Theorem 3.2 there is an edge e of P or Q, with vertices p and q, such that the line ℓ_e containing e separates P from Q. If e is an edge of P then p and q are red by using the above observation. Thus we have that $\{\nabla(S_{\ell_e}^+), \nabla(S_{\ell_e}^-)\} = \{d, d-2\}$. A symmetric argument works when e belongs to Q. $\square$

Proposition 3.4. *Let $S = R \cup B$, then we have,*

$$\max\left\{1, \lfloor\frac{|r-b|}{2}\rfloor\right\} \leq d_2(R \cup B) \leq \max\left\{\lfloor\frac{|r-b|}{2}\rfloor, \min\{r, b\}\right\}.$$

Furthermore, both bounds are tight.

Proof. Suppose w.l.o.g. that $r \geq b$. By the *Ham Sandwich Cut Theorem* [10] there exists a line ℓ passing through at most one red point and at most one blue point such that $|S_{\ell^+} \cap R| = |S_{\ell^-} \cap R| = \lfloor\frac{r}{2}\rfloor$ and $|S_{\ell^+} \cap B| = |S_{\ell^-} \cap B| = \lfloor\frac{b}{2}\rfloor$. Four cases arise depending on the parities of r and b. We show only the case when r and b are even. The other cases can be solved in a similar way.

If $r = 2a$ and $b = 2c$ then ℓ contains no point of S, and $\nabla(S_{\ell^+}) = \nabla(S_{\ell^-}) = a - c = \lfloor\frac{r-b}{2}\rfloor$. Thus $\lfloor\frac{r-b}{2}\rfloor = d(S, \Pi_\ell) \leq d_2(S)$.

If $|r-b| \geq 2$ then $d_2(S) \geq \lfloor\frac{|r-b|}{2}\rfloor \geq 1$ thus it is missing to prove that $d_2(S) \geq 1$ when $|r-b| \leq 1$. Suppose w.l.o.g. that $b \leq r \leq b+1$. If there is a blue point p in the convex hull of S take a line ℓ separating p from $S - \{p\}$ and suppose that $p \in S_{\ell^+}$, then $\nabla(S_{\ell^+}) = 1$, $\nabla(S_{\ell^-}) = r - b + 1 \geq 1$ and $d_2(S) \geq d(S, \Pi_l) = 1$. If no such p exists then there are two consecutive red points p and q in the convex hull of S, then take a line ℓ separating p and q from $S - \{p, q\}$ and suppose that $p, q \in S_{\ell^+}$, then $\nabla(S_{\ell^+}) = 2$, $\nabla(S_{\ell^-}) = b - r + 2 \geq 1$ and $d_2(S) \geq d(S, \Pi_l) = 1$. This proves the lower bound.

We show now that this lower bound is tight. Suppose w.l.o.g. that $r > b$ and let X be a set composed by r red points and r blue points, and let Y be a set of $r - b$ red points. Put the elements of X on an alternating convex chain and the elements of Y in the interior of the convex hull of X in such a way there is a line ℓ_e such that $d(X, \Pi_{\ell_e}) = 0$ and ℓ_e splits Y into two subsets of cardinality $\lfloor\frac{|Y|}{2}\rfloor$ and $\lceil\frac{|Y|}{2}\rceil$ respectively. Let $S = X \cup Y$ and observe that $d(S, \Pi_{\ell_e}) = \lfloor\frac{|Y|}{2}\rfloor = \lfloor\frac{r-b}{2}\rfloor$. For any line ℓ we have that $d(X, \Pi_\ell) \in \{0, 1\}$ (by Lemma 2.3). If $d(X, \Pi_\ell) = 0$ then $d(S, \Pi_\ell) = d(X \cup Y, \Pi_\ell) = d(Y, \Pi_\ell) \leq \lfloor\frac{|Y|}{2}\rfloor = \lfloor\frac{r-b}{2}\rfloor$. If $d(X, \Pi_\ell) = 1$ then $d(X \cup Y, \Pi_\ell) = \min\{x-1, (r-b)-x+1\}$ where x is such that l splits Y in x and $(r-b)-x$ points respectively. It is easy to prove that $\min\{x-1, (r-b)-x+1\} \leq \lfloor\frac{r-b}{2}\rfloor$. Then $d_2(S) = d(S, \Pi_{\ell_e}) = \lfloor\frac{r-b}{2}\rfloor$.

To prove the upper bound suppose w.l.o.g. that $r \geq b$ (i.e. $b = \min\{r, b\}$). We have to show that $d(S, \Pi_\ell) > b \Rightarrow d(S, \Pi_\ell) \leq \lfloor\frac{r-b}{2}\rfloor$ for every line ℓ. Let ℓ be a line such that $d(S, \Pi_\ell) > b$. Then we have that $\nabla'(S_{\ell^+}) > 0$ and $\nabla'(S_{\ell^-}) > 0$. In fact, suppose that $\nabla'(S_{\ell^+}) < 0$, then $\nabla(S_{\ell^+}) = |S_{\ell^+} \cap B| - |S_{\ell^+} \cap R| \leq |S_{\ell^+} \cap B| \leq b$ thus $d(S, \Pi_\ell) \leq b$, a contradiction. Now, $\nabla'(S_{\ell^+}) > 0$ and $\nabla'(S_{\ell^-}) > 0$ imply that $d(S, \Pi_\ell) \leq \lfloor\frac{r-b}{2}\rfloor$. In fact, suppose the contrary, $\nabla(S_{\ell^+}) \geq \lfloor\frac{r-b}{2}\rfloor + 1$ and

$\nabla(S_{\ell^-}) = (r-b) - \nabla(S_{\ell^+}) \geq \lfloor\frac{r-b}{2}\rfloor + 1$, thus $r - b \geq 2\lfloor\frac{r-b}{2}\rfloor + 2$, a contradiction. If $\lfloor\frac{r-b}{2}\rfloor \leq b$ the upper bound is tight if we take separable sets R and B. If $\lfloor\frac{r-b}{2}\rfloor > b$ we have shown above how to build a set of points S with $d_2(S) = \lfloor\frac{r-b}{2}\rfloor$. $\square$

Corollary 3.5. *Let $S = R \cup B$ such that $|r-b| \geq 2$. If $r \geq 3b$ or $b \geq 3r$ then $d_2(R \cup B) = \lfloor\frac{|r-b|}{2}\rfloor$.*

Proof. Suppose that $r \geq 3b$, then $r - b \geq 2b \Rightarrow \frac{r-b}{2} \geq b \Rightarrow \lfloor\frac{r-b}{2}\rfloor \geq b$. Thus the upper and lower bounds of $d_2(R \cup B)$ in Proposition 3.4 are equal. $\square$

3.1 Hardness

Theorem 3.6. *Given an integer $d \geq 1$ it is 3SUM-hard to decide if $d_2(S) = d$.*

Proof. We will use a reduction from the 3SUM-problem similar to the 3SUM-hardness proof of the 3-POINTS-ON-LINE-problem [9]. Consider the set $X = \{x_1, \ldots, x_n\}$ of n integer numbers (positive's and negative's) an instance of 3SUM-problem and assume w.l.o.g. that $x_1 < \ldots < x_j < 0 < x_{j+1} < \ldots < x_n$ $(1 \leq j < n)$. Let $M = \max\{|x_1|, |x_n|\}$. If $d = 1$ put a blue point in $(-2M, 0)$ and a red point in $(2M, 0)$. If $d > 2$ then for each $1 \leq i \leq d-2$ put a red point in $(-2M - i + 1, 0)$ and a blue point in $(2M + i - 1, 0)$. Let ε be a small real positive number such that $\varepsilon < \frac{1}{6M}$. For each $1 \leq i \leq n$ put a red point r_i in $(x_i - \varepsilon, x_i^3)$ and a blue point b_i in $(x_i + \varepsilon, x_i^3)$ (see Fig. 4). We can prove that there exists a line separating three distinct pairs (r_i, b_i), (r_j, b_j) and (r_k, b_k) if and only if (x_i, x_i^3), (x_j, x_j^3) and (x_k, x_k^3) are collinear (i.e. $x_i + x_j + x_k = 0$). Let S be the set of red an blue points as above. We have that $d_2(S) \geq d$ because $d_2(S, \Pi_\ell) = d$ for every line ℓ separating exactly two distinct pairs (r_i, b_i) and (r_j, b_j). If $d_2(S) > d$ then there is a line separating more than two pairs implying that three elements in X sum to zero. Therefore, three elements in X sum to zero if and only if $d_2(S) \neq d$. $\square$

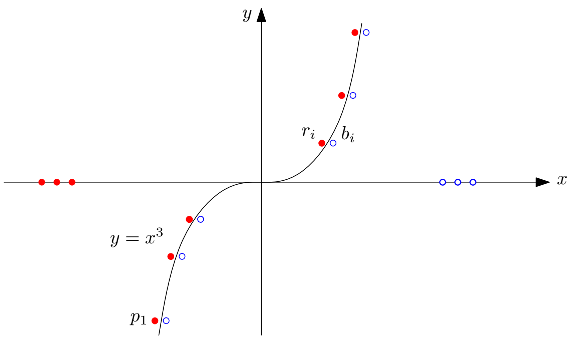


Figure 4: Reduction from 3SUM-problem when $d = 5$.

Theorem 3.7. *Computing the linear discrepancy of a bichromatic point set is 3SUM-hard and it can be done in $O(n^2)$ time.*

Proof. The hardness is an implication of Theorem 3.6 and we can use duality for computing $d_2(S)$. $\square$

3.2 The Weak Separator problem

Given a bichromatic set of points in the plane the *Weak Separator problem* (WS-problem) looks for a line that maximizes the number of blue points on one of its sides plus the number of the red ones on the other. The WS-problem can be solved in $O(n^2)$ [11] or in $O(nk \log k + n \log n)$ [8] where k is the number of misclassified points. Recently an $O((n + k^2) \log n)$ expected time algorithm has been presented in [3]. We now prove that the WS-problem is 3SUM-hard.

Lemma 3.8. *Let $S = R \cup B$ such that $r = b$. Solving the WS-problem for S is equivalent to finding a line ℓ such that $d(S, \Pi_\ell) = d_2(S)$.*

Proof. Let ℓ be any line such that $d(S, \Pi_\ell) = d_2(S)$. Since $r = b$ then $\nabla'(S_{\ell^+}) = -\nabla'(S_{\ell^-})$. Suppose w.l.o.g. that $d_2(S, \Pi_\ell) = \nabla'(S_{\ell^+}) = |S_{\ell^+} \cap R| - |S_{\ell^+} \cap B| > 0$. We have that $|S_{\ell^+} \cap R| + |S_{\ell^-} \cap B| = |S_{\ell^+} \cap R| + |B| - |S_{\ell^+} \cap B| = b + |S_{\ell^+} \cap R| - |S_{\ell^+} \cap B|$. Hence $|S_{\ell^+} \cap R| + |S_{\ell^-} \cap B|$ is maximum if and only if $|S_{\ell^+} \cap R| - |S_{\ell^+} \cap B| = d_2(S)$ is maximum. □

The next result follows:

Theorem 3.9. *The WS-problem is 3SUM-hard.*

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