Some Open Problems

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1 Introduction

In this paper we present a collection of problems which have defied solution for some time. We hope that this paper will stimulate renewed interest in these problems, leading to solutions to at least some of them.

1.1 Points and Circles

In 1985, in a joint paper with V. Neumann-Lara, the following result was proved: Any set P_n of n points on the plane contains two points $p,\ q \in P_n$ such that any circle containing them contains at least $\frac{n-2}{60}$ elements of P_n. This result was first improved to $\frac{n}{30}$ [5], then to $\lfloor \frac{n}{27} \rfloor + 2$ [10], to $\lceil \frac{5(n-3)}{84} \rceil$ [11], and to approximately $\frac{n}{4.7}$ [9]. Our first conjecture presented here is:

Conjecture 1 *Any set P_n of n points on the plane contains two elements such that any circle containing them contains at least $\frac{n}{4} \pm c$ elements of P_n.*

An example exists with $4n$ points, due to Hayward, Rappaport and Wenger [10], such that for any pair of points of P_n there is a circle containing them that contains at most $n - 1$ elements of P_n, see Figure 1.

This problem has been studied for point sets in convex position (that is when the point set is the set of vertices of a convex polygon), for point sets in higher dimensions, and for families whose elements are not points, but convex sets [1, 4, 5]. For point sets in convex position, the problem was settled by Hayward, Rappaport and Wenger [10], who proved that the tight bound for this case is $\lceil \frac{n}{3} \rceil + 1$.

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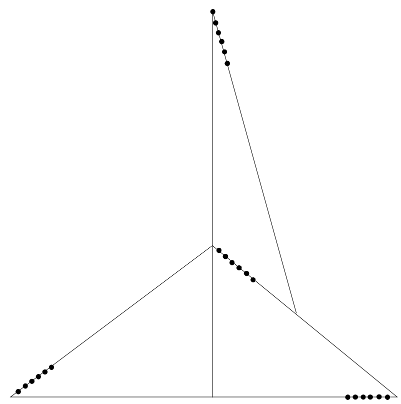

Figure 1: A point set P_{4n} with $4n$ points such that for any two points p, q of P_{4n} there is a circle containing them that contains at most $n-1$ additional points of P_{4n}.

1.2 Convex partitionings of the convex hull of a point set

The following problem arose some years ago during a series of meetings in Madrid with M. Abellanas, G. Hernandez, P. Ramos and the author. We were studying problems on quadrilaterizations of point sets. Given a set of points P_n in general position, a collection $\mathcal{F}= \{Q_1, \ldots, Q_m\}$ of convex polygons with disjoint interiors is callead a convex decomposition of the convex hull $Conv(P_n)$ of P_n, a convex decomposition of P_n for short if:

1. The union of the elements of $\mathcal{F}$ is $Conv(P_n)$
2. No element of $\mathcal{F}$ contains an element of P_n in its interior.

If all the elements of $\mathcal{F}$ are quadrilaterals (resp. triangles), $\mathcal{F}$ is called a convex quadrilaterization (triangulation) of P_n. It is well known that not all point sets admit a convex quadrilaterization (even if they contain the right number of points in their convex hull and their interior). It is easy to see that if a point set with n points, k on the boundary of its convex hull, has a convex quadrilaterization, it contains exactly $\frac{(n+k-2)}{2}$elements. We observed that although convex quadrilaterizations of point sets do not always exist, we were always able to obtain convex partitionings of all point sets with at most $n+1$ elements that we tried. Thus we conjectured:

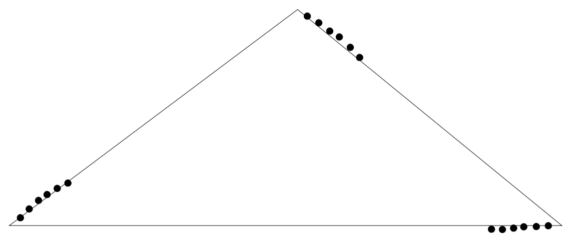

Figure 2: A point set P_{3n} of $3n$ points in convex position such that for any two points in P_{3n} there is a circle containing them which contains at most $n-1$ extra elements of P_{3n}.

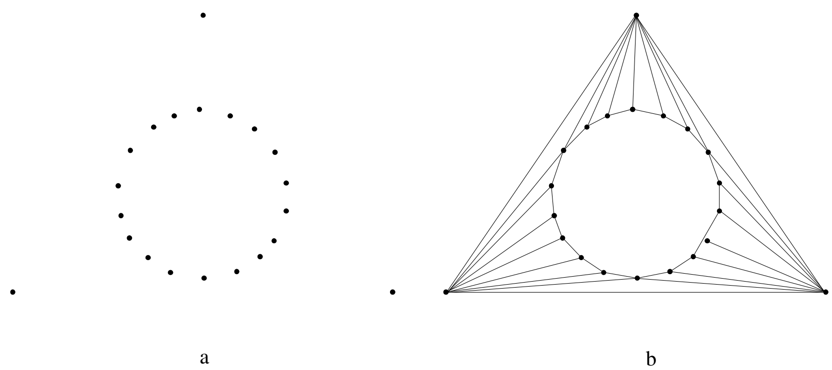

Figure 3: A point set with $3n$ elements such that any convex partitioning of it has at least $3n+1$ polygons.

Conjecture 2 *Any set P_n of n in general position has a convex decomposition with at most $n+1$ elements.*

A set of points achieving this bound is shown in Figure 4. Our conjecture was proved false in 2001 by O.Aichholzer and H.Krasser [2]. They were able to construct a point set P_n such that any convex partitioning of it contains at least $n+2$ elements.

It is known [16] that any point set P_n has a convex partitioning with at most $\lceil \frac{3n-2k}{2} \rceil$ elements, where k is, as before, the number of elements of P_n on the boundary of its convex hull. A convex partition with at most that number of elements can be obtained as follows. First calculate a triangulation of P_n.

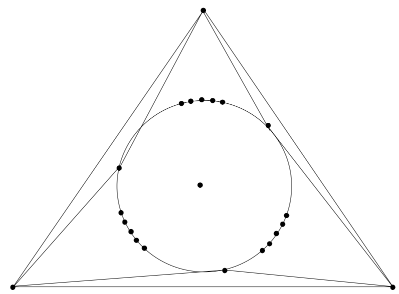

Figure 4: Aichholzer and Krasser's point set. All the inner points are cocircular, except for the center of the circle on which the points lie, which is also in the point set.

Then in a greedy way, delete as many edges as possible from our triangulation, making sure that the remaining edges induce a convex partitioning of P_n. It is proved in [16] that the remaining edges induce a convex partitioning of P_n with at most $\lceil \frac{3n-2k}{2} \rceil$ elements; see Figure 5.

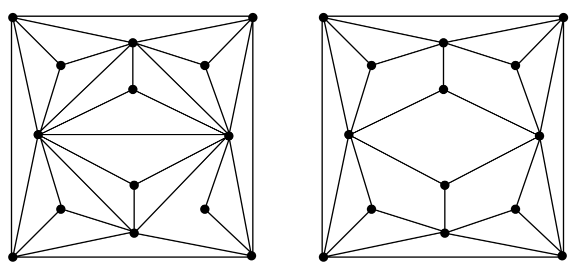

Figure 5: Consider the triangulation on the left of the point set P_n with fourteen points. Then we can remove at most five edges of its edges to obtain a convex partitioning of P_n. The removal of exactly five edges produces a convex partitioning of this point set with $\lceil \frac{3(14)-2*4}{2} \rceil = 14$ elements.

1.3 Problems on line segments

The next problem is at least ten years old.

Let $\mathcal{F}= \{l_1, \ldots, l_n\}$ be a family of disjoint closed segments. A simple alternating path of $\mathcal{F}$ is a non intersecting polygonal chain with $2k + 1$ line segments l_i, $i = 1, \ldots,\ 2k+1$ such that for all i, $k = 0, \ldots, k$, l_{2i+1} belongs to $\mathcal{F}$. Segments l_{2i} are not allowed to intersect other elements of $\mathcal{F}$, $i = 0, \ldots, k$.

Conjecture 3 *Any set $\mathcal{F}= \{l_1, \ldots, l_n\}$ of n disjoint line segments has a simple alternating path with at least $O(\ln n)$ elements.*

In Figure 6 we show a family of $2^n - 1$ segments such that any alternating path has $O(n)$ elements. The family consists of $2^n - 1$ segments with endpoints on a circle such that l_{2i} is visible from $\{l_j : j = i,\ 2i+1,\ 4i,\ 4i+1\}$, $i = 1, \ldots,\ 2^{n-1} - 1$. Moreover l_{2i+1} is visible only from $\{l_j : j = i,\ 2i+2,\ 4i+2,\ 4i+3\}$, $i = 0, \ldots,\ 2^{n-1} - 1$.

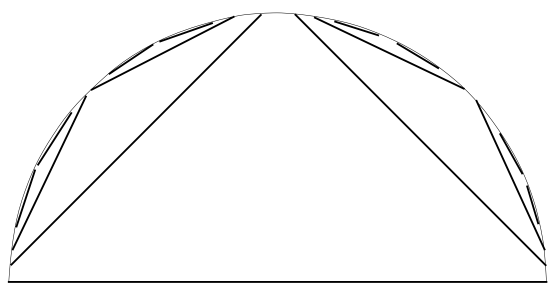

Figure 6: Any alternating path of this set of line segments has $O(\ln n)$ elements.

The following is a related problem. Given a a family $\mathcal{F}$ of n disjoint closed line segments, find a subset of it that admits a simple alternating path. That is, in our previous problem, remove the restriction that the segments l_{2i} are not allowed to intersect other elements of $\mathcal{F}$, $i = 0, \ldots,\ k$. In this version of the problem, it is not difficult to prove that any family with n line segments contains a subset with at least $n^{\frac{1}{5}}$ elements that admits a simple alternating path. This can be proved using techniques similar to those used in [14]. Observe that if $\mathcal{F}$ has $n^{\frac{1}{5}}$ elements with disjoint projections on the x-axis, this subset admits a simple alternating path, and we are done. If this is not the case, then there is a vertical line $\mathcal{L}$ that intersects at least $k \geq n^{\frac{4}{5}}$ segments of $\mathcal{F}$. Let $\mathcal{S}= \{m_1, \ldots, m_k\}$ be the sequence containing the slopes of the line segments in $\mathcal{F}$ intersected by $\mathcal{L}$, according to the order in which they are intersected by $\mathcal{L}$. Then by a well known result of Erdös-Szekeres, $\mathcal{S}$ contains an increasing or decreasing subsequence $\mathcal{S}'$ with at least $n^{\frac{2}{5}}$ elements. Suppose w.l.o.g. that the elements of $\mathcal{S}'$ are in increasing order. Let $\mathcal{F}'$ be the subset of $\mathcal{F}$ corresponding to the elements of $\mathcal{S}^{'}$. Let $\mathcal{Y}$ be the sequence defined by the second coordinates of the left endpoints of the elements in $\mathcal{F}^{'}$. Once more, there is an increasing or decreasing subsequence of $\mathcal{Y}$ with $n^{\frac{1}{5}}$ elements. It is easy to see now that the line segments of $\mathcal{F}$ corresponding to these elements admit a simple alternating path. This bound, however, seems to be far from optimal. We believe that the correct value is $O(n^{\frac{1}{2}})$. There are collections of n^2 line segments such that any subset of them that admits a simple alternating path contains at most $2n$ segments. In Figure 7 we illustrate how to obtain such family for $n = 5$. This construction is easily generalizable for any $n \geq 3$.

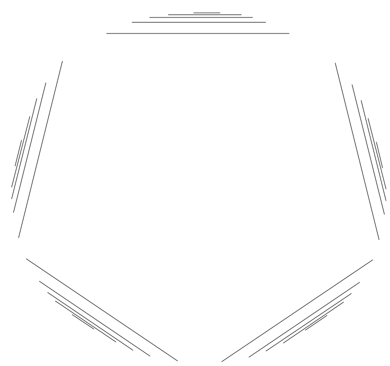

Figure 7: A family with $5^2 = 25$ segments such that any subset of it admits a simple alternating path with at most $2 \times 5 = 10$ elements.

1.4 Separability

Given two disjoint closed convex sets, we say that a line l separates them if the convex sets are such that one is contained in each of the open half-planes defined by l. H. Tverberg [20] studied the following problem. Let $K_d(r,s) = k$ be the smallest integer k such that given n disjoint convex sets $C_1, \cdots, C_k$, there exists a closed half-plane containing at least r convex sets, and its complement contains s of them. Tverberg proved that $K_2(r,1)$ always exists. Examples found by K. Villanger show that $K_2(2,2)$ does not exist. Villanger's example consists of an arbitrarily large number of non-collinear line segments such that the convex hull of any pair of them contains the point $(1, 1)$.

It is known that $K_2(r, 1) \leq 12(r-1)$; see [12, 7]. There are families of line segments $\mathcal{F}$ with $3m$ elements such that no element of $\mathcal{F}$ can be separated from more than $m+1$ elements [7, 8].

Conjecture 4 *Any family $\mathcal{F}$ of n disjoint closed convex sets has an element that can be separated with a single line from at least $\lfloor \frac{n}{3} \rfloor \pm c$ elements of $\mathcal{F}$.*

It is known [3] that for any family $\mathcal{C}$ of congruent disks with $O(m^2 \ln m)$ elements, there always exists a direction α such that any line with direction α intersects at most m elements of $\mathcal{C}$. It then follows that there is a line that leaves at least $\frac{(m^2 \ln m)-m}{2}$ elements of $\mathcal{C}$ in each of the semiplanes which it defines. For families of n circles, not necessarily of the same size, it is known [6] that there is a line that separates a circle from at least $\frac{n-c}{2}$ other circles.

1.5 Illumination

One of my favourite areas is that of illumination. Here I will mention some open problems related to this area of research. A more extensive list of open problems and results in this area can be found in [21].

An α-floodlight is a light source that illuminates within an angular region of size α. For example, a $\frac{\pi}{3}$-floodlight illuminates an angular wedge of the plane with angular width $\frac{\pi}{3}$. The source of illumination is located at the apex of the angular region. Given a simple polygon, a floodlight is called a *vertex floodlight* if its source is located at a vertex of the polygon. The following old conjecture of mine was believed to be true up to December, 2001, when I found a counterexample:

Conjecture 5 *Any simple polygon with n vertices can be illuminated with $\lceil \frac{3n}{5} \rceil - 1$ vertex π-floodlights. We do not allow more than one floodlight on any vertex of the polygon.*

A family of polygons that requires $\lceil \frac{3n}{5} \rceil - 1$ π-vertex floodlights was obtained by F. Santos; see Figure 8.

A family of polygons with $9 + 8k$ vertices which require $5(k + 1)$ vertex π-floodlights to illuminate them can be constructed by using the star shown

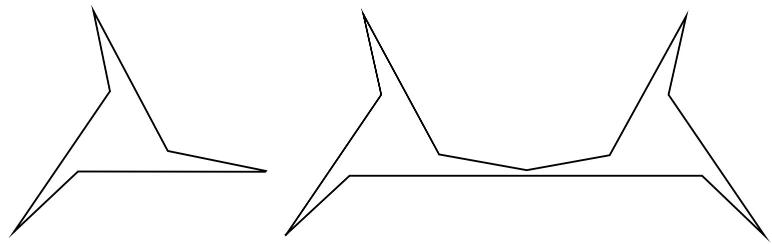

Figure 8: A polygon with $5n + 1$ vertices which requires $3n$ vertex π-floodlights can be obtained by pasting n copies of the star polygon on the left.

in Figure 9. It is tempting to conjecture that the correct bound for the previous conjecture is $\frac{5(n-1)}{8} \pm c$. Recently Speckman and Töth proved that any polygon with n vertices, k of which are convex, can be illuminated with $\lfloor \frac{2n-k}{3} \rfloor$ vertex π-floodlights. We close this section with two long standing conjectures on illumination, the first one due to T. Shermer:

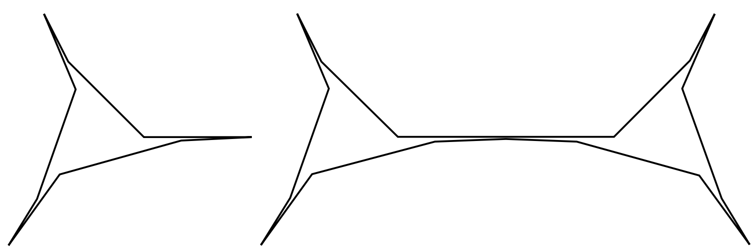

Figure 9: Constructing a family of polygons with $8n+1$ vertices that require $5n$ vertex π-floodlights to illuminate them.

Conjecture 6 *Any polygon with n vertices and h holes can be illuminated with $\lfloor \frac{n+h}{3} \rfloor$ vertex guards.*

In this case, the guards can illuminate all around themselves. The second conjecture is due to G. Toussaint, and was first stated in 1981:

Conjecture 7 *There is an n_0 such that any polygon $\mathcal{P}$ with $n \geq n_0$ vertices can be illuminated with $\lfloor \frac{n}{4} \rfloor$ edge guards. That is, any polygon $\mathcal{P}$ with $n \geq n_0$ vertices has a subset of $\lfloor \frac{n}{4} \rfloor$ edges such that any point in $\mathcal{P}$ is visible from one of these edges.*

Two surveys in this area [18, 21] and a book by O'Rourke [17] contain most of the information concerning illumination and these problems up to 2000.

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