A note on minimally 3-connected graphs*

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Abstract

If G is a minimally 3-connected graph and C is a double cover of the set of edges of G by irreducible walks, then $|E(G)| \geq 2|C| - 2$.

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1 Introduction

A *walk* α in a simple graph G is a sequence $w_0, w_1, \ldots, w_s$ of vertices of G, not necessarily different, such that $w_{i-1}w_i$ is an edge of G for $i = 1, 2, \ldots, s$. An edge e of G is said to be *traversed* in a walk α if its vertices are consecutive in α; an edge may be traversed more than once in a given walk.

A walk α in a graph G is *irreducible* if $a \neq b$ for every pair a, b of edges which are traversed consecutively in α. A set C of irreducible closed walks in a graph G is a *walk double cover* of G if each edge of G is traversed exactly two times, either once in two different walks in C or twice in the same walk in C.

For any simple graph G and any edge $e = uv$ of G we denote by $G - e$ the graph obtained from G by deleting the edge e, and by $G \cdot e$ the simple graph obtained from G by identifying the vertices u and v and deleting loops and multiple edges. A *minimally* 3-*connected* graph is a 3-connected graph G such that, for every edge e of G, the graph $G - e$ is no longer 3-connected.

Whenever possible we follow the terms and notation given in [1]. A *wheel* W_t is a graph with $t + 1$ vertices, obtained from a cycle C_t with t vertices

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by adding a new vertex w adjacent to each vertex in C_t. The cycle C_t and the vertex w are called the rim and the hub of W_t, respectively. In this note we prove the following result.

Theorem 1.1. *Let G be a minimally 3-connected graph with m edges. If C is a walk double cover of G with k walks, then $m \geq 2k-2$. Moreover if $m \leq 2k-1$, then G is a planar graph and C is the set of planar faces of G; in particular if $m = 2k-2$, then G is a wheel.*

2 Proof of Theorem 1

The following result due to R. Halin [2] will be used in the proof of Theorem 1.

Theorem 2.1. *If $e = uv$ is an edge of a minimally 3-connected graph G with $\min\{d(u), d(v)\} \geq 4$, then e lies in no cycle of G of length 3 and $G \cdot e$ is also minimally 3-connected.*

For any graph G and any walk double cover C of G, we denote by $m(G)$ and by $k(C)$ the number of edges of G and the number of walks in C, respectively.

Remark 1. *Let G be a 3-connected graph and C be a walk double cover of G. If two edges uw and wv are consecutive edges in two walks in C, then the degree of w is at least 4.*

Proof of Theorem 1. The smallest 3-connected graph is the wheel W_3 which is planar and has 6 edges. Since each irreducible walk has at least 3 edges, no walk double cover of W_3 has more than 4 walks. Moreover, the only walk double cover of W_3 with 4 walks consists of the planar faces of W_3.

We proceed by induction assuming $m \geq 7$ and that the result holds for every minimally 3-connected graph with less than m edges.

If G has an edge $e = uv$ with $\min\{d(u), d(v)\} \geq 4$, then by Halin's theorem, $G \cdot e$ is also minimally 3-connected. Let $C \cdot e$ denote the set of k walks of $G \cdot e$ obtained from the walks in C by contracting the edge e.

Also by Halin's theorem, the edge e lies in no cycle of G of length 3; this implies that all walks in $C \cdot e$ are irreducible. Because C is a walk double cover of G and e is not an edge of $G \cdot e$, $C \cdot e$ is a walk double cover of $G \cdot e$. By induction, $m(G \cdot e) \geq 2k(C \cdot e) - 2$; therefore $m \geq 2k-1$, since $m(G \cdot e) = m-1$ and $k(C \cdot e) = k$.

If $m = 2k - 1$, then $m(G \cdot e) = 2k(C \cdot e) - 2$; by induction $G \cdot e$ is a wheel W_t and $C \cdot e$ is the set of planar faces of W_t. Let x be the vertex of W_t obtained by identifying u and v. Since u and v have degree at least 4 in G, the vertex x must be the hub of W_t; let $w_0, w_1, \ldots, w_{t-1}$ be the rim of W_t.

Since e is in no cycle of G of length 3, G is a graph consisting of the cycle $w_0, w_1, \ldots, w_{t-1}$, the two adjacent vertices u and v, and one edge joining each vertex w_i to either u or v.

Suppose there are distinct integers a, b and c such that w_a, w_{b+1} and w_c are adjacent to u in G and w_{a+1}, w_b and w_{c+1} are adjacent to v in G. The walks w_a, x, w_{a+1}, w_b, x, w_{b+1} and w_c, x, w_{c+1} lie in C, since they are faces of $G \cdot e$. This implies that w_a, u, v, w_{a+1}, w_b, v, u, w_{b+1} and w_c, u, v, w_{c+1} are walks in C which is not possible, since the edge $e = uv$ cannot lie in three walks in C.

Therefore there are integers i and j such that $w_i, w_{i+1}, \ldots, w_{j-1}$ are adjacent to u in G and $w_j, w_{j+1}, \ldots, w_{i-1}$ are adjacent to v in G. This shows that G is a planar graph.

Since $C \cdot e$ is the set of faces of $G \cdot e = W_t$ and each walk in $C \cdot e$ is either a walk in C or is obtained from a walk in C by contracting the edge e, the set C must be the set of faces of G .

We can now assume that each edge of G has at least one end with degree 3. If C contains no cycle of length 3, then $2m \geq 4k$ and $m \geq 2k$. Therefore we can also assume that C contains at least one cycle of length 3. Let C_3 be the set of cycles in C of length 3; two cases are considered.

Case 1.- *There is a cycle α in C_3 such that no pair of edges of α are traversed consecutively in any other walk in C.*

Let u, v and w be the vertices of α. Since each edge of G has an end with degree 3, without loss of generality, we can assume $d_G(u) = d_G(v) = 3$. Let u_1 and v_1 denote the third vertex of G adjacent to u and the third vertex of G adjacent to v, respectively; notice that $u_1 \neq v_1$, since G is 3-connected and has at least 5 vertices.

Subcase 1.1.- If $d_G(w) = 3$, let w_1 denote the third vertex of G adjacent to w; as above $u_1 \neq w_1 \neq v_1$. Let G' be the graph obtained from G by contracting the cycle α to a single point x. We claim that G' can also be obtained from G by a *delta* to *wye* transformation (see Figure 1), and therefore it is also a 3-connected graph.

Since $d_{G'}(x) = 3$ and $d_{G'}(z) = d_G(z)$ for each vertex $z \neq x$ of G', every edge of G' has an end with degree 3; therefore G' is minimally 3-connected.

Let C' be the set of $k-1$ walks of G' obtained from the walks in $C \setminus \{\alpha\}$

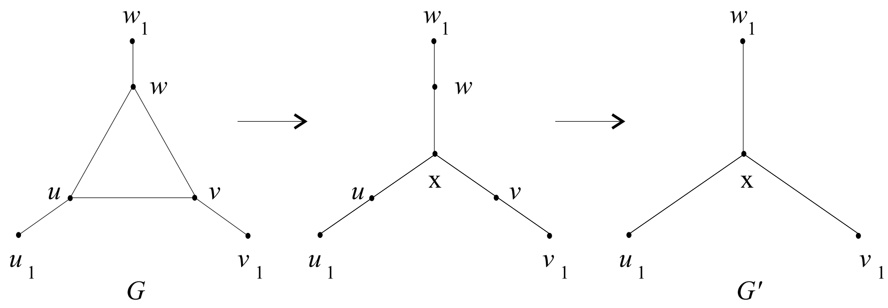

Figure 1:

by contracting the edges uv, vw and wu. Since no pair of edges of α are consecutive edges in any walk in $C\backslash\{\alpha\}$, all walks in C' are irreducible. Moreover, C' is a walk double cover of G', since C is a walk double cover of G and uv, vw and wu are not edges of G'.

By induction $m(G') \geq 2k(C')-2$; hence $m \geq 2k-1$, since $m(G') = m-3$ and $k(C') = k-1$. If $m = 2k-1$, then $m(G') = 2k(C')-2$. Again by induction $G \cdot e$ is a wheel W_t and C' is the set of planar faces of W_t. Since x has degree 3 in G', we can assume without loss of generality that x lies in the rim of $G' = W_t$ and that w_1 is the hub; this implies that G is a graph as in Figure 2 and therefore it is a planar graph in which α is a face.

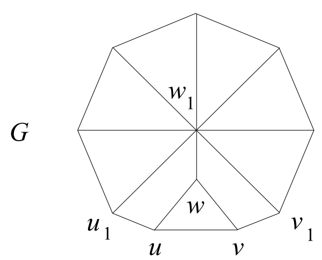

Figure 2:

Since C' is the set of faces of G' and every walk in C' is either a walk in $C\backslash\{\alpha\}$ or is obtained from a walk in $C\backslash\{\alpha\}$ by contracting some of the edges uv, vw and wu, the set C must be the set of planar faces of G.

Subcase 1.2.- If $d_G(w) \geq 4$, we consider the graph $G \cdot uv$. We claim that u and v cannot be contained in a 3-vertex cut of G and, therefore, $G \cdot uv$ is 3 connected.

Since $d_{G \cdot uv}(x) = 3$ and $d_{G \cdot uv}(z) \leq d_G(z)$ for each vertex $z \neq x$ of $G \cdot uv$, every edge of $G \cdot uv$ has an end with degree 3; therefore $G \cdot uv$ is minimally

3-connected.

Let $C \cdot uv$ be the set of $k-1$ walks of $G \cdot uv$ obtained from the walks in $C \setminus \{\alpha\}$ by contracting the edge uv to a vertex x and substituting each of the edges uw and vw by the edge xw. Each walk in $C \cdot uv$ is irreducible, because no pair of edges of α are traversed consecutively in any other walk in C. Since C is a walk double cover of G and uv is not an edge of $G \cdot uv$, the set $C \cdot uv$ is a walk double cover of $G \cdot uv$.

By induction $m(G \cdot uv) \geq 2k(C \cdot uv) - 2$; hence $m \geq 2k-2$, since $m(G \cdot uv) = m - 2$ and $k(C \cdot uv) = k - 1$. If $m \leq 2k-1$, then $m(G \cdot uv) \leq 2k(C \cdot uv) - 1$; again by induction, $G \cdot uv$ is a planar graph and $C \cdot uv$ is the set of planar faces of $G \cdot uv$.

Since $G \cdot uv$ is 3-connected, there is a planar drawing $\overline{G \cdot uv}$ of $G \cdot uv$ in which x is an interior vertex. Let R be the region formed by the three faces of $\overline{G \cdot uv}$ in which x is a vertex. Since w, u_1 and v_1 lie in the boundary of R and x is in the interior of R, a planar drawing $\overline{G}$ of G can be obtained from $\overline{G \cdot uv}$ by replacing (within the interior of R) the vertex x with two adjacent vertices u and v, and the edges wx, u_1x and v_1x with the edges wu, wv, u_1u and v_1v as in Figure 3.

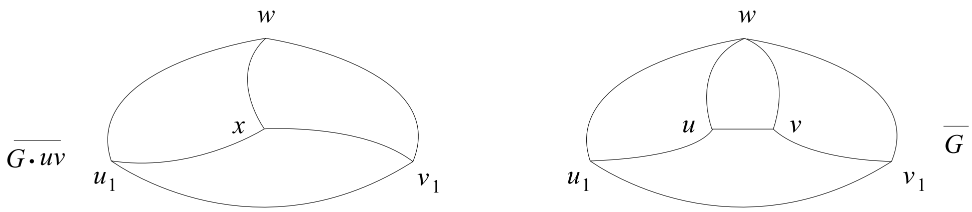

Figure 3:

Therefore G is a planar graph and α is a face of G. Furthermore, C is the set of faces of G, since $C \cdot uv$ is the set of planar faces of $G \cdot uv$ and each walk in $C \cdot uv$ is either a walk in $C \setminus \{\alpha\}$ or is obtained from a walk in $C \setminus \{\alpha\}$ by contracting the edge uv to the vertex x and substituting each of the edges uw and vw by the edge xw.

If $m = 2k-2$, then $m(G \cdot uv) = 2k(C \cdot uv) - 2$; again by induction, $G \cdot uv$ is a wheel W_t. Since $d_{G \cdot uv}(x) = 3$, we can assume that x lies in the rim of $G \cdot uv$.

If w is the hub of $G \cdot uv$, then G is the wheel W_{t+1}, also with hub w. If u_1 is the hub of $G \cdot uv$, then G is a graph as in Figure 4. Notice that if $t > 3$, then $G - u_1w$ is 3-connected which is not possible since G is minimally 3-

connected. Therefore $t = 3$ and G is the wheel W_4 with hub w. Analogously, if v_1 is the hub of $G \cdot uv$, then G is the wheel W_4.

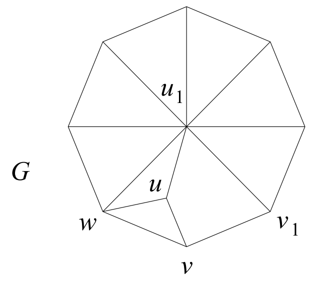

Figure 4:

Case 2.- *For every cycle $\alpha \in C_3$ there is walk $\sigma_\alpha \neq \alpha$ in C such that two edges of α are traversed consecutively in σ_α.*

For this case, we shall prove that the average length of the walks in C is at least 4 and therefore $2m \geq 4k$ and $m \geq 2k$.

For each $\alpha \in C_3$ let u_α, w_α and v_α denote the vertices of α. Without loss of generality we assume that $u_\alpha w_\alpha$ and $w_\alpha v_\alpha$ are traversed consecutively in σ_α. Notice that the walk σ_α is uniquely determined since C is a walk double cover of G.

By Remark 1, $d_G(w_\alpha) \geq 4$; therefore $d_G(u_\alpha) = d_G(v_\alpha) = 3$, since every edge of G has an end with degree 3. Let u'_α and v'_α denote the third vertex of G adjacent to u_α and the third vertex of G adjacent to v_α, respectively.

Again by Remark 1, the edges $w_\alpha u_\alpha$ and $u_\alpha v_\alpha$ are not traversed consecutively in σ_α; therefore σ_α must traverse the edge $u_\alpha u'_\alpha$; analogously σ_α traverses the edge $v_\alpha v'_\alpha$. If $u'_\alpha = v'_\alpha$, then u_α and v_α are adjacent only to $u'_\alpha = v'_\alpha$, to w_α and to each other which is not possible since G is a 3-connected graph with at least 5 vertices; therefore σ_α has length at least 5 for each $\alpha \in C_3$. For each $\tau \in C$ let $l(\tau)$ denote the length of τ.

Consider the equivalence relation in C_3 given by $\beta \sim \gamma$ if and only if $\sigma_\beta = \sigma_\gamma$. For $\alpha \in C_3$ let $[\alpha]$ denote the equivalence class of α.

Let β and γ be two distinct cycles in $[\alpha]$ and assume, without loss of generality, that the edges $u_\beta w_\beta$, $w_\beta v_\beta$, $u_\gamma w_\gamma$ and $w_\gamma v_\gamma$ are traversed in $\sigma_\alpha = \sigma_\beta = \sigma_\gamma$ in that relative order. The edges $u_\beta w_\beta$ and $w_\beta v_\beta$ are not edges of γ since they are traversed in β and by $\sigma_\beta \neq \beta$; analogously $u_\gamma w_\gamma$ and $w_\gamma v_\gamma$ are not edges of β.

Suppose that $w_\beta v_\beta$ and $u_\gamma w_\gamma$ are traversed consecutively in σ_α. Then $v_\beta = u_\gamma$ and $w_\beta \neq w_\gamma$, since σ_α is an irreducible walk. Moreover, $u_\beta = v_\gamma$

since $d_G(v_\beta = u_\gamma) = 3$ and w_β, w_γ, u_β and v_γ are all adjacent to $v_\beta = u_\gamma$. This implies that the vertices $v_\beta = u_\gamma$ and $u_\beta = v_\gamma$ are adjacent in G only to w_β, to w_γ and to each other which is not possible since G is 3-connected and has at least 5 vertices.

Therefore, no edges of two distinct cycles in $[\alpha]$ are traversed consecutively in σ_α. This implies that σ_α has at least $3\,|[\alpha]|$ edges.

By the above arguments

$$\tfrac{l(\sigma_\alpha)+l(\alpha)}{2} \geq \tfrac{5+3}{2} = 4$$

for each $\alpha \in C_3$ with $|[\alpha]| = 1$, and

$$\tfrac{l(\sigma_\alpha)+\sum_{\beta\in[\alpha]} l(\beta)}{|[\alpha]|+1} \geq \tfrac{3|[\alpha]|+3|[\alpha]|}{|[\alpha]|+1} = \tfrac{6|[\alpha]|}{|[\alpha]|+1} \geq 4$$

for each $\alpha \in C_3$ with $|[\alpha]| \geq 2$.

Since all walks in C which are not in C_3 have length at least 4, the average length in C must also be at least 4.

Corollary 2.2. *Let G be a minimally 3-connected graph with n vertices. If C is a walk double cover of G with k walks, then $k \leq \frac{3n-4}{2}$.*

Proof. Let m denote the number of edges in G. W. Mader proved in [3] that $m \leq 3n - 6$; by Theorem 1, $k \leq \frac{m+2}{2} \leq \frac{(3n-6)+2}{2} = \frac{3n-4}{2}$. □

Corollary 2.3. *If G is a minimally 3-connected planar graph with n vertices, then G has at most n faces. Moreover if G has exactly n faces, then G is a wheel.*

Proof. Since G is 3-connected, its set of faces is a walk double cover. By Theorem 1, $m \geq 2r - 2$, where m and r are the number of edges and faces of G, respectively. Since $n - m + r = 2$, it follows $r \leq n$.

Also by Theorem 1, if G is not a wheel, then $m \geq 2r - 1$, in which case $r \leq n - 1$. □

Corollary 2.4. *If G is a minimally 3-connected graph with n vertices embedded in a closed surface S with Euler characteristic $\chi \neq 2$, then G has at most $n - \chi$ faces.*

Proof. As in Corollary 4, the set of faces of G is a walk double cover of G. Since S is not the sphere, C is not the set of planar faces of G. By Theorem 1, $m \geq 2r$, where m and r are the number of edges and faces of G, respectively. Since $\chi = n - m + r$, it follows $r \leq n - \chi$. □

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