Singularities in dynamics
A catastrophic viewpoint*

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to Vladimir Igorevich Arnol'd
in memory of René Thom

An unworthy disciple of René Thom, I feel that his vision of mathematics is being partly forgotten[1] (not by V.I. Arnol'd and his school, though).

This is particularly obvious in dynamics, where the rather superficial belief that "the structural stability program has failed" certainly prevented the full development of singularity theory. Of course, the latter is by no means as beautiful and general a theory as for smooth maps: almost from the beginning, one has to crawl from hole to hole, each crowded with a fauna rich enough to deserve a few years of study[2]. To convey the idea of it, I shall develop the beginning of "catastrophe theory" for dynamical systems (including quite recent results. . .) and compare the statements to those available for smooth maps[3].

Of course, most of what follows—except my mistakes—is well-known to V.I. Arnol'd and his school, but the story is told a little differently.

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[1]Thom was perhaps alluding to this phenomenon thirty years ago when he quoted Max Planck (?): "Science progresses by the death of the old".

[2]Entomologists are not all interested in butterflies. Thom had warned me to beware of aesthetics.

[3]The idea of structural stability is essential here. Intuitively, a situation is structurally stable when all nearby situations look like it. This notion is closely related to the implicit function theorem and its intrinsic avatar: transversality. Even though this is as essential a tool in dynamics as in the theory of smooth maps, we shall see that the situation is not as nice, at least if one believes that "looking alike" should be a global equivalence relation: in order not to *a priori* exclude structural stability, one must use an equivalence relation both too coarse and too fine to be of much practical interest—anyway, not coarse enough for structural stability to be generic, except in very low dimension (the "failure" mentioned before. Since no cure has been found so far, this text may well be criticized for its "furious lack of concepts", as Thom used to put it).

1 Generic stationary points.

Given a separable n–dimensional manifold M, we consider a system modelled by[4] a "potential" $V : M \to \mathbf{R}$, a vector field X on M or a map $g : M \to M$. These three objects bear the common name f in the sequel. A *stationary point* or *rest point* of f is a point at which the system is as little dynamical as possible, i.e. a critical point of V, a zero of X or a fixed point of g.

1.1 What Thom's transversality lemma tells us

Setting $k = 1$ in the case of potentials and $k = 0$ otherwise, a is a stationary point of f if and only if the jet[5]

$$j^k f(a) := \begin{cases} j^1 V(a) := (a, V(a), d_a V) & \text{in the case of potentials,} \\ j^0 X(a) := X(a) & \text{in the case of vector fields} \\ j^0 g(a) := \big(a, g(a)\big) & \text{in the case of maps.} \end{cases}$$

belongs to the submanifold Σ of

$$J^k := \begin{cases} J^1(M, \mathbf{R}) & \text{in the case of potentials,} \\ J^0(TM) = TM & \text{in the case of vector fields.} \\ J^0(M, M) = M \times M & \text{in the case of maps} \end{cases}$$

consisting of those $j^k\varphi(a)$ such that a is a stationary point of φ, namely
− for potentials, the zero section of $J^1(M, \mathbf{R})$ as a vector bundle over $J^0(M, \mathbf{R}) = M \times \mathbf{R}$ with projection $j^1\varphi(x) \mapsto j^0\varphi(x)$
− for vector fields, the zero section of TM
− for maps, the diagonal of $M \times M$.

By Thom's transversality lemma, the map $j^k f$ is almost surely ("generically") transversal to Σ. In other words:

Proposition 1.1.1 *(i) Almost every potential V on M is a* Morse function, *meaning that all its critical points are* nondegenerate*: for each critical point a of V, the Hessian $D_a^2 V$ of V at a is a nondegenerate quadratic form on $T_a M$.*

(ii) For almost every vector field X on M, all the zeros of X are nondegenerate*: for every zero a of X, the differential $d_a X$ of X at a is an automorphism of $T_a M$.*

(iii) For almost every map g of M into itself, all the fixed points of g are nondegenerate*: for every fixed point a of g, the differential $d_a g$ of g at a does not have 1 as an eigenvalue.*

1.2 Geometric interpretation in function space

Denote by $\mathcal{F}$ the space $C^\infty(M, \mathbf{R})$ of all smooth potentials on M, the space $C^\infty(TM)$ of all smooth vector fields on M or the space $C^\infty(M, M)$ of all smooth maps of M into itself.

Forgetting about technical details, the reason for Thom's transversality lemma is that, for all ℓ, the map $j^\ell : (f, x) \mapsto j^\ell f(x)$ of $\mathcal{F} \times M$ into J^ℓ is a submersion. Therefore, for every submanifold W of J^ℓ, the inverse image $(j^\ell)^{-1}(W)$ is a submanifold of $\mathcal{F}$ with the same codimension as W.

[4] Unless otherwise specifed, all maps are C^∞.
[5] The systematic use of jet spaces is one of Thom's fundamental ideas in singularity theory.

The fundamental observation[6] is that the set of those (f,x)'s such that $j^\ell f$ is not transversal to W at x is the critical set of the restricted projection $(j^\ell)^{-1}(W) \ni (f,x) \overset{\pi}{\longmapsto} f$. Therefore, by Sard's theorem, the image of this critical set by π (i.e. the set of those f's such that $j^\ell f$ is not transversal to W) is neglectible.

Notes Of course, *there are* technical details[7]. When M is compact, the above argument can be applied *stricto sensu* [2] provided C^∞ is replaced by C^r with $r \in \mathbf{N}$ large enough, more precisely $r - \ell > \max\{\dim M - \operatorname{codim} W, 0\}$). Indeed, the function space $C^r(M,\mathbf{R})$, $C^r(TM)$ or $C^r(M,M)$ is a separable Banach manifold, the map j^ℓ is $C^{r-\ell}$ and Sard's theorem [24, 2] has been extended by Smale [25, 2] to Fredholm maps like π. At the limit when r goes to infinity, we get Thom's theorem. Moreover, still assuming M compact, it is true that $(j^\ell)^{-1}(W)$ is a Fréchet submanifold of the Fréchet manifold $\mathcal{F} \times M$ with the same codimension as W.

The word "neglectible" also requires an explanation: in Thom's original paper [27] as well as in [25, 2], it meant "meager", i.e. contained in the intersection of countably many closed subsets with empty interior—the "fat" complementary subset of a meager subset is dense, as our function space is Baire. However, even though the definition of a measure on this infinite dimensional Baire space is problematic, the definition of a neglectible subset is not[8], and it can be proven [17] that the set of critical values of π is neglectible in that sense too.

For noncompact M, the geometrically significant topology is Whitney's fine C^∞ topology, which has the Baire property but is certainly not a Fréchet topology, as it does not even make $C^\infty(M,\mathbf{R})$ and $C^\infty(TM)$ into topological vector spaces[9]. It is often fruitful to avoid the problem by considering that a space "is a place where one can wander"[10]: in $\mathcal{F}$, a *smooth family with p parameters* is a map $u \mapsto f_u$ of some open subset U of $\mathbf{R}^p$ into $\mathcal{F}$ such that $(u,x) \mapsto f_u(x)$ is smooth. Similarly, a map j of $\mathcal{F}$ into another space of smooth functions $\mathcal{G}$ is smooth when, for every smooth family $u \mapsto f_u$ in $\mathcal{F}$ with p parameters, the family $u \mapsto j(f_u)$ is smooth. For example, j^ℓ is smooth.

Proposition 1.2.1 *With the notation of paragraph 1.1, we have the following:*

(i) The closed subset $\widetilde{\Sigma} := (j^k)^{-1}(\Sigma)$ of $\mathcal{F} \times M$ consisting of those (f,a) such that a is a rest point of f is a smooth submanifold modelled on $\mathcal{F}$, whose codimension is the dimension n of M.

(ii) The subset $\widetilde{\Sigma}_0$ of $\widetilde{\Sigma}$ consisting of those (f,a) such that a is a nondegenerate *rest point of f is open and dense in $\widetilde{\Sigma}$. Its complement $\widetilde{\Sigma} \setminus \widetilde{\Sigma}_0$ is the union of finitely many mutually disjoint submanifolds, all of which have positive codimension in $\widetilde{\Sigma}$.*

(iii) We have $(f,a) \in \widetilde{\Sigma}_0$ if and only if the restriction $\widetilde{\Sigma} \xrightarrow{\pi} \mathcal{F}$ of the projection $\mathcal{F} \times M \to \mathcal{F}$ is a local diffeomorphism at (f,a). Thus, in the neighbourhood of each $(f,a) \in \widetilde{\Sigma}_0$, the submanifold $\widetilde{\Sigma}$ is the graph of a smooth "implicit function" Φ defined in an open neighbourhood of f, with values in M.

Proof Assertions (i) and (iii) are what we have just said about Thom's transversality lemma and its proof, in the particular case of the submanifold Σ of J^k. To get (ii), just note that $\widetilde{\Sigma} = \widetilde{\Sigma}_0 \cup \cdots \cup \widetilde{\Sigma}_n$,

[6] Generalising the following fact: if b is a regular value of $\rho \in C^\infty(\mathbf{R}^2,\mathbf{R})$, then the set of those $(x,y) \in \rho^{-1}(b)$ at which the curve $\rho^{-1}(b)$ has a vertical tangent is defined by the equation $\partial_y \rho(x,y) = 0$.

[7] Thom's transversality lemma was astounding even to experts like Whitney, for example.

[8] See [17].The idea goes back to Kolmogorov.

[9] The multiplication $\mathbf{R} \times \mathcal{F} \ni (\lambda, f) \mapsto \lambda f$ is not continuous. It should be noted, however, that the map j^ℓ, near each $(f,x) \in \mathcal{F} \times M$, involves only the restriction of the elements of $\mathcal{F}$ to neighbourhoods of x, making the question of the global topology of $\mathcal{F}$ somewhat irrelevant.

[10] This approach is fundamental in singularity theory, where the fact that "infinitesimal stability implies stability", for example, is proved by integrating differential equations along paths in $\mathcal{F}$. Jean-Marie Souriau has formalised it in his theory of "diffeological" spaces [18].

where $\widetilde{\Sigma}_\kappa = (j^{k+1})^{-1}(\Sigma_\kappa)$ and Σ_κ denotes the set of those $j_x^{k+1}\varphi \in J^{k+1}$ such that $j_x^k\varphi \in \Sigma$ and
− in the case of potentials, the Hessian $D_x^2\varphi$ has corank κ
− in the case of vector fields, the differential $d\varphi_x$ has corank κ
− in the case of maps, the difference $d\varphi_x - \mathrm{Id}$ has corank κ.

Each Σ_κ is a submanifold, whose codimension ($n + \kappa^2$ in the last two cases) increases with κ and equals n if $\kappa = 0$. Moreover, $\Sigma_\kappa \cup \cdots \cup \Sigma_n$ is closed for $0 \leq \kappa \leq n$. □

Let us now express Proposition 1.2.1 (iii) in purely finite dimensional and local terms.

1.3 Local deformations and unfoldings

A *local deformation with* p *parameters* of f at $a \in M$ is a map $(u, x) \mapsto V_u(x) \in \mathbf{R}$, $(u, x) \mapsto X_u(x) \in TM$ or $(u, x) \mapsto g_u(x) \in M$ defined in an open neighbourhood of $(0, a)$ in $\mathbf{R}^p \times M$, such that, near a, $V_0 = V$, $X_0 = X$, $g_0 = g$ and that, in the second case, each X_u is a vector field.

We set $\tilde{V}(u, x) := \big(u, V_u(x)\big) \in \mathbf{R}^p \times \mathbf{R}$, $\tilde{X}(u, x) := \big(0, X_u(x)\big) \in \mathbf{R}^p \times T_xM$ and $\tilde{g}(u, x) := \big(u, g_u(x)\big) \in \mathbf{R}^p \times M$, and call the mapping $\tilde{V}$, the vector field $\tilde{X}$ and the mapping $\tilde{g}$ the *unfoldings* of V, X and g associated to the deformation. As before, we use the common notation $\tilde{f}$ and f_u for all three cases.

Proposition 1.3.1 *Given such a deformation, if* a *is a nondegenerate stationary point of* f*, there exist an open subset* $U \ni 0$ *of* $\mathbf{R}^p$*, an open subset* $\Omega \ni a$ *of* M *and a mapping* $\varphi : U \to \Omega$ *of class* C^∞ *such that, for every* $u \in U$*,* $\varphi(u)$ *is the only stationary point of* f_u *contained in* Ω*. Therefore, in terms of unfoldings, the graph of* φ *consists of the stationary points*[11] *of* $\tilde{f}$ *contained in* $U \times \Omega$*.*

Proof Given a chart $c : (M, a) \to (\mathbf{R}^n, 0)$, just apply the implicit function theorem at $(0, 0) \in \mathbf{R}^k \times \mathbf{R}^n$ to the equation $D(c_*V_u)(x) = 0$, $X_{c,u}(x) = 0$ or $c_*g_u(x) - x = 0$, where $c_*X_u(x) = \big(x, X_{c,u}(x)\big) \in T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$. □

Remark Even though the notion of a nondegenerate stationary point admits an "intrinsic" formulation on a manifold M, the equation $d_xV_u = 0$, $(X_u)_x = 0$ or $g_u(x) = x$ cannot be solved using the implicit function theorem, except in a chart, since its right-hand side depends also on $x \in M$ in the first two cases and, in the third, the equation $g_u(x) - x = 0$ does not mean anything: on manifolds, the langage of transversality is hardly avoidable.

In Proposition 1.3.1, $\mathbf{R}^p$ can be replaced by a Banach space and even by $\mathcal{F}$, the deformation being in that case the *tautological deformation* $(\varphi, x) \mapsto \varphi(x)$. This yields the implicit function in Proposition 1.2.1 (iii).

From now on, *we work in function space and leave the corresponding finite dimensional statements to the reader (despite the global language, most of what we are going to say has to do with local situations).*

[11] A "stationary point" of $\tilde{V}$ being a critical point, i.e. a point at which the differential does not have maximal rank.

1.4 Miracle! the nondegenerate critical points of potentials are structurally stable

The notion of "likeness" considered here is quite reasonable: two potentials V and W look alike if one passes from one to the other by a (nonlinear, but C^∞) change of coordinates and the addition of a real constant (in physics, a potential is defined up to a constant).

Theorem 1.4.1 (Morse lemma with parameters) *For every $(V,a) \in \widetilde{\Sigma}_0$, there exists a smooth local family $\big(\mathcal{F} \times M, (V,a)\big) \ni (W,x) \mapsto c_W(x) \in (\mathbf{R}^n, 0)$ of local charts of M such that each $c_{W*}W := W \circ c_W^{-1}$ is given by $c_{W*}W = W\big(\Phi(W)\big) + Q$, where $\Phi : (\mathcal{F}, V) \to (M,a)$ is the implicit function of Proposition 1.2.1 (iii) and $Q(x_1,\dots,x_n) = \pm x_1^2 \pm \cdots \pm x_n^2$. Thus, in the image of $c_V^{-1} \circ c_W$, the image of W by $c_V^{-1} \circ c_W$ equals V up to the addition of a constant.*

Proof Choose a chart $C : (M,a) \to (\mathbf{R}^n, 0)$ such that $\frac{1}{2} D^2(C_*V)(0)x^2 = Q(x)$ and set $C_W(x) := C(x) - C\big(\Phi(W)\big)$. By Taylor's formula, $C_{W*}W(x) = W\big(\Phi(W)\big) + B_W(x)x^2$, where $B_W(x)$ belong to the space $L_s^2(\mathbf{R}^n, \mathbf{R})$ of symmetric bilinear forms on $\mathbf{R}^n$ and equals $\int_0^1 (1-t) D^2(C_{W*}W)(tx)dt$. As $B_0 := B_V(0)$ is nondegenerate, it follows from the inverse mapping theorem (or the Gauss decomposition algorithm) that there exists an analytic local map $A : \big(L_s^2(\mathbf{R}^n, \mathbf{R}), B_0\big) \to \big(\mathbf{GL}_n(\mathbf{R}), I_n\big)$ such that $A(B)^* B_0 := B_0 \circ \big(A(B) \times A(B)\big)$ equals B for all $B \in \operatorname{dom} A$. This yields $C_{W*}W(x) = W\big(\Phi(W)\big) + B_0\Big(A\big(B_W(x)\big)x\Big)^2$, hence our lemma with $C_W \circ c_W^{-1}(x) := A\big(B_W(x)\big)x$. □

Remarks This result has the following global version: if M is compact, the potentials $V : M \to \mathbf{R}$ which are structurally stable for bilateral equivalence[12] are Thom's *excellent* Morse functions, i.e. the Morse functions which take different values at different critical points. By the transversality lemma[13], they form a *dense* open subset.

In the case of dynamical systems, we shall see that what happens near the "big stratum" $\widetilde{\Sigma}_0$ is already too complicated to be completely described, even locally.

From the Morse lemma with parameters, we deduce a key result of singularity theory:

Corollary 1.4.2 (splitting lemma) *For* every $(V,a) \in \widetilde{\Sigma}$*, there exists a smooth family of local charts $\big(\mathcal{F} \times M, (V,a)\big) \ni (W,x) \mapsto c_W(x) \in (\mathbf{R}^\nu \times \mathbf{R}^r, 0)$ such that $c_{W*}W(y,z) = R_W(y) + Q(z)$, where r is the rank of the Hessian $D_a^2 V$ and $Q(z) = \pm z_1^2 \pm \cdots \pm z_r^2$.*

Proof Using a chart, we may assume $(M,a) = (\mathbf{R}^n, 0)$. We then choose in $\mathbf{R}^n$ a supplementary subspace S of the kernel K of $D^2V(a)$ and identify $\mathbf{R}^n$ to $K \times S$, K to $\mathbf{R}^\nu$ and S to $\mathbf{R}^r$. For V close to W, we can then consider $V(y,z)$ as a function V_y of the variable $z \in \mathbf{R}^r$, depending on the parameter $y \in \mathbf{R}^\nu$ and apply the Morse lemma with parameters at $(V_0, 0) \in C^\infty(\mathbf{R}^r, \mathbf{R}) \times \mathbf{R}^r$. □

Remark This result diminishes in a sometimes spectacular way the number of significant variables, for it reduces the study of the W's near a to that of the R_W's near 0, the behaviour of $c_{W*}W(y,z)$ with respect to z being that of Q. In particular, if V reaches a local minimum at a, which implies that Q is positive definite, the local study of the minima of the potential W is reduced to that of the minima of the potential R_W, which depends on fewer variables (often much fewer)[14].

[12] Two functions V and W are equivalent if there exist two diffeomorphisms $\phi : M \to M$ and $\psi : \mathbf{R} \to \mathbf{R}$ such that $W = \psi \circ V \circ \phi^{-1}$.

[13] Applied to the submersive "bijet" map ${}_2j^1 : \mathcal{F} \times {}_2M \ni (f,x,y) \mapsto \big(j^1 f(x), j^1 f(y)\big) \in (J^1)^2$ and the submanifold $\big\{\big(j^1 f(x), j^1 f(y)\big) \in \Sigma^2 \,:\, f(x) = f(y)\big\}$, where ${}_2M := M^2 \setminus \Delta M$.

[14] It is not difficult to see that this "residual singularity", up to changes of variables, is unique: it does not depend on the choices we have made to obtain it.

1.5 In dynamics, the miracle is not immediate and has a high cost

In this paragraph and the following one, we let $f = X$ or g. The notion of "likeness" which would seem reasonable in dynamics is the following: we would say that f looks like $f_1 = X_1$ or g_1 locally if one passes from one to the other by a differentiable change of local coordinates: there exists a C^∞ *local conjugacy between the two systems*, i.e. a local diffeomorphism Ψ of M such that $X = \Psi^* X_1 := T\Psi^{-1} \circ X_1 \circ \Psi$ or $g = \Psi^* g_1 := \Psi^{-1} \circ g_1 \circ \Psi$. In the first case, this means that Ψ sends the local solutions of $\frac{dx}{dt} = X(x)$ onto those of $\frac{dy}{dt} = X_1(y)$.

This equivalence relation is too fine to yield structural stability: if a is a stationary point of f, then $d_a\Psi$ realises a conjugacy between $d_a f$ and $d_{\Psi(a)} f_1$, which therefore have the same eigenvalues. These are *moduli*, i.e. numerical invariants which can vary continuously from one equivalence class to another. As this phenomenon takes place as soon as Ψ is C^1, one considers *topological conjugacy*, where Ψ is merely a local homeomorphism[15]. Even then, one must exclude some nondegenerate stationary points:

Proposition 1.5.1 *Let $a \in M$ be a nondegenerate stationary point of f where at least one eigenvalue of $d_a X$ (resp. $d_a g$) is pure imaginary (resp. of modulus 1). Then f admits a local deformation $\mathbf{R} \times M \ni (u, x) \mapsto f_u(x)$ at a such that, denoting by φ the implicit function of Proposition 1.3.1, there exists no local topological conjugacy $\Psi : \big(M, \varphi(u)\big) \to \big(M, \varphi(-u)\big)$ between f_u and f_{-u} for small positive u. In particular, the system is not structurally stable in the neighbourhood of a.*

Proof Using a local chart, we may assume that $M = \mathbf{R}^n$ and $a = 0$. It is then enough to take $X_u(x) := X(x) + ux$ (resp. $g_u(x) := e^u g(x)$): one has $\varphi(u) = 0$ and, if ν is the number of pure imaginary eigenvalues (resp. eigenvalues of modulus 1) of $DX(0)$ (resp. $Dg(0)$), counted with their multiplicities, $DX_u(0)$ (resp. $Dg_u(0)$) has no pure imaginary eigenvalue (resp. no eigenvalue of modulus 1) for $u \neq 0$ close to 0 and, for small enough positive $u > 0$, $DX_u(0)$ (resp. $Dg_u(0)$) has ν eigenvalues with negative real part (resp. of modulus < 1) fewer than $DX_{-u}(0)$ (resp. $Dg_{-u}(0)$), the eigenvalues being still counted with their multiplicity.
Now, the number of eigenvalues with negative real part (resp. of modulus < 1) of $DX_u(0)$ (resp. $Dg_u(0)$) is the dimension of the *stable manifold* W_u^+ of X_u (resp. g_u) at $0 = \varphi(u)$, which is the submanifold germ at 0 admitting the following topological definition[16]:
− in the case of X_u, it is the germ at 0 of the set of those points x such that the integral curve of X_u which passes through x at time $t = 0$ tends to 0 when $t \to +\infty$;
− in the case of g_u, it is the germ at 0 of the set of those points x such that $g_u^n(x)$ tends to 0 when $n \to +\infty$.

Any homeomorphism germ $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ sending X_u (resp. g_u) onto X_{-u} (resp. g_{-u}) would satisfy $h(W_u^+) = W_{-u}^+$, hence $\dim W_u^+ = \dim W_{-u}^+$, a contradiction. □

Definition *A stationary point a of X (resp. g) is* hyperbolic *when $d_a X$ (resp. $d_a g$) has no eigenvalue on the imaginary axis (resp. on the unit circle or 0).*

Proposition 1.5.2 *For almost every f, all the zeros of f are hyperbolic. More precisely, the subset $\widetilde{\Sigma}_h$ of $\widetilde{\Sigma}_0$ consisting of those (f, a) such that a is a hyperbolic rest point of f is open and dense in $\widetilde{\Sigma}$ and the set of those f all of whose rest points are hyperbolic is open and dense in $\mathcal{F}$.*

[15] In the case of vector fields, the relation $X = \Psi^* X_1$ means by definition that Ψ sends the local solutions of $\frac{dx}{dt} = X(x)$ (or local *integral curves* of X) onto those of $\frac{dy}{dt} = X_1(y)$.

[16] Choosing a representative of f_u defined in a small enough neighbourhood of the origin.

Proof Let us treat the case of vector fields, the other one being analogous. To apply the transversality lemma in jet spaces, one must be somewhat cautious, for the set Σ_{nh} of those $j^1_a Y \in J^1(TM)$ such that $Y(a) = 0$ and d_aY is not hyperbolic is closed, but it is not a submanifold: its fibre Σ_{nha} over each $a \in M$ is *semi-algebraic* (i.e. defined by finitely many polynomial inequalities), being the image of the (algebraic) set of those $(Y_a, d_aY, \alpha) \in T_aM \times \mathbf{gl}(T_aM) \times \mathbf{R}$ such that $Y_a = 0$ and $\det(d_aY - i\alpha) = 0$ by the projection $(Y_a, d_aY, \alpha) \mapsto (Y_a, d_aY)$; one can therefore *stratify* Σ_{nha}, i.e. describe it as a finite union of disjoint submanifolds ("strata"), in a canonical enough way to obtain a stratification of Σ_{nh}; the strata being of codimension strictly greater than $n = \dim M$ (the non-hyperbolicity is added to the n conditions $Y_a = 0$), for almost every X, the 1–jet j^1X is transversal to each one of them, i.e. meets none of them. Openness follows from the fact Σ_{nh} is closed. Finally, $\widetilde{\Sigma}_h$ is open and dense in $\widetilde{\Sigma}$ since its complement $(j^1)^{-1}(\Sigma_{nh})$ is closed and the union of finitely many submanifolds of codimension at least 1. □

We can at last state a structural stability result more or less analogous to the Morse Lemma with parameter. Its proof can be found in [3]:

Theorem 1.5.3 (Grobman-Hartman theorem with parameter) *Let a be a hyperbolic stationary point of f. There exists a continuous family $\big(\mathcal{F} \times M, (f, a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (T_aM, 0)$ of local homeomorphisms such that $c_{\varphi *}\varphi \equiv d_af$. The image of φ by the local homeomorphism $c_f^{-1} \circ c_\varphi$ is therefore equal to f in the neighbourhood of a.*

However, topological conjugacy is a very coarse equivalence relation: for example, in phase space, it does not make any difference between those systems which oscillate while they approach the equilibrium and those which don't. In many problems, the good local equivalence relation is C^k conjugacy with $k \geq 1$, even though there are moduli:

1.6 Normal forms and C^k conjugacy, $k \geq 1$

Definition (Poincaré and Siegel domains) The open and dense subset $\widetilde{\Sigma}_h$ of $\widetilde{\Sigma}$ consisting of all (f, a)'s with a hyperbolic is the union of two disjoint open subsets:

– The *Poincaré domain* $\widetilde{\Sigma}_{hp}$, set of those (X, a) (resp. (g, a)) in $\widetilde{\Sigma}$ such that all the eigenvalues of d_aX (resp. d_ag) lie in the same connected component of $\mathbf{C} \setminus i\mathbf{R}$ (resp. $\mathbf{C} \setminus (\mathbf{S}^1 \cup \{0\})$).

– The (hyperbolic) *Siegel domain* $\widetilde{\Sigma}_{hs} := \widetilde{\Sigma}_h \setminus \widetilde{\Sigma}_{hp}$.

Theorem 1.6.1 (linearisation in the Poincaré domain) *For every (f, a) in the Poincaré domain, there exists a C^1 family $\big(\mathcal{F} \times M, (f, a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{R}^n, 0)$ of local charts such that each $c_{\varphi *}\varphi$ is linear. Moreover, the set of those (f, a)'s for which this family of local charts can be chosen C^∞ is open and dense in the Poincaré domain (if $\dim M = 1$, it equals the whole of the Poincaré domain, which coincides with $\widetilde{\Sigma}_h$).*

Despite its apparent simplicity, the first half is trickier than the second. It shows that, in the Poincaré domain, the only C^1 conjugacy invariant is the obvious one: the linear conjugacy class of the differential at the stationary point, i.e., generically, its eigenvalues. Before giving more explanations, let us state the analogous result in the Siegel domain:

Theorem 1.6.2 (linearisation in the Siegel domain) *For every positive integer k, the Siegel domain $\widetilde{\Sigma}_{hs}$ has an open and dense subset $\mathcal{L}_k$ of points (f, a) for which there exists a C^k family $\big(\mathcal{F} \times M, (f, a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{R}^n, 0)$ of local charts such that each $c_{\varphi *}\varphi$ is linear.*

Thus, in $\mathcal{L}_k$, the only C^k conjugacy invariants are the obvious, linear ones. The problem of finding the largest possible $\mathcal{L}_k$'s is probably unsolved and difficult. Moreover, in contrast with the Poincaré domain, the Siegel domain does not contain *any* (f,a) for which there exists a C^∞ family of local charts $\big(\mathcal{F}\times M,(f,a)\big)\ni(\varphi,x)\mapsto c_\varphi(x)\in(\mathbf{R}^n,0)$ such that each $c_{\varphi*}\varphi$ is linear: in fact, as we shall see at the end of this subsection, there is a dense subset of points (f,a) in $\widetilde{\Sigma}_{hs}$ such that the germ of f at a is not even *formally* linearisable.

For simplicity, we shall restrict the theory to the set $\widetilde{\Sigma}_\sigma$ of those $(f,a)\in\widetilde{\Sigma}_0$ such that d_af is simple (i.e. has only simple eigenvalues).

Proposition 1.6.3 *The set $\widetilde{\Sigma}_\sigma$ is open and dense in $\widetilde{\Sigma}_0$, and $\widetilde{\Sigma}_0\smallsetminus\widetilde{\Sigma}_\sigma$ is stratified, of codimension* 1. *For $(f,a)\in\widetilde{\Sigma}_\sigma$, denoting by r the number of real eigenvalues of d_af and setting $2c=n-r$, there is a C^∞ family $\big(\mathcal{F}\times M,(f,a)\big)\ni(\varphi,x)\mapsto c_\varphi(x)\in(\mathbf{R}^r\times\mathbf{C}^c,0)$ of local charts such that the derivative of each $c_{\varphi*}\varphi$ at $0=c_\varphi\big(\Phi(\varphi)\big)$ is* diagonal*: there exist smooth functions $\lambda_1,\dots,\lambda_r:(\mathcal{F},f)\to\mathbf{R}$ and $\lambda_{r+1},\dots,\lambda_{r+c}:(\mathcal{F},f)\to\mathbf{C}\smallsetminus\mathbf{R}$ such that $D(c_{\varphi*}\varphi)(0)x=\big(\lambda_1(\varphi)x_1,\dots,\lambda_{r+c}(\varphi)x_{r+c}\big)$.*

Proof We have that $\widetilde{\Sigma}_\sigma=(j^1)^{-1}(\Sigma_\sigma)$, where $\Sigma_0\smallsetminus\Sigma_\sigma$ is the set of those $j^1f(a)\in\Sigma_0$ such that the discriminant of the characteristic polynomial of d_af is zero. Thus, the restriction to $\Sigma_0\smallsetminus\Sigma_\sigma$ of the projection $j^1f(a)\to a$ is a locally trivial fibration with algebraic fibre. Stratifying the latter in a canonical way and considering the inverse image by j^1 of the resulting stratification of $\Sigma_0\smallsetminus\Sigma_\sigma$, we obtain the first assertion.

To get the rest, we shall use the following

Lemma 1.6.4 *If B_0 is a simple endomorphism of $\mathbf{R}^n$, there exists an analytic mapping A, defined in the neighbourhood of B_0 in the space $\mathbf{gl}_n(\mathbf{R})$ of endomorphisms of $\mathbf{R}^n$, with values in the space of isomorphisms of $\mathbf{R}^n$ onto $\mathbf{R}^r\times\mathbf{C}^c$ (where r is the number of real eigenvalues of B_0 and $r+2c=n$) such that, for every $B\in\operatorname{dom}A$, the endomorphism $A(B)_*B$ of $\mathbf{R}^r\times\mathbf{C}^c$ is diagonal.*

The family c_φ is obtained as follows: given a local chart $c:(M,a)\to(\mathbf{R}^n,0)$, apply the lemma to $B_0:=c_*d_af$ and take $c_\varphi:=A(c_*d_{\Phi(\varphi)}\varphi)\circ\big(c-c\big(\Phi(\varphi)\big)\big)$, where c_* denotes conjugacy by $d_{\Phi(\varphi)}c$ and Φ is the implicit function of Proposition 1.2.1 (iii).

Proof of the lemma. Diagonalise the adjoint $L(\mathbf{R}^n,\mathbf{C})\ni A\mapsto A\circ B_0$ so that the real eigenvalues $\alpha_1,\dots,\alpha_r$ of B_0 appear first, that its eigenvalues $\alpha_{r+1},\dots,\alpha_{r+c}$ with, say, positive imaginary part appear next and that the first r vector of the corresponding basis $(z_1,\dots,z_n)$ of $L(\mathbf{R}^n,\mathbf{C})$ are *real* linear forms. Then, $A(B_0):=(z_1,\dots,z_{r+c})$ is an isomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^r\times\mathbf{C}^c$ sending B_0 onto a diagonal endomorphism and we can get $A(B)=\big(x_1(B),\dots,x_{r+c}(B)\big)$ as follows: for $1\le j\le r+c$, the linear form $x_j(B)$ is the second component of the implicit function $\big(\beta_j(B),x_j(B)\big)$ obtained by solving the implicit equation $F(B,\beta_j,x_j):=x_j\circ B-\beta_jx_j=0$ near (B_0,α_j,z_j) in $\mathbf{gl}_n(\mathbf{R})\times\mathbf{K}\times H_j$, where

$$\mathbf{K}=\begin{cases}\mathbf{R} & \text{for } 1\le j\le r,\\ \mathbf{C} & \text{otherwise.}\end{cases}$$

and H_j is the affine hyperplane of $L(\mathbf{R}^n,\mathbf{K})$ consisting of those x_j's whose j–th component in the basis $(z_1,\dots,z_n)$ of $L(\mathbf{R}^n,\mathbf{C})$ equals 1. □

The following important result goes back essentially to Poincaré:

Theorem 1.6.5 (normal forms) *For every positive integer ℓ and every $(f,a)\in\widetilde{\Sigma}_\sigma$, there exists a C^∞ family $\big(\mathcal{F}\times M,(f,a)\big)\ni(\varphi,x)\mapsto c^\ell_\varphi(x)\in(\mathbf{R}^r\times\mathbf{C}^c,0)$ of local charts c_φ as in Proposition 1.6.3* *with the additional property that the ℓ–th order Taylor polynomials N^ℓ_φ of the $c^\ell_{\varphi*}\varphi$'s at 0 are in* normal form*, i.e. commute*[17] *with $DN^\ell_f(0) = D(c^\ell_{f*}f)(0)$. Setting $x_j := \bar{x}_{j-c}$ and $\lambda_j := \bar{\lambda}_{j-c}$ for $r+c < j \le c+2r$, this amounts to saying that the components $R^\ell_{\varphi,1}, \dots, R^\ell_{\varphi,r+c}$ of $R^\ell_\varphi := N^\ell_\varphi - DN^\ell_\varphi(0)$ are of the form*

$$R^\ell_{\varphi,j}(x_1,\dots,x_{r+c}) \quad = \quad \sum_{\alpha\in P^\ell_j(f)} a_\alpha(\varphi)x^\alpha,$$

$$\text{where} \quad P^\ell_j(f) \quad := \quad \begin{cases} \{\alpha\in \mathbf{N}^n \,:\, 1<|\alpha|\le \ell, \alpha\lambda(f)=\lambda_j(f)\} & \text{for vector fields,} \\ \{\alpha\in \mathbf{N}^n \,:\, 1<|\alpha|\le \ell, \lambda(f)^\alpha=\lambda_j(f)\} & \text{for mappings,}\end{cases}$$

using the standard notation for multi-indices: $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $|\alpha| := \alpha_1+\cdots+\alpha_n$ and $\alpha\lambda := \alpha_1\lambda_1+\cdots+\alpha_n\lambda_n$.

Proof By induction on ℓ. If $\ell = 1$, this is just Proposition 1.6.3. Otherwise, assuming the $c^{\ell-1}_\varphi$'s constructed, we obtain c^ℓ_φ by composing $c^{\ell-1}_\varphi$ with a polynomial change of variables of degree ℓ in $x_1,\dots,x_n$, of the form "identity plus homogeneous polynomial $\big(\sum_{|\alpha|=\ell} Q_{j,\alpha}(\varphi)x^\alpha\big)_{1\le j\le r+c}$ of degree ℓ" killing the monomials of degree ℓ which prevent the Taylor polynomial at order ℓ of $c^{\ell-1}_{\varphi*}\varphi$ from being in normal form. Denoting by $\big(\sum_{|\alpha|=\ell}\varphi_{j,\alpha}x^\alpha\big)_{1\le j\le r+c}$ the homogeneous term of degree ℓ of the latter Taylor polynomial, this is expressed by the relations

$$Q_{j,\alpha}(\varphi) = \begin{cases} \big(\lambda_j(\varphi)-\alpha\lambda(\varphi)\big)^{-1}\varphi_{j,\alpha} & \text{for } \alpha\notin P^\ell_j(f,a) \text{ in the case of vector fields,} \\ \big(\lambda_j(\varphi)-\lambda(\varphi)^\alpha\big)^{-1}\varphi_{j,\alpha} & \text{for } \alpha\notin P^\ell_j(f,a) \text{ in the case of maps,}\end{cases} \tag{1}$$

the simplest choice for $\alpha\in P^\ell_j(f,a)$ being $Q_{j,\alpha}(\varphi) = 0$. □

Note that (1) makes sense only if we have $P^\ell_j(\varphi)\subset P^\ell_j(f,a)$, which may force the domain of c^ℓ_f to shrink when ℓ increases. In the Poincaré domain, it stops shrinking after some time:

Theorem 1.6.6 (Sternberg-Chen theorem with parameter, Poincaré domain) *Let a be a stationary point of f where d_af is simple and in the Poincaré domain. Then, there exists a C^∞ family of local charts $\big(\mathcal{F}\times M,(f,a)\big)\ni(\varphi,x)\mapsto c_\varphi(x)\in(\mathbf{R}^r\times\mathbf{C}^c,0)$ such that $(\varphi,x)\mapsto c_{\varphi*}\varphi$ is a polynomial normal form, whose degree has an upper bound $\ell(f,a)$ determined by the eigenvalues of d_af.*

Proof With the notation of Proposition 1.6.3 and Theorem 1.6.5, as the real parts (resp. the logarithms of the absolute values) of the $\lambda_j(f)$'s all have the same sign, the sets $P^\ell_j(f,a)$ are the same for every $\ell\ge\ell(f,a)$, where (denoting by $[\cdot]$ the integer part)

$$\ell(f,a) := \begin{cases} \max\,[\Re\mathrm{e}\,\lambda_j(f)/\,\Re\mathrm{e}\,\lambda_k(f)] & \text{for vector fields,} \\ \max\,[\log|\lambda_j(f)|/\log|\lambda_k(f)|] & \text{for mappings.}\end{cases} \tag{2}$$

One then proves the following

Lemma 1.6.7 *There exists a unique germ $\mathcal{F}\times(\mathbf{R}^r\times\mathbf{C}^c)\ni(\varphi,x)\mapsto h_\varphi(x)\in\mathbf{R}^r\times\mathbf{C}^c$ of a C^∞ family such that each h_φ has contact of order $\ell(f,a)$ with the identity at 0 and sends $c^{\ell(f,a)}_{\varphi*}\varphi$ onto $N^{\ell(f,a)}_\varphi$.*

[17] For vector fields, this means that the flows commute, i.e. that the Lie bracket is 0.

The theorem follows with $c_\varphi := h_\varphi \circ c_\varphi^{\ell(f,a)}$.

Proof of the lemma[18]. In the case of mappings, the relation $h^\ell_{\varphi *} c^{\ell(f,a)}_{\varphi *}\varphi = N_\varphi^{\ell(f,a)}$ can be written $h_\varphi = u_\varphi(h_\varphi)$, where

$$u_\varphi(h) := \begin{cases} N_\varphi^{\ell(f,a)} \circ h \circ (c^{\ell(f,a)}_{\varphi *}\varphi)^{-1} & \text{if one has } |\lambda_j(f)| > 1 \text{ for every } j, \\ (N_\varphi^{\ell(f,a)})^{-1} \circ h \circ (c^{\ell(f,a)}_{\varphi *}\varphi) & \text{if one has } |\lambda_j(f)| < 1 \text{ for every } j. \end{cases}$$

For every φ close enough to f and every integer $k > \ell(f,a)$, this defines a C^∞ strict contraction u_φ of the space of those C^k functions on a ball $B = B_\rho(0)$ with small enough radius ρ which have contact of order $\ell(f,a)$ with the identity at 0, and this contraction depends in a C^∞ way on φ. Its unique fixed point is the required h_φ (to see that it is C^∞ though the radius ρ may decrease when k increases, use the conjugacy relation to extend the h_φ's to a fixed domain).

In the case of vector fields, this argument applies to the values of the involved flows for positive times t, yielding an h_φ which depends continuously on t. By uniqueness, it is the same for every rational $t \in (0,1]$ and therefore, by continuity, for every $t \in (0,1]$. □

Proposition 1.6.8 *With the notation of Theorem 1.6.5, for every positive integer* ℓ*,*

(i) the set $\widetilde{\Sigma}^\ell_\sigma$ *of those* $(f,a) \in \widetilde{\Sigma}_\sigma$ *such that all the* $P^\ell_j(f,a)$*'s are empty*[19] *is open and dense in* $\widetilde{\Sigma}$*, and its complement* $\widetilde{\Sigma} \smallsetminus \widetilde{\Sigma}^\ell_\sigma$ *is a finite union of submanifolds of codimension at least* 1 *in* $\widetilde{\Sigma}$

(ii) the set $\mathcal{U}^\ell$ *of those* $f \in \mathcal{F}$ *all of whose rest points a satisfy* $(f,a) \in \widetilde{\Sigma}^\ell_\sigma$ *is open and dense.*

Proof The set Σ^ℓ_σ of those $j^1 f(a) \in \Sigma_\sigma$ such that all the $P^\ell_j(f,a)$'s are empty being obviously open in the open subset Σ_σ of Σ_0, it follows that $\widetilde{\Sigma}^\ell_\sigma = (j^1)^{-1}(\Sigma^\ell_\sigma)$ is open in $\widetilde{\Sigma}_\sigma$and therefore in $\widetilde{\Sigma}$. Moreover, denoting by $\Sigma^1 = \Sigma_0 \cup \cdots \cup \Sigma_n$ the set of all $j^1 f(a)$ with $j^0 f(a) \in \Sigma$, the set $\mathcal{U}^\ell$ of those f such that $j^1 f$ takes all of its values in the open subset $J^1 \smallsetminus (\Sigma^1 \smallsetminus \Sigma^\ell_\sigma)$ is open. To prove that it is dense, we just have to prove that $\Sigma^1 \smallsetminus \Sigma^\ell_\sigma$ is a finite union of submanifolds of codimension at least 1 in Σ^1 and apply the transversality lemma to them. Taking inverse images by j^1, this will also yield the rest of (i).

In order to get those submanifolds, as we already know that $\Sigma^1 \smallsetminus \Sigma_0 = \Sigma_1 \cup \cdots \cup \Sigma_n$ and $\Sigma_0 \smallsetminus \Sigma_\sigma$ are stratified into such submanifolds, we should prove that $\Sigma_\sigma \smallsetminus \Sigma^\ell_\sigma$ is a finite union of submanifolds of codimension at least 1 in Σ. A relatively easy way is to consider, for each $\alpha \in \mathbf{N}^n$ with $1 < |\alpha| \le \ell$, the set of those $\big(\lambda_1, \ldots, \lambda_n, j^1 f(a)\big) \in \mathbf{C}^n \times \Sigma_\sigma$ which satisfy the equations $\alpha\,\lambda = \lambda_1$ in the case of vector fields, $\lambda^\alpha = \lambda_1$ in the case of maps, and $\det(d_a f - \lambda_k) = 0$ for $1 \le k \le n$. This is a sub-bundle of the natural bundle (with smooth semi-algebraic fibre) $\mathbf{C}^n \times \Sigma_\sigma \to \Sigma$ whose (smooth) semi-algebraic fibre has codimension $2n+1$ where λ_1 is real, $2n+2$ elsewhere. Therefore, its projection Σ^α_σ onto Σ_σ is a sub-bundle of $\Sigma_\sigma \to \Sigma$ whose fibre is semi-algebraic, of codimension at least 1. It follows that Σ^α_σ can be stratified and the codimension of its strata in Σ_σ is at least 1. As Σ^ℓ_σ is the union of all Σ^α_σ's with $1 < |\alpha| \le \ell$, this proves what we wanted. □

We can now establish the last assertion of Theorem 1.6.1 in $\widetilde{\Sigma}_\sigma$:

Corollary 1.6.9 *The set of those* $(f,a) \in \widetilde{\Sigma}_{hp} \cap \widetilde{\Sigma}_\sigma$ *such that the* $c_{\varphi *}\varphi$*'s of Theorem 1.6.6 are linear (diagonal) is open and dense in the Poincaré domain.*

[18] See [26, 22, 6] and, for another approach, [8].

[19] Or, in other words, such that the germ of φ at b is formally linearisable at order ℓ for every $(\varphi, b) \in \widetilde{\Sigma}$ close enough to (f,a).

Proof For each $(f,a) \in \widetilde{\Sigma}_{hp} \cap \widetilde{\Sigma}_\sigma$, the definition (2) of $\ell(f,a)$ implies that, in a neighbourhood of (f,a), one has $\ell(\varphi,b) \le \ell(f,a)$ and $P_j^\ell(\varphi,b) = P_j^{\ell(f,a)}(\varphi)$ for all $\ell \ge \ell(f,a)$. Therefore, there exists an open neighbourhood of (f,a) in $\widetilde{\Sigma}_\sigma$ whose intersection with the dense subset $\bigcap_\ell \widetilde{\Sigma}_\sigma^\ell$ equals its intersection with the *open* and dense subset $\widetilde{\Sigma}_\sigma^{\ell(f,a)}$.

It follows that the subset $\cap_\ell \widetilde{\Sigma}_\sigma^\ell \cap \widetilde{\Sigma}_{hp}$ is open and dense in $\widetilde{\Sigma}_{hp} \cap \widetilde{\Sigma}_\sigma$. Now, it is precisely the set of those $(f,a) \in \widetilde{\Sigma}_{hp} \cap \widetilde{\Sigma}_\sigma$ such that the $c_{\varphi*}\varphi$'s of Theorem 1.6.6 are linear. Indeed, every neighbourhood of any other $(f,a) \in \widetilde{\Sigma}_{hp} \cap \widetilde{\Sigma}_\sigma$ contains points (φ,b) such that the germ of φ at b is not even formally linearisable. □

Remarks When d_af is not simple, the notion of a normal form is heavier but the idea is the same.

To prove the first assertion of Theorem 1.6.1, one can follow the lines of the proof of Hartman's C^1 linearisation theorem[20] [15, 1] to get a parametric version of it. This works in the $C^{1+\text{Lipschitz}}$ category. In the smooth category, another method consists in getting the linearisation explicitly from the normal form, in the spirit of [7] (see also [6], Appendice 8 and [5]), yielding a linearisation whose derivative is Hölderian for *every* Hölder exponent less than 1.

When M is analytic, choosing an analytic chart c in the proof of Theorem 1.6.3, it is quite easy to see (using complexification) that the chart c_φ constructed in the proof of Theorem 1.6.6 is analytic when φ is. This will *not* be the case in the Siegel domain.

Theorem 1.6.10 (Sternberg-Chen theorem with parameter, Siegel domain) *Let a be a stationary point of f where d_af is simple, hyperbolic, in the Siegel domain. Then, for each positive integer k, there is a C^k family $\big(\mathcal{F} \times M, (f,a)\big) \ni (\varphi,x) \mapsto c_\varphi(x) \in (\mathbf{R}^r \times \mathbf{C}^c, 0)$ of local charts such that $(\varphi,x) \mapsto c_{\varphi*}\varphi$ is a polynomial normal form, whose degree has an upper bound $\ell_k(f,a)$, upper semi-continuous with respect to (f,a), determined by k and the eigenvalues of d_af.*

Proof This result is more difficult than the previous ones and the smallest possible integer $\ell_k(f,a)$ is not known in general. With the notation of Theorem 1.6.5, let us sketch a proof [26, 22]. First remark that, for every integer ℓ and every $\varphi \in \mathcal{F}$ close to f, the unstable manifold W^- of the normal form N_φ^ℓ is that of $DN_f^\ell(0)$, i.e. the direct sum of the characteristic subspaces of $DN_f^\ell(0)$ associated to the λ_j's with $\Re\mathrm{e}\,\lambda_j > 0$ (resp. $|\lambda_j| > 1$): this follows easily from the fact that N_φ^ℓ commutes with $DN_f^\ell(0)$. We then prove the following generalisation of the previous lemma:

Lemma 1.6.11 *For every integer $\ell > 0$, there exist an integer $d = d_\ell(f,a) \ge \ell$ and a C^∞ local family $\big(\mathcal{F} \times (\mathbf{R}^r \times \mathbf{C}^c), (f,0)\big) \ni (\varphi,x) \mapsto h_\varphi^\ell(x) \in (\mathbf{R}^r \times \mathbf{C}^c, 0)$ such that each h_φ^ℓ has contact of order ℓ with the identity at 0 and that $h_{\varphi*}^\ell c_{\varphi*}^d\varphi$ has contact of order ℓ with N_φ^d along W^-.*

Proof of the lemma. For mappings, the equation $\big(j^\ell(h_{\varphi*}^\ell c_{\varphi*}^d\varphi)\big)\big|_{W^-} = (j^\ell N_\varphi^d)|_{W^-}$ we should solve writes $(j^\ell(h_\varphi^\ell)^{-1})|_{W^-} = \big(j^\ell(\varphi_d \circ (h_\varphi^\ell)^{-1} \circ (N_\varphi^d)^{-1})\big)\big|_{W^-}$, where $\varphi_d := c_{\varphi*}^d\varphi$. Thus, $(j^\ell(h_\varphi^\ell)^{-1})|_{W^-}$ must be a fixed point of $\Phi_\ell : (j^\ell h)|_{W^-} \mapsto \big(j^\ell(\varphi_d \circ h \circ (N_\varphi^d)^{-1})\big)\big|_{W^-}$ in a space of sections of the source projection $J^\ell(\mathbf{R}^r \times \mathbf{C}^c, \mathbf{R}^r \times \mathbf{C}^c) \to \mathbf{R}^r \times \mathbf{C}^c$ over a ball B^- of W^- centered at 0 and of small enough radius. It happens that, for d large enough[21], Φ_ℓ is a strict contraction if one restricts to those sections having contact of large enough order at 0 with $(j^\ell \mathrm{Id})|_{W^-}$, hence

$$(j^\ell(h_\varphi^\ell)^{-1})|_{B^-} = \lim_{k\to\infty} \big(j^\ell(\varphi_d^k \circ (N_\varphi^d)^{-k})\big)\big|_{B^-},$$

[20] Which, unlike the Grobman-Hartman theorem, is *false* in infinite dimensions [23].
[21] Defining our $d_\ell(f,a)$, which we do not claim is the best possible one.

implying that the unique fixed point of Φ_ℓ is indeed the jet along B_- of a (far from unique) map $(h^\ell_\varphi)^{-1}$.

In the case of vector fields, apply the above to the time 1 of the flows of $\varphi_d := c^d_{\varphi*}\varphi$ and of the normal form, and proceed as in the proof of the previous lemma to show that the $(j^\ell(h^\ell_\varphi)^{-1})|_{B^-}$'s thus obtained do satisfy $(j^\ell h^\ell_{\varphi*}\varphi_d)|_{B^-} = (j^\ell N^d_\varphi)|_{B^-}$.

In both cases, dependence with respect to φ is not a problem.

End of the proof. Here, Nelson's idea [22] is to proceed as in the Poincaré domain, replacing the stationary point 0 by the invariant manifold W^-. For that purpose, we use *the same* bump function to extend $N^{d_\ell(f,a)}_\varphi$ and $h^\ell_{\varphi*}c^{d_\ell(f,a)}_{\varphi*}\varphi$ to the whole of $\mathbf{R}^r\times\mathbf{C}^c$ so both equal $DN^{d_\ell(f,a)}_\varphi(0)$ off a compact subset and leave W^- globally invariant. These two extensions $\tilde N^{d_\ell(f,a)}_\varphi$ and $\tilde\varphi_{d_\ell(f,a)}$ have contact of order ℓ along W^-.

In the case of mappings, one then shows the existence of an integer $m_k(f,a)\ge k$ such that, for $\ell\ge m_k(f,a)$, the sequence $\big(\tilde\varphi^{-m}_{d_\ell(f,a)}\circ(\tilde N^{d_\ell(f,a)}_\varphi)^m\big)_{m\in\mathbf{N}}$ converges in the C^k sense to a diffeomorphism $\tilde h_\varphi$, which is therefore tangent at order k to the identity along W^- and of course satisfies $\tilde h_\varphi\circ\tilde N^{d_\ell(f,a)}_\varphi = \varphi_{d_\ell(f,a)}\circ\tilde h_\varphi$. The theorem follows with $c_u = \tilde h^{-1}_\varphi\circ h^\ell_\varphi\circ c^{d_\ell(f,a)}_\varphi$ and $\ell_k(f,a) = d_{m_k(f,a)}(f,a)$. One passes from maps to vector fields as usual. □

We can now establish Theorem 1.6.2:

Corollary 1.6.12 *For every positive integer k, the set $\mathcal{L}_k$ of those $(f,a)\in\widetilde\Sigma_{hs}\cap\widetilde\Sigma_\sigma$ such that the $c_{\varphi*}\varphi$'s of Theorem 1.6.10 are linear (diagonal) is open and dense in $\widetilde\Sigma_{hs}$.*

Proof As ℓ_k is upper semi-continuous, each $(f,a)\in\widetilde\Sigma_{hs}\cap\widetilde\Sigma_\sigma$ possesses an open neighbourhood $\mathcal{N}_k(f,a)$ consisting of points (φ,b) with $\ell_k(\varphi,b)\le\ell_k(f,a)$. With the notation of Proposition 1.6.8, the dense open subset $\mathcal{N}_k(f,a)\cap\widetilde\Sigma^{\ell_k(f,a)}_\sigma$ of $\mathcal{N}_k(f,a)$ consists of points of $\mathcal{L}_k$, which is easily seen to equal $\bigcup_{(f,a)}\mathcal{N}_k(f,a)\cap\widetilde\Sigma^{\ell_k(f,a)}_\sigma$. □

Remarks As mentioned before, the set of those (f,a)'s such that the germ of f at a is not *formally* linearisable is dense in Σ_{hs}. Indeed, as $\mathbf{Q}$ is dense in $\mathbf{R}$, any pair (λ,μ) of real numbers with $\lambda<0<\mu$ is the limit of a sequence (λ_n,μ_n) such that there exist positive integers p_n,q_n with $p_n\lambda_n+q_n\mu_n=0$. Applying this remark to the eigenvalues of d_af or their real parts, one can see that every $(f,a)\in\widetilde\Sigma_{hs}\cap\widetilde\Sigma_\sigma$ is the limit of a sequence (f_n,a_n) such that *all* the sets $\cup_\ell P^\ell_j(f_n,a_n)$ are infinite. Arbitrarily close to any such (f_n,a_n), there are points (φ_n,b_n) such that the germ of φ_n at b_n is not formally linearisable.

The C^∞ Sternberg-Chen theorem states that two hyperbolic germs are smoothly conjugate if and only if they are formally conjugate. It can be proved [6] that, in the Siegel domain, the set of all possible germs of such conjugacies is infinite dimensional, in sharp contrast with the fact that, when the two germs are analytic, there may be *no* analytic conjugacy between them, due to the presence of small denominators—the denominators on the right-hand side of (1)—preventing convergence.

2 Birth of dynamics out of statics.

In this section, with the notation of the previous one, we first describe the simplest part of the stratified subset $\widetilde\Sigma\setminus\widetilde\Sigma_0$ consisting of those (f,a)'s such that a is a degenerate rest point of f.

We consider for the last time potentials as well as dynamical systems and do not go beyond the singularity theory of mappings.

Then, for $\mathcal{F} = C^\infty(M, M)$, we turn to the specifically dynamical problem of describing what happens near the simplest part of the stratified set $\widetilde{\Sigma}_0 \smallsetminus \widetilde{\Sigma}_h$ of those (f, a)'s such that a is a *non-hyperbolic* nondegenerate rest point of f. This yields the *period doubling bifurcation.*

For vector fields, it implies two rather different phenomena: the *Hopf* (or *Hopf-Andronov*) *bifurcation*, another example of the birth of dynamics out of statics, and the period doubling bifurcation, familiar to flutists and even violonists.

We then turn briefly to the somewhat discordant period tripling bifurcation, in the general setting of the p–upling bifurcation and its relationship with the Arnol'd "tongues".

We consider next a subtler bifurcation: the so-called *Hopf bifurcation for maps*, more rightly known as the *Naimark-Sacker bifurcation.* In the case of vector fields, it implies the *periodic-quasiperiodic bifurcation*, which has received much attention as a model of transition to turbulence.

Finally, we show that the Naimark-Sacker circles are *very* far from being the only compact invariant manifolds that can arise from partially elliptic rest points.

2.1 The fold catastrophe and such

Recall that $\widetilde{\Sigma}$ is the disjoint union of $\widetilde{\Sigma}_0, \dots, \widetilde{\Sigma}_n$, where $\widetilde{\Sigma}_\kappa = (j^{k+1})^{-1}(\Sigma_\kappa)$ and Σ_κ denotes the set of those $j^{k+1}\varphi(x) \in J^{k+1}$ such that $j^k\varphi(x) \in \Sigma$ and, in the case of potentials (resp. vector fields, maps), $D_x^2\varphi$ (resp. $d\varphi_x$, $d\varphi_x - \mathrm{Id}$) has corank κ. Here, we only study $\widetilde{\Sigma}_1$.

Theorem 2.1.1 *The submanifold $\widetilde{\Sigma}_1$ has the following properties:*

(i) For each of its points (f, a), there exists a smooth family of local charts $\big(\mathcal{F} \times M, (f, a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{R} \times \mathbf{R}^{n-1}, 0)$ such that, near (f, a), the submanifold $\widetilde{\Sigma}$ is the set of those $(\varphi, x) \in \mathcal{F} \times M$ such that $(y, z) := c_\varphi(x)$ satisfies the equations $z = 0$ and $F(\varphi, y) = 0$, where $F : \big(\mathcal{F} \times \mathbf{R}, (f, 0)\big) \to (\mathbf{R}, 0)$ is a smooth function with $\partial_y F(f, 0) = 0$ such that, for each integer k satisfying $\partial_y F(f, 0) = \cdots = \partial_y^{k+1} F(f, 0) = 0$, the differentials $\partial_\varphi \partial_y^j F(f, 0)$ with $0 \le j \le k$ are linearly independent.

(ii) A decreasing sequence $(\widetilde{\Sigma}_{1_{k+1}})_{k \in \mathbf{N}}$ of smooth submanifolds of $\widetilde{\Sigma}_1$ can be defined inductively from $\widetilde{\Sigma}_{1_1} := \widetilde{\Sigma}_1$ as follows: $\widetilde{\Sigma}_{1_{k+2}}$ is the (closed) set of those points of $\widetilde{\Sigma}_{1_{k+1}}$ at which the differential of the restricted projection $\widetilde{\Sigma}_{1_{k+1}} \to \mathcal{F}$ is not injective. Thus,

- *(a) near every $(f, a) \in \widetilde{\Sigma}_1$, with the notation of (i), each $\widetilde{\Sigma}_{1_{k+1}}$ is defined by the conditions $z = 0$ and $\partial_y^j F(\varphi, y) = 0$ for every integer $j \le k + 1$, hence $\widetilde{\Sigma}_{1_{k+1}}$ has codimension $n + k + 1$ in $\mathcal{F} \times M$ and, therefore, codimension k in $\widetilde{\Sigma}_1$*
- *(b) in particular, $\widetilde{\Sigma}_{1_{k+1}}$ is the set of those $(f, a) \in \widetilde{\Sigma}_1$ such that the function F in (i) satisfies $\partial_y^j F(f, 0) = 0$ for $1 \le j \le k + 1$*
- *(c) the closed subset $\widetilde{\Sigma}_{1,\infty} = \widetilde{\Sigma}_{1_\infty} := \bigcap_k \widetilde{\Sigma}_{1_{k+1}}$, which consists of those $(f, a) \in \widetilde{\Sigma}_1$ such that the function F introduced in (i) satisfies $\partial_y^j F(f, 0) = 0$ for every integer j, has infinite codimension: for every integer k, almost every smooth k–parameter family (f_u) of elements of $\mathcal{F}$ satisfies $(f_u, x) \notin \widetilde{\Sigma}_{1,\infty}$ for all (u, x).*

(iii) It follows that the submanifold $\widetilde\Sigma_1$ is the disjoint union of the subset $\widetilde\Sigma_{1,\infty}$ and the submanifolds $\widetilde\Sigma_{1,k} := \widetilde\Sigma_{1_{k+1}} \smallsetminus \widetilde\Sigma_{1_{k+2}}$, $k \in \mathbf{N}$, each of which has codimension k in $\widetilde\Sigma_1$ and does not intersect the closure $\widetilde\Sigma_{1_{\ell+1}}$ of $\widetilde\Sigma_{1,\ell}$ for $\ell > k$. Moreover, for $(f,a) \in \widetilde\Sigma_{1,k}$, $k < \infty$,

(a) the charts c_φ and the function F in (i) can be chosen so that

$$F(\varphi, y) = y^{k+2} + \sum_{j=1}^{k+1} \alpha_j(\varphi)\, y^{j-1},$$

where $\alpha_1, \dots, \alpha_{k+1} : (\mathcal{F}, f) \to (\mathbf{R}, 0)$ have independent differentials at f

(b) in the case of potentials, the charts c_φ in (i) can be chosen so that

$$c_{\varphi *}\varphi(y,z) = \pm\left(\frac{y^{k+3}}{k+3} + \sum_{j=1}^{k+1} \alpha_j(\varphi)\,\frac{y^j}{j} + \alpha_0(\varphi)\right) \pm z_1^2 \pm \cdots \pm z_{n-1}^2,$$

where $\alpha_1, \dots, \alpha_{k+1}$ are as in (a) and $\alpha_0 : (\mathcal{F}, f) \to \big(\mathbf{R}, f(a)\big)$ is a smooth local function whose differential at f is independent from those of $\alpha_1, \dots, \alpha_{k+1}$.

Proof (i) In the case of potentials, by the splitting lemma p. 5, there exists a smooth family of local charts $\big(\mathcal{F} \times M, (f,a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{R} \times \mathbf{R}^{n-1}, 0)$ such that $c_{\varphi *}\varphi(y,z) = R_\varphi(y) \pm z_1^2 \pm \cdots \pm z_{n-1}^2$, where each R_φ is a function of one variable and $R_f'(0) = R_f''(0) = 0$. This implies at once (i) with $F(y,\varphi) := R_\varphi'(y)$. The differentials $\partial_\varphi \partial_y^j F(f,0)$ with $j \le k+1$ are independent because, for each such j, there is a smooth path $\varepsilon \mapsto \varphi_\varepsilon$ in $\mathcal{F}$ such that $\varphi_0 = 0$ and $c_{f*}\varphi_\varepsilon(y,z) = c_{f*}f(y,z) + \varepsilon y^{j+1}$ near 0, hence $\frac{d}{d\varepsilon}\partial_y^j F(\varphi_\varepsilon, 0)|_{\varepsilon=0} = (j+1)!$ and $\frac{d}{d\varepsilon}\partial_y^i F(\varphi_\varepsilon, 0)|_{\varepsilon=0} = 0$ for $0 \le i \le k$ and $i \ne j$. This makes sense because the conditions are stable by changes of variables preserving the projection onto $\mathcal{F}$.

In the case of vector fields (resp. maps), let $c_0 : (M,a) \to (\mathbf{R}^n, 0)$ be a smooth chart sending the kernel of $d_a f$ (resp. $d_a f - \mathrm{Id}$) onto $\mathbf{R} \times \{0\}$. Setting $f_0 := c_{0*}f$ and $\varphi_0 := c_{0*}\varphi$ (resp. $f_0 := c_{0*}f - \mathrm{Id}$ and $\varphi_0 := c_{0*}\varphi - \mathrm{Id}$), let A be an automorphism of $\mathbf{R}^n$ sending the image of $Df_0(0)$ onto $\{0\} \times \mathbf{R}^{n-1}$ and let $A_1 : \mathbf{R}^n \to \mathbf{R}$, $A_2 : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be its components. By the inverse mapping theorem, the smooth mapping

$$\begin{array}{rcl} h : (\varphi, y, z) & \mapsto & \big(\varphi, y, A_2\varphi_0(y,z)\big) =: \big(\varphi, h_\varphi(y,z)\big) \\ \big(\mathcal{F} \times \mathbf{R} \times \mathbf{R}^{n-1}, (f,0)\big) & \to & \big(\mathcal{F} \times \mathbf{R} \times \mathbf{R}^{n-1}, (f,0)\big) \end{array}$$

is a local diffeomorphism such that $A \circ \varphi_0 \circ h_\varphi^{-1}(y,z)$ is of the form $\big(F_1(\varphi, y, z), z\big)$, hence (i) with $c_\varphi = h_\varphi \circ c_0$ and $F(\varphi, y) = F_1(\varphi, y, 0)$. As before, the differentials $\partial_\varphi \partial_y^j F(f,0) \ne 0$ with $j \le k$ are independent because, for each such j, there is a smooth path $\varepsilon \mapsto \psi_\varepsilon$ in $\mathcal{F}$ such that $\psi_0 = f$ and $A \circ c_{0*}\psi_\varepsilon \circ h_f^{-1}(y,z) = A \circ c_{0*}f \circ h_f^{-1}(y,z) + (\varepsilon y^j, 0)$ near 0, hence $\frac{d}{d\varepsilon}\partial_y^j F(\psi_\varepsilon, 0)|_{\varepsilon=0} = j!$ and $\frac{d}{d\varepsilon}\partial_y^i F(\psi_\varepsilon, 0)|_{\varepsilon=0} = 0$ for $0 \le i \le k$ and $i \ne j$. The reason why this condition is intrinsic is the same as before.

(ii) With the notation of (i), setting $\widetilde\Sigma_{1_0} := \widetilde\Sigma$, assume that $\widetilde\Sigma_{1_k}$, for some $k \in \mathbf{N}$, is defined near $(f,a) \in \widetilde\Sigma_1$ by the equations $z = 0$ and $\partial_y^j F(\varphi, y) = 0$ for $0 \le j \le k$ (which, by (i), is the case if $k = 0$). Let us show that these equations are independent near (f,a): if they are not satisfied at (f,a), then they are not satisfied near (f,a); if they are, then, by (i), the differentials $\partial_\varphi \partial_y^j F(f,0)$ with $0 \le j < k$ are linearly independent and two cases can occur:

– if $\partial_y^{k+1}F(\varphi,y) = 0$, then, by (i), the differentials $\partial_\varphi \partial_y^j F(f,0)$ with $0 \le j \le k$ are linearly independent, hence the equations $z = 0$ and $\partial_y^j F(\varphi,y) = 0$ $(0 \le j \le k)$ defining $\widetilde{\Sigma}_{1_k}$ near (f,a) are independent;

– for $\partial_y^{k+1}F(\varphi,y) \neq 0$, this is also the case since the linear forms $d\partial_y^j F(f,0) = \partial_\varphi \partial_y^j F(f,0)\, d\varphi$ with $0 \le j < k$ and $d\partial_y^k F(f,0) = \partial_y^{k+1}F(\varphi,y)\, dy + \partial_\varphi \partial_y^k F(f,0)\, d\varphi$ are independent.

This proves that the subset $\widetilde{\Sigma}_{1_k}$ is indeed a smooth submanifold of codimension $n+k$ in $\mathcal{F} \times M$.

It follows that a point $(\varphi,x) \in \widetilde{\Sigma}_{1_k}$ close enough to (f,a) belongs to $\widetilde{\Sigma}_{1_{k+1}}$ if and only if, setting $c_\varphi(x) =: (y,z) \in \mathbf{R} \times \mathbf{R}^{n-1}$ there exists $(\delta y, \delta z) \in \mathbf{R} \times \mathbf{R}^{n-1} \setminus \{0\}$ such that $\delta z = 0$ and $\partial_y^{j+1}F(\varphi,y)\,\delta y = 0$ for $0 \le j \le k$, which is the case if and only if (φ,y) fulfils the additional condition $\partial_y^{k+1}F(\varphi,y) = 0$. This proves (a), from which (b) follows immediately. To get (c), just notice that, by transversality (see the following remarks), almost every smooth k–parameter family (f_u) of elements of $\mathcal{F}$ satisfies $(f_u,x) \notin \widetilde{\Sigma}_{1,k+1}$ for all (u,x).

(iii) With the notation of (i), it follows from (ii) that (f,a) belongs to $\widetilde{\Sigma}_{1,k}$ if and only if we have $\partial_y^j F(f,0) = 0$ for every integer $j \le k+1$ and $\partial_y^{k+2}F(f,0) \neq 0$. Then, by (an easy extension of) the Malgrange preparation theorem, there exist smooth local functions $u : \big(\mathcal{F} \times \mathbf{R}, (f,0)\big) \to \mathbf{R} \setminus \{0\}$ and $\beta_1, \dots, \beta_{k+2} : (\mathcal{F}, f) \to (\mathbf{R}, 0)$ such that

$$F(\varphi,y) = u(\varphi,y) \left(y^{k+2} + \sum_{j=1}^{k+2} \beta_j(\varphi)\, y^{j-1} \right);$$

as we are only interested in the zeros of F, we can divide it by u and replace the variable y by $y + \frac{1}{k+2}\beta_{k+2}(\varphi)$, proving (a). Assertion (b) is a consequence of the preparation theorem (*universal unfolding* of the function germ $y \mapsto y^{k+2}$ in Thom's terminology). □

Example (the fold catastrophe) It occurs at points of the "big stratum" $\widetilde{\Sigma}_{1,0}$ of $\widetilde{\Sigma}_1$. For each $(f,a) \in \widetilde{\Sigma}_{1,0}$, we are in the situation of Theorem 2.1.1 (iii)(a) with $k = 0$. Therefore, the equation $F(\varphi,y) = 0$ writes $y^2 + \alpha_1(\varphi) = 0$, implying that φ has the two rest points $c_\varphi^{-1}\left(\pm\sqrt{-\alpha_1(\varphi)}, 0\right)$ for $\alpha_1(\varphi) < 0$ and no rest point near a for $\alpha_1(\varphi) > 0$. Thus, if $s \mapsto \varphi_s$ is a smooth path $(\mathbf{R},0) \to (\mathcal{F},a)$, generic in the sense that $\frac{d}{ds}\alpha_1(\varphi_s)|_{s=0}$ is nonzero, then, according to its sign, one observes for $s = 0$ the birth or death at a of two rest points of φ_s.This is the *fold catastrophe*, a first example of the birth of dynamics out of statics as there does not remain any rest point near a after the collision.

Remarks Thus, the simplest thing that can happen to systems depending on a parameter is a catastrophe. This shows the superiority of Thom's global viewpoint over that of former specialists in bifurcation theory, who postulated the persistence of a rest point for all values of the parameter: arriving at the collision with their nose on one of the two colliding points, they could not predict the catastrophe[22].

Theorem 2.1.1 is an example of a general feature of singularity theory for mappings: locally, except on a subset of infinite codimension, one can choose smooth coordinates so that everything becomes algebraic (in dynamics, the meaning of "everything" here is much more modest than for potentials).

[22]And, sometimes, drew improbable nose-saving bifurcation diagrams…

For integer k, the notation $\widetilde{\Sigma}_{1_{k+1}}$ is compatible with our previous conventions, as we do have that $\widetilde{\Sigma}_{1_{k+1}} = (j^\ell)^{-1}(\Sigma^\ell_{1_{k+1}})$ for large enough ℓ, where $\Sigma^\ell_{1_{k+1}} \subset J^\ell$ is a smooth submanifold of codimension $n+k+1$. Even though this definition in terms of jets is an essential part of singularity theory, the more geometric functional definition we chose is simpler: one can think of $\widetilde{\Sigma}$ as a huge "surface" in $\mathcal{F} \times M$, whose local equations can be used to describe the points at which it projects badly into $\mathcal{F}$. The construction of the $\widetilde{\Sigma}_{1,k}$'s is the starting point of the so-called *Thom-Boardman stratification.*

In the case of vector fields (resp. maps), each stratum $\widetilde{\Sigma}_{1,k}$ with $k > 0$ contains dynamically different kinds of points (f, a), namely points at which 0 is a simple eigenvalue of $A := d_a f$ (resp. $A := d_a f - \mathrm{Id}$) and f is degenerate in the direction of $\mathrm{Ker}\, A$, and points at which 0 is a *geometrically simple* multiple eigenvalue of A, meaning that $\mathrm{Ker}\, A$ is a line.

The more complicated study of $\widetilde{\Sigma}_\kappa$ for $\kappa > 1$ reduces to the essentially well-known theory of singularities of real functions (in the case of potentials) or maps between spaces of the same dimension (in the dynamical cases).

2.2 Periodic orbits of maps

Assuming $\mathcal{F} = C^\infty(M, M)$, for each $f \in \mathcal{F}$, and each positive integer $p > 1$, let $f^p := f \circ \cdots \circ f$ denote the iterate of p times f (inductively, $f^0 = \mathrm{Id}$ and $f^{p+1} := f \circ f^p$). If $(f, a) \in \mathcal{F} \times M$ satisfies $f^p(a) = a$, the point a is called a *periodic point of period* p, or p*–periodic point*, of f, and $\{a, f(a), \dots, f^{p-1}(a)\}$ is the corresponding *periodic orbit of period* p, or p*–periodic orbit.* If this orbit contains exactly p points, a is called a periodic point of *minimal* period p and, of course, so is every point of the orbit.

Let $\widetilde{\Sigma}^{(p)}$ be the set of those $(\varphi, x) \in \mathcal{F} \times M$ such that x is a periodic point of minimal period p of φ, hence $\widetilde{\Sigma}^{(1)} = \widetilde{\Sigma}$. The set $\widehat{\Sigma}^{(p)}$ of those (φ, x) such that $\varphi^q(x) = \varphi^{q'}(x)$ for some pair of integers satisfying $0 \le q < q' < p$ is closed. We can now generalize Theorem 2.1.1 to periodic points of minimal period $p > 1$:

Theorem 2.2.1 *Given an integer* $p > 1$*, the subset* $\widetilde{\Sigma}^{(p)}$*, being the inverse image of* $\Delta M \subset M \times M$ *by the submersion* $(\mathcal{F} \times M) \smallsetminus \widehat{\Sigma}^{(p)} \ni (\varphi, x) \to \big(x, \varphi^p(x)\big)$*, is a smooth submanifold of codimension* n*. This submanifold is the disjoint union of smooth submanifolds* $\widetilde{\Sigma}^{(p)}_\kappa$ *of codimension* κ^2*,* $0 \le \kappa \le n$*, each of which consists of all* $(f, a) \in \widetilde{\Sigma}^{(p)}$ *such that* $d(f^p)_a - \mathrm{Id}$ *has corank* κ*. The open and dense subset* $\widetilde{\Sigma}^{(p)}_0$ *is the set of those points at which the projection* $\widetilde{\Sigma}^{(p)} \ni (\varphi, x) \mapsto \varphi \in \mathcal{F}$ *is a local diffeomorphism, whereas the hypersurface* $\widetilde{\Sigma}^{(p)}_1$ *has the following properties:*

(i) For each of its points (f, a)*, there exists a smooth family of local charts* $\big(\mathcal{F} \times M, (f, a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{R} \times \mathbf{R}^{n-1}, 0)$ *such that, near* (f, a)*, the submanifold* $\widetilde{\Sigma}^{(p)}$ *is the set of those* $(\varphi, x) \in \mathcal{F} \times M$ *at which* $(y, z) := c_\varphi(x)$ *satisfies the equations* $z = 0$ *and* $F_p(\varphi, y) = 0$*, where* $F_p : \big(\mathcal{F} \times \mathbf{R}, (f, 0)\big) \to (\mathbf{R}, 0)$ *is a smooth function with* $\partial_y F_p(f, 0) = 0$ *such that, for each integer* k *satisfying* $\partial_y F_p(f, 0) = \cdots = \partial_y^{k+1} F_p(f, 0) = 0$*, the differentials* $\partial_\varphi \partial_y^j F_p(f, 0)$ *with* $0 \le j \le k$ *are linearly independent.*

(ii) A decreasing sequence $\big(\widetilde{\Sigma}^{(p)}_{1_{k+1}}\big)_{k \in \mathbf{N}}$ *of smooth submanifolds of* $\widetilde{\Sigma}^{(p)}_1$ *can be defined inductively from* $\widetilde{\Sigma}^{(p)}_{1_1} := \widetilde{\Sigma}^{(p)}_1$ *as follows:* $\widetilde{\Sigma}^{(p)}_{1_{k+2}}$ *is the (closed) set of those points of* $\widetilde{\Sigma}^{(p)}_{1_{k+1}}$ *at which the differential of the restricted projection* $\widetilde{\Sigma}^{(p)}_{1_{k+1}} \to \mathcal{F}$ *is not injective. Thus,*

(a) *near every* $(f,a) \in \widetilde{\Sigma}_1^{(p)}$, *with the notation of (i), each* $\widetilde{\Sigma}_{1_{k+1}}^{(p)}$ *is defined by the conditions* $z = 0$ *and* $\partial_y^j F_p(\varphi, y) = 0$ *for every integer* $j \leq k+1$, *hence* $\widetilde{\Sigma}_{1_{k+1}}^{(p)}$ *has codimension* $n+k+1$ *in* $\mathcal{F} \times M$ *and, therefore, codimension* k *in* $\widetilde{\Sigma}_1^{(p)}$

(b) *in particular,* $\widetilde{\Sigma}_{1_{k+1}}^{(p)}$ *is the set of those* $(f,a) \in \widetilde{\Sigma}_1^{(p)}$ *such that the function* F_p *in (i) satisfies* $\partial_y^j F_p(f,0) = 0$ *for* $1 \leq j \leq k+1$

(c) *the closed subset* $\widetilde{\Sigma}_{1,\infty}^{(p)} = \widetilde{\Sigma}_{1_\infty}^{(p)} := \bigcap_k \widetilde{\Sigma}_{1_{k+1}}^{(p)}$, *consisting of all* $(f,a) \in \widetilde{\Sigma}_1^{(p)}$ *such that the function* F_p *introduced in (i) satisfies* $\partial_y^j F_p(f,0) = 0$ *for every integer* j, *has infinite codimension.*

(iii) *It follows that the submanifold* $\widetilde{\Sigma}_1^{(p)}$ *is the disjoint union of* $\widetilde{\Sigma}_{1,\infty}^{(p)}$ *and the submanifolds* $\widetilde{\Sigma}_{1,k}^{(p)} := \widetilde{\Sigma}_{1_{k+1}}^{(p)} \smallsetminus \widetilde{\Sigma}_{1_{k+2}}^{(p)}$, $k \in \mathbf{N}$, *each of which has codimension* k *in* $\widetilde{\Sigma}_1^{(p)}$ *and does not intersect the closure* $\widetilde{\Sigma}_{1_{\ell+1}}^{(p)}$ *of* $\widetilde{\Sigma}_{1,\ell}^{(p)}$ *for* $\ell > k$. *Moreover, for* $(f,a) \in \widetilde{\Sigma}_{1,k}^{(p)}$, $k < \infty$, *the charts* c_φ *and the function* F_p *in (i) can be chosen so that*

$$F_p(\varphi, y) = y^{k+2} + \sum_{j=1}^{k+1} \alpha_j(\varphi)\, y^{j-1},$$

where $\alpha_1, \ldots, \alpha_{k+1} : (\mathcal{F}, f) \to (\mathbf{R}, 0)$ *have independent differentials at* f.

Proof This is easier if we restrict to the open and dense subset $\mathcal{U}_p$ of $\mathcal{F} \times M$ consisting of those (f,a) such that $d(f^p)_a$ is invertible (which would be the case if $\mathcal{F}$ were the space of diffeomorphisms of M onto itself), as the map $(f,a) \mapsto j^k f^p(a)$ is[23] a submersion of $\mathcal{U}_p \smallsetminus \widehat{\Sigma}^{(p)}$ into J^k for every integer k. Therefore, the map $(f,a) \mapsto (f^p, a)$ of $\mathcal{U}_p \smallsetminus \widehat{\Sigma}^{(p)}$ into $\mathcal{F} \times M$ is evidently transversal not only to $\widetilde{\Sigma}$, but also to the submanifolds $\widetilde{\Sigma}_\kappa$ and $\widetilde{\Sigma}_{1,k}$ of Theorem 2.1.1, whose inverse images are the manifolds $\widetilde{\Sigma}_\kappa^{(p)}$ and $\widetilde{\Sigma}_{1,k}^{(p)}$. The rest of the theorem is essentially obvious in that case, the function F_p being defined from the function F of Theorem 2.1.1 by the formula $F_p(\varphi, y) = F(\varphi^p, y)$. The case $(f,a) \notin \mathcal{U}_p$ can be treated as in the proof of Theorem 2.1.1. □

2.3 The period doubling bifurcation for maps and such

As we have just seen, the set $\widetilde{\Sigma}^{(2)}$ of those $(\varphi, x) \in \mathcal{F} \times M$ such that x is a periodic point of minimal period 2 of φ, is a submanifold of codimension n. Denoting by $\mathrm{cl}(\widetilde{\Sigma}^{(2)})$ the closure of $\widetilde{\Sigma}^{(2)}$ in $\mathcal{F} \times M$, the closed subset $\widetilde{\Sigma}^{(1,2)} := \mathrm{cl}(\widetilde{\Sigma}^{(2)}) \cap \widetilde{\Sigma} = \mathrm{cl}(\widetilde{\Sigma}^{(2)}) \smallsetminus \widetilde{\Sigma}^{(2)}$ of $\widetilde{\Sigma}$ is of particular interest, being the birth place of periodic points of minimal period 2:

Proposition 2.3.1 *The closed subset* $\widetilde{\Sigma}^{(1,2)}$ *is the set of those* (f,a) *in* $\widetilde{\Sigma}$ *such that* -1 *is an eigenvalue of* $d_a f$. *It is the disjoint union of the submanifolds* $\widetilde{\Sigma}_1^{(1,2)}, \ldots, \widetilde{\Sigma}_n^{(1,2)}$ *defined as follows:* $\widetilde{\Sigma}_\kappa^{(1,2)}$ *is the set of those* $(f,a) \in \widetilde{\Sigma}$ *such that* $d_a f + \mathrm{Id}$ *has corank* κ. *In particular,* $\widetilde{\Sigma}_1^{(1,2)}$ *is a hypersurface*[24] *of* $\widetilde{\Sigma}$.

[23]Because f varies near $f^{p-1}(a)$ independently from its variations near the $f^j(a)$'s with $0 \leq j < p-1$ and, moreover, there is no constraint on its higher order jet since the differential of f at every point of the orbit is invertible.

[24]As before, it should be noted that it contains points at which -1 is a multiple but *geometrically simple* eigenvalue of $d_a f$. This makes a huge difference dynamically, but not from the limited viewpoint adopted here.

Proof We use the following obvious result:

Lemma 2.3.2 *For each $(f,a)\in\widetilde\Sigma$ and each chart $c:(M,a)\to(\mathbf{R}^n,0)$, one defines a smooth local map $G:\big(\mathcal{F}\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{S}^{n-1},(f,0,0)\times\mathbf{S}^{n-1}\big)\to\mathbf{R}^{2n}$ whose partial derivative $\partial_\varphi G(f,0,0,u)$ is onto for all $u\in\mathbf{S}^{n-1}$ by the formula $G(\varphi,x,r,u):=\big(c_*\varphi(x)-(x+ru),G^{(1)}(\varphi,x,r,u)+u\big)$, where*

$$G^{(1)}(\varphi,x,r,u):=\begin{cases}r^{-1}\big(c_*\varphi(x+ru)-c_*\varphi(x)\big) & \text{for } r\neq 0,\\ D(c_*\varphi)(x)u & \text{if } r=0.\end{cases}$$

Under the hypotheses of Lemma 2.3.2, if the sequence (φ_m,a_m) in $\widetilde\Sigma^{(2)}$ converges to (f,a), then, for large m, both a_m and $\varphi_m(a_m)$ lie in $\operatorname{dom} c$. Setting $x_m:=c(a_m)$ and $x_m+r_mu_m:=c\big(\varphi_m(a_m)\big)$ with $r_m>0$ and $u_m\in\mathbf{S}^{n-1}$, the relation $(\varphi_m,a_m)\in\widetilde\Sigma^{(2)}$ reads $G(\varphi_m,x_m,r_m,u_m)=0$. Replacing (φ_m,a_m) by a subsequence, we may assume that (u_m) converges to some $u\in\mathbf{S}^{n-1}$, in which case (φ_m,x_m,r_m,u_m) converges to $(f,0,0,u)$, implying that $G(f,0,0,u)=0$, hence $D(c_*f)(0)u=-u$, i.e. $d_af(dc_a^{-1}u)=-dc_a^{-1}u$.

Conversely, still under the hypotheses of Lemma 2.3.2, assume that there exists $u_0\in\mathbf{S}^{n-1}$ such that $D(c_*f)(0)u_0=-u_0$, i.e. $G(f,0,0,u_0)=0$. As the partial derivative $\partial_\varphi G(f,0,0,u_0)$ is onto, the implicit function theorem implies that, arbitrarily close to $(f,0,0,u_0)$, there exist points (φ,x,r,u) with $r\neq 0$ such that $G(\varphi,x,r,u)=0$, i.e. $c_*\varphi(x)=x+ru$ and $c_*\varphi(x+ru)=x$, hence $\big(\varphi,c^{-1}(x)\big)\in\widetilde\Sigma^{(2)}$, proving the first assertion of Proposition 2.3.1.

The rest follows from the fact that $\widetilde\Sigma^{(1,2)}=(j^1)^{-1}(\Sigma^{(1,2)})$ and $\widetilde\Sigma^{(1,2)}_\kappa=(j^1)^{-1}(\Sigma^{(1,2)}_\kappa)$, where $\Sigma^{(1,2)}$ (resp. $\Sigma^{(1,2)}_\kappa$) is the set of those $j^1f(a)\in J^1(M,M)$ such that $f(a)=a$ and -1 is an eigenvalue of d_af (resp. and $d_af+\mathrm{Id}$ has corank κ). Clearly, $\Sigma^{(1,2)}$ is the union of the $\Sigma^{(1,2)}_\kappa$'s, which, as in the proof of Proposition 1.2.1, are submanifolds, the first of which has codimension $n+1$. □

Remarks Basically, the idea of this proof is that every tangent is the limit of secants. It can be used ([6], Appendice 4) to "fill in the gap over the diagonal" in the spaces ${}_2J^k(M,N):=\big\{\big(j^kf(a),j^kg(b)\big)\in J^k(M,N)^2:a\neq b\big\}$ of bijets of mappings of M into a manifold N, providing an algebraic description of the relationship between conflict and bifurcation, in Thom's terminology. The case of general multijets is not fully understood.

Proposition 2.3.1 is typically simpler in our functional setting than for a particular deformation, which might involve an exotic "slice" of the general situation.

Under the hypothesis of Lemma 2.3.2, near (f,a), the subset $\mathrm{cl}(\widetilde\Sigma^{(2)})$ is the image of the smooth submanifold $G^{-1}(0)$ by the projection $(\varphi,x,r,u)\mapsto\big(\varphi,c^{-1}(x)\big)$. Here are cases where this projected set is itself a smooth manifold:

Theorem 2.3.3 *The intersection of $\widetilde\Sigma^{(1,2)}_1$ with $\widetilde\Sigma_0$ has the following properties:*

(i) Near every $(f,a)\in\widetilde\Sigma^{(1,2)}_1\cap\widetilde\Sigma_0$, the subset $\mathrm{cl}(\widetilde\Sigma^{(2)})$ is a smooth n–codimensional submanifold of $\mathcal{F}\times M$ intersecting $\widetilde\Sigma$ at $\widetilde\Sigma^{(1,2)}_1$. More precisely, there exists a smooth family of local charts $\big(\mathcal{F}\times M,(f,a)\big)\ni(\varphi,x)\mapsto c_\varphi(x)\in(\mathbf{R}\times\mathbf{R}^{n-1},0)$ such that, near (f,a),

(a) $\widetilde\Sigma$ is the set of those (φ,x) such that $c_\varphi(x)=0$

(b) $\mathrm{cl}(\widetilde\Sigma^{(2)})$ is the set of those (φ,x) such that $(y,z):=c_\varphi(x)$ satisfies the equations $z=0$ and $F(\varphi,y)=0$, where $F:\big(\mathcal{F}\times\mathbf{R},(f,0)\big)\to(\mathbf{R},0)$ is a smooth local function satisfying $\partial_\varphi F(f,0)\neq 0$ and $F(\varphi,-y)=F(\varphi,y)$

(c) *in* $\operatorname{dom} c_\varphi$*, the restriction of* φ *to the set of its* 2*–periodic points is given by the formula* $c_{\varphi *}\varphi(y,0) = (-y,0)$

(d) *for each integer* k *satisfying* $\partial_y F(f,0) = \cdots = \partial_y^{2k+1} F(f,0) = 0$*, the differentials* $\partial_\varphi \partial_y^{2j} F(f,0)$ *with* $0 \le j \le k$ *are linearly independent.*

(ii) *A decreasing sequence* $\big(\widetilde{\Sigma}^{(1,2)}_{1_{k+1}}\big)_{k\in\mathbf{N}}$ *of smooth submanifolds of* $\widetilde{\Sigma}^{(1,2)}_1 \cap \widetilde{\Sigma}_0 =: \widetilde{\Sigma}^{(1,2)}_{1_1}$ *can be defined as follows:* $\widetilde{\Sigma}^{(1,2)}_{1_{k+2}}$ *is the (closed) set of those points of* $\widetilde{\Sigma}^{(1,2)}$ *which lie in the closure of the subset* $\widetilde{\Sigma}^{(2)}_{1_{k+1}}$ *of Theorem 2.2.1. Thus,*

(a) *near every* $(f,a) \in \widetilde{\Sigma}^{(1,2)}_1$*, with the notation of (i),*

– *each* $\widetilde{\Sigma}^{(2)}_{1_{k+1}}$ *is defined by the conditions* $z = 0$*,* $y \ne 0$ *and* $\partial_y^j F(\varphi, y) = 0$ *for every integer* $j \le k+1$

– *each* $\widetilde{\Sigma}^{(1,2)}_{1_{k+1}}$ *is defined by the conditions* $c_\varphi(x) = 0$ *and* $\partial_y^{2j} F_p(\varphi,0) = 0$ *for every integer* $j \le k$*, hence* $\widetilde{\Sigma}^{(1,2)}_{1_{k+1}}$ *has codimension* $n+k+1$ *in* $\mathcal{F}\times M$ *and, therefore, codimension* k *in* $\widetilde{\Sigma}^{(1,2)}$

(b) *in particular,* $\widetilde{\Sigma}^{(1,2)}_{1_{k+1}}$ *is the set of those* $(f,a) \in \widetilde{\Sigma}^{(1,2)}_1$ *such that the function* F *in (i) satisfies* $\partial_y^{2j} F(f,0) = 0$ *for* $1 \le j \le k$

(c) *the closed subset* $\widetilde{\Sigma}^{(1,2)}_{1,\infty} = \widetilde{\Sigma}^{(1,2)}_{1_\infty} := \bigcap_k \widetilde{\Sigma}^{(1,2)}_{1_{k+1}}$*, consisting of all* $(f,a) \in \widetilde{\Sigma}^{(1,2)}_1$ *such that the function* F *introduced in (i) satisfies* $\partial_y^j F(f,0) = 0$ *for every integer* j*, has infinite codimension.*

(iii) *It follows that the submanifold* $\widetilde{\Sigma}^{(1,2)}_1$ *is the disjoint union of the subset* $\widetilde{\Sigma}^{(1,2)}_{1,\infty}$ *and the submanifolds* $\widetilde{\Sigma}^{(1,2)}_{1,k} := \widetilde{\Sigma}^{(1,2)}_{1_{k+1}} \smallsetminus \widetilde{\Sigma}^{(1,2)}_{1_{k+2}}$*,* $k \in \mathbf{N}$*, each of which has codimension* k *in* $\widetilde{\Sigma}^{(1,2)}_1$ *and does not intersect the closure* $\widetilde{\Sigma}^{(1,2)}_{1_{\ell+1}}$ *of* $\widetilde{\Sigma}^{(1,2)}_{1,\ell}$ *for* $\ell > k$*. Moreover, for* $(f,a) \in \widetilde{\Sigma}^{(1,2)}_{1,k}$*,* $k < \infty$*, the function* F *in (i) can be chosen of the form*

$$F(\varphi, y) = y^{2k+2} + \sum_{j=0}^{k} \alpha_j(\varphi)\, y^{2j},$$

where $\alpha_0, \ldots, \alpha_k : (\mathcal{F}, f) \to (\mathbf{R}, 0)$ *are smooth local functions whose differentials at* f *are independent.*

Proof (i) Since (f,a) belongs to $\widetilde{\Sigma}_0$, the submanifold $\widetilde{\Sigma}$, near (f,a), is the graph of an implicit function $x = \Phi(\varphi)$. If $c : (M,a) \to (\mathbf{R}^n, 0)$ is a local chart, the formula $\chi_\varphi(x) := c(x) - c\big(\Phi(\varphi)\big)$ defines a family of local charts such that $\widetilde{\Sigma}$, near (f,a), is the set of those (φ, x) satisfying $\chi_\varphi(x) = 0$.

It follows that, near (f,a), the equation $\varphi\circ\varphi(x) = x$ defining $\widetilde{\Sigma} \cup \operatorname{cl}(\widetilde{\Sigma}^{(2)})$ can be written $(\chi_{\varphi *}\varphi)\circ(\chi_{\varphi *}\varphi)(\xi) - \xi = 0$ with $\xi = \chi_\varphi(x)$, that is

$$\big(A(\varphi,\xi) - I_n\big)\xi = 0\,, \quad \text{where} \quad A(\varphi,\xi) := \int_0^1 D\big((\chi_{\varphi *}\varphi)\circ(\chi_{\varphi *}\varphi)\big)(t\xi)\, dt. \tag{3}$$

As (f,a) belongs to $\widetilde{\Sigma}_0$, the kernel of $A(f,0)-I_n = \big(D(c_*f)(0)-I_n\big)\circ\big(D(c_*f)(0)+I_n\big)$ is $\operatorname{Ker}\big(D(c_*f)(0)+I_n\big)$, which has dimension 1 since (f,a) belongs to $\widetilde{\Sigma}_1^{(1,2)}$. Hence, there exist invertible matrices Q_1, Q_2 such that

$$Q_2\big(A(f,0)-I_n\big)Q_1^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

and therefore a smooth local map $P : \big(\mathcal{F}\times\mathbf{R}^n,(f,0)\big)\to\big(\mathbf{GL}(n,\mathbf{R}),I_n\big)$ such that

$$P(\varphi,\xi)Q_2\big(A(\varphi,\xi)-I_n\big)Q_1^{-1}P(\varphi,\xi)^{-1} = \begin{pmatrix} \lambda(\varphi,\xi) & 0 \\ 0 & B(\varphi,\xi) \end{pmatrix} \tag{4}$$

with $\lambda(\varphi,\xi)\in\mathbf{R}$ and $B(\varphi,\xi)\in\mathbf{GL}(n-1,\mathbf{R})$. Thus, the formula

$$h_\varphi(\xi) := P(\varphi,\xi)Q_1\xi$$

defines a smooth family $\big(\mathcal{F}\times\mathbf{R}^n,(f,0)\big)\to(\mathbf{R}\times\mathbf{R}^{n-1},0)$ of local diffeomorphisms such that, setting $(y,z) := h_\varphi(\xi)$, equation (3) reads

$$\begin{array}{rcl} \lambda\big(\varphi,h_\varphi^{-1}(y,z)\big)\,y & = & 0 \\ B\big(\varphi,h_\varphi^{-1}(y,z)\big)\,z & = & 0. \end{array} \tag{5}$$

As $B\big(\varphi,h_\varphi^{-1}(y,z)\big)$ is invertible, the second equation is equivalent to $z=0$. Therefore, setting

$$F_0(\varphi,y) := \lambda\big(\varphi,h_\varphi^{-1}(y,0)\big)\,,$$

(5) writes

$$\begin{array}{rcl} y\,F_0(\varphi,y) & = & 0 \\ z & = & 0. \end{array} \tag{6}$$

This proves (i) with $F=F_0$ and $c_\varphi = c_{0\varphi} := h_\varphi\circ\chi_\varphi$ if we forget about (c) and the condition $F(\varphi,-y)=F(\varphi,y)$. Indeed, near (f,a), the obvious solution $(y,z)=0$ of (6) defines $\widetilde{\Sigma}$, whereas $\widetilde{\Sigma}^{(2)}$ is defined by the conditions $z=0$, $y\neq 0$ and $F_0(\varphi,y)=0$. Now, $\partial_\varphi F_0(f,0)$is nonzero, as it is the derivative at f of the eigenvalue $\lambda(\varphi,0)$ of $Q_2\big(A(\varphi,0)-I_n\big)Q_1^{-1}$. Moreover, by Proposition 2.3.1, there exists a sequence (φ_m,y_m,δ_m) in $\mathcal{F}\times(\mathbf{R}^*)^2$, converging to $(f,0,0)$, such that $\varphi_m\big(c_{0\varphi}^{-1}(y_m,0)\big) = c_{0\varphi}^{-1}(y_m+\delta_m,0)$ and $\varphi_m\big(c_{0\varphi}^{-1}(y_m+\delta_m,0)\big) = c_{0\varphi}^{-1}(y_m,0)$, hence $F_0(\varphi_m,y_m)=F_0(\varphi_m,y_m+\delta_m)=0$ and

$$0 = \frac{F_0(\varphi_m,y_m+\delta_m)-F_0(\varphi_m,y_m)}{\delta_m} = \int_0^1 \partial_y F_0(\varphi_m,y_m+t\delta_m)\,dt\,,$$

yielding $\partial_y F_0(f,0)=0$ at the limit. It does follow that, near (f,a), the subset $\mathrm{cl}(\widetilde{\Sigma}^{(2)})$ is the set of those $\big(\varphi,c_{0\varphi}^{-1}(y,0)\big)$ such that $F_0(\varphi,y)=0$, and therefore a smooth n–codimensional submanifold of $\mathcal{F}\times M$ intersecting $\widetilde{\Sigma}$ at $\widetilde{\Sigma}_1^{(1,2)}$ (this last point is due to Proposition 2.3.1 and the fact that $\widetilde{\Sigma}_1^{(1,2)}$ is open in $\widetilde{\Sigma}^{(1,2)}$).

To modify the family $c_{0\varphi}$ and F_0 so that (c) and the condition $F(\varphi,-y)=F(\varphi,y)$ hold, we observe that $\mathrm{cl}(\widetilde{\Sigma}^{(2)})$ is equipped with the involution $\tau : (\varphi,x)\mapsto\big(\varphi,\varphi(x)\big)$, whose fixed point set is $\mathrm{cl}(\widetilde{\Sigma}^{(2)})\cap\widetilde{\Sigma} = \mathrm{cl}(\widetilde{\Sigma}^{(2)})\smallsetminus\widetilde{\Sigma}^{(2)}$.

Let us first assume that $M = \mathbf{R}^n$, $a = 0$ and $c = \mathrm{Id}$. Then, if we choose a complement L of the closed hyperplane $K := \mathrm{Ker}\,\partial_\varphi F_0(f,0)$ in the vector space $\mathcal{F} = T_f\mathcal{F}$, we can write each $\varphi \in \mathcal{F}$ in a unique fashion as $f + \varphi_1 + \varphi_2$ with $\varphi_1 \in K$ and $\varphi_2 \in L$. Thus, the equation $F_0(\varphi, y) = 0$ defines near $(f,0)$ the graph of an implicit function $\varphi_2 = \Psi(\varphi_1, y)$ and we can take φ_1, y as local coordinates on $\mathrm{cl}(\widetilde{\Sigma}^{(2)})$. In these coordinates, the involution τ reads $(\varphi_1, y) \mapsto \big(\varphi_1, \tau_{\varphi_1}(y)\big)$ with $\tau_{\varphi_1}(0) = 0$ and $\Psi\big(\varphi_1, \tau_{\varphi_1}(y)\big) = \Psi(\varphi_1, y)$. As τ is not the identity near (f,a), we have that $\tau'_{\varphi_1}(0) = -1$, implying that $H : (\varphi_1, y) \mapsto \big(\varphi_1, \frac{1}{2}\big(y - \tau_{\varphi_1}(y)\big)\big) =: \big(\varphi_1, H_{\varphi_1}(y)\big)$ is a local diffeomorphism tangent to the identity at $(0,0)$. Since[25] $H_{\varphi_1}\big(\tau_{\varphi_1}(y)\big) = -H_{\varphi_1}(y)$, we get the required properties if $c_\varphi \circ c_{0,\varphi}^{-1}(y,z) := \big(H_{\varphi_1}(y), z\big)$ and

$$F(\varphi, y) := \varphi_2 - \Psi \circ H^{-1}(\varphi_1, y)\,.$$

The general case is along the same lines (as usual, the fact that c_φ and $F_0(\varphi, y)$ are determined by the local map $c_*\varphi$ and not by the whole of φ makes the proof rather indifferent to the topology of $\mathcal{F}$). Recall that $T_f\mathcal{F}$ is the space of smooth vector fields over f, i.e. smooth sections of the vector bundle f^*TM over M whose fibre over x is $T_{f(x)}M$. Given a complete Riemanian metric on M, one defines a smooth local diffeomorphism $\exp_f : (T_f\mathcal{F}, 0) \to (\mathcal{F}, f)$, tangent to $\mathrm{Id}_{T_f\mathcal{F}}$ at 0, by $(\exp_f X)(x) := \exp_{f(x)} X(x)$. Choosing a complement L of $K := \mathrm{Ker}\,\partial_\varphi F_0(f,0)$ in the vector space $T_f\mathcal{F}$, each $\varphi \in \mathcal{F}$ close enough to f can be written in a unique fashion as $\exp_f(\varphi_1 + \varphi_2)$ with $\varphi_1 \in K$ and $\varphi_2 \in L$. We then proceed as before.

Proof of (ii). As $F(\varphi, -y) = F(\varphi, y)$, the differentiable Newton theorem (with parameter φ) implies that there exists a smooth local function $\bar{F} : \big(\mathcal{F} \times \mathbf{R}, (f,0)\big) \to (\mathbf{R}, 0)$ such that $F(\varphi, y) = \bar{F}(\varphi, y^2)$. If (φ_n, x_n) is a sequence in $\widetilde{\Sigma}^{(2)}_{1_{k+1}}$ tending to (f,a), then $c_{\varphi_n}(x_n) = (0, y_n)$ with $\partial_y^j F(\varphi_n, y_n) = 0$ for $0 \le j \le k+1$. In other words, $\partial_y^j \bar{F}(\varphi_n, y_n^2) = 0$ for $0 \le j \le k+1$, hence, inductively, $\partial_2^j \bar{F}(\varphi_n, y_n^2) = 0$ for $0 \le j \le k+1$ and therefore $\partial_2^j \bar{F}(f,0) = 0$ for $0 \le j \le k+1$, yielding $\partial_y^j F(f,0) = 0$ for $0 \le j \le 2k+2$. Conversely, if this is the case, then, as the differentials $\partial_\varphi \partial_2^j \bar{F}(f,0)$ with $0 \le j \le k$ are independent by (i), the differentials $d\partial_2^j \bar{F}(f,0)$ with $0 \le j \le k+1$ are independent both if $\partial_2^{k+2} \bar{F}(\varphi_n, y_n^2) = 0$ and otherwise, hence the equations $z = 0$ and $\partial_2^j \bar{F}(f,y) = 0$ with $0 \le j \le k+1$ define a $(k+2)$–codimensional submanifold, which must have other points in every neighbourhood of (f,a).

Proof of (iii). As in the proof of Theorem 2.1.1, if $\bar{y} \mapsto \bar{F}(f, \bar{y})$ vanishes exactly at order k at 0, applying the preparation theorem to $\bar{F}$, we may assume $F(\varphi, y) = y^{2k+2} + \alpha_k(\varphi)\, y^{2k} + \cdots + \alpha_0(\varphi)$, where $\alpha_0, \ldots, \alpha_k : (\mathcal{F}, f) \to (\mathbf{R}, 0)$ are smooth local functions, hence (a)—the fact that the linear maps $d\alpha_j(f,0)$ are independent can be obtained as in the proof of Theorem 2.1.1. □

Remark I do not know if the structure of $\mathrm{cl}(\widetilde{\Sigma}^{(2)})$ near $\widetilde{\Sigma}^{(1,2)}_\kappa$ is understood for $\kappa > 1$.

Proposition 2.3.1 and Theorem 2.3.3 generalize immediately from fixed points to periodic orbits:

Proposition 2.3.4 *Denoting by* $\mathrm{cl}(\widetilde{\Sigma}^{(2p)})$ *the closure of* $\widetilde{\Sigma}^{(2p)}$ *in* $\mathcal{F} \times M$*, the closed subset* $\widetilde{\Sigma}^{(p,2p)} := \mathrm{cl}(\widetilde{\Sigma}^{(2p)}) \cap \widetilde{\Sigma}^{(p)}$ *of* $\widetilde{\Sigma}^{(p)}$ *is the set of those* $(f,a) \in \widetilde{\Sigma}^{(p)}$ *such that* -1 *is an eigenvalue of* $d_a f^p$*. It is the disjoint union of the submanifolds* $\widetilde{\Sigma}^{(p,2p)}_\kappa$*,* $1 \le \kappa \le n$*, each of which is the set of those* $(f,a) \in \widetilde{\Sigma}^{(p)}$ *such that* $d_a f^p + \mathrm{Id}$ *has corank* κ*. Thus,* $\widetilde{\Sigma}^{(p,2p)}_1$ *is a hypersurface of* $\widetilde{\Sigma}^{(p)}$*.*

[25] A particular case of Bochner's linearization theorem for compact Lie group actions near a fixed point [6].

Theorem 2.3.5 *The intersection of* $\widetilde{\Sigma}_1^{(p,2p)}$ *with* $\widetilde{\Sigma}_0^{(p)}$ *has the following properties:*

(i) Near every $(f,a) \in \widetilde{\Sigma}_1^{(p,2p)} \cap \widetilde{\Sigma}_0^{(p)}$*, the subset* $\mathrm{cl}(\widetilde{\Sigma}^{(2p)})$ *is a smooth* n*–codimensional submanifold of* $\mathcal{F} \times M$ *intersecting* $\widetilde{\Sigma}^{(p)}$ *at* $\widetilde{\Sigma}_1^{(p,2p)}$*. More precisely, there exists a smooth family* $\big(\mathcal{F} \times M, (f,a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{R} \times \mathbf{R}^{n-1}, 0)$ *of local charts such that, near* (f,a)*,*

- *(a)* $\widetilde{\Sigma}^{(p)}$ *is the set of those* (φ, x) *such that* $c_\varphi(x) = 0$
- *(b)* $\mathrm{cl}(\widetilde{\Sigma}^{(2p)})$ *is the set of those* (φ, x) *such that* $(y,z) := c_\varphi(x)$ *satisfies the equations* $z = 0$ *and* $F(\varphi, y) = 0$*, where* $F : \big(\mathcal{F} \times \mathbf{R}, (f,0)\big) \to (\mathbf{R}, 0)$ *is a smooth local function satisfying* $\partial_\varphi F(f,0) \neq 0$ *and* $F(\varphi, -y) = F(\varphi, y)$
- *(c) in* $\mathrm{dom}\, c_\varphi$*, the restriction of* φ^p *to the set of its* 2*–periodic points is given by the formula* $c_{\varphi *}\varphi^p(y, 0) = (-y, 0)$
- *(d) for each integer* k *satisfying* $\partial_y F(f,0) = \cdots = \partial_y^{2k+1} F(f,0) = 0$*, the differentials* $\partial_\varphi \partial_y^{2j} F(f,0)$ *with* $0 \leq j \leq k$ *are linearly independent.*

(ii) A decreasing sequence $\big(\widetilde{\Sigma}_{1_{k+1}}^{(p,2p)}\big)_{k \in \mathbf{N}}$ *of smooth submanifolds of* $\widetilde{\Sigma}_1^{(p,2p)} \cap \widetilde{\Sigma}_0^{(p)} =: \widetilde{\Sigma}_{1_1}^{(p,2p)}$ *can be defined as follows:* $\widetilde{\Sigma}_{1_{k+2}}^{(p,2p)}$ *is the (closed) set of those points of* $\widetilde{\Sigma}^{(p,2p)}$ *which lie in the closure of the subset* $\widetilde{\Sigma}_{1_{k+1}}^{(2p)}$ *of Theorem 2.2.1. Thus,*

- *(a) near every* $(f,a) \in \widetilde{\Sigma}_1^{(p,2p)}$*, with the notation of (i),*
  - *each* $\widetilde{\Sigma}_{1_{k+1}}^{(2p)}$ *is defined by the conditions* $z = 0$*,* $y \neq 0$ *and* $\partial_y^j F(\varphi, y) = 0$ *for every integer* $j \leq k+1$
  - *each* $\widetilde{\Sigma}_{1_{k+1}}^{(p,2p)}$ *is defined by the conditions* $c_\varphi(x) = 0$ *and* $\partial_y^{2j} F_p(\varphi, y) = 0$ *for every integer* $j \leq k$*, hence* $\widetilde{\Sigma}_{1_{k+1}}^{(p,2p)}$ *has codimension* $n+k+1$ *in* $\mathcal{F} \times M$ *and, therefore, codimension* k *in* $\widetilde{\Sigma}^{(p,2p)}$
- *(b) in particular,* $\widetilde{\Sigma}_{1_{k+1}}^{(p,2p)}$ *is the set of those* $(f,a) \in \widetilde{\Sigma}_1^{(p,2p)}$ *such that the function* F *in (i) satisfies* $\partial_y^{2j} F(f,0) = 0$ *for* $1 \leq j \leq k$
- *(c) the closed subset* $\widetilde{\Sigma}_{1,\infty}^{(p,2p)} = \widetilde{\Sigma}_{1_\infty}^{(p,2p)} := \bigcap_k \widetilde{\Sigma}_{1_{k+1}}^{(p,2p)}$*, consisting of all* $(f,a) \in \widetilde{\Sigma}_1^{(p,2p)}$ *such that the function* F *introduced in (i) satisfies* $\partial_y^j F(f,0) = 0$ *for every integer* j*, has infinite codimension.*

(iii) It follows that the submanifold $\widetilde{\Sigma}_1^{(p,2p)}$ *is the disjoint union of the subset* $\widetilde{\Sigma}_{1,\infty}^{(p,2p)}$ *and the submanifolds* $\widetilde{\Sigma}_{1,k}^{(p,2p)} := \widetilde{\Sigma}_{1_{k+1}}^{(p,2p)} \smallsetminus \widetilde{\Sigma}_{1_{k+2}}^{(p,2p)}$*,* $k \in \mathbf{N}$*, each of which has codimension* k *in* $\widetilde{\Sigma}_1^{(p,2p)}$ *and does not intersect the closure* $\widetilde{\Sigma}_{1_{\ell+1}}^{(p,2p)}$ *of* $\widetilde{\Sigma}_{1,\ell}^{(p,2p)}$ *for* $\ell > k$*. Moreover, for* $(f,a) \in \widetilde{\Sigma}_{1,k}^{(p,2p)}$*,* $k < \infty$*, the function* F *in (i) can be chosen of the form*

$$F(\varphi, y) = y^{2k+2} + \sum_{j=0}^{k} \alpha_j(\varphi)\, y^{2j},$$

where $\alpha_0, \ldots, \alpha_k : (\mathcal{F}, f) \to (\mathbf{R}, 0)$ *are smooth local functions whose differentials at* f *are independent.*

Example (the period doubling bifurcation) It occurs at points of the "big stratum" $\widetilde{\Sigma}_{1,0}^{(p,2p)}$ of $\widetilde{\Sigma}_1^{(p,2p)} \cap \widetilde{\Sigma}_0^{(p)}$. For each $(f,a) \in \widetilde{\Sigma}_{1,0}^{(p,2p)}$, we are in the situation of Theorem 2.3.5 (iii) with $k=0$. Therefore, the equation $F(\varphi,y)=0$ writes $y^2+\alpha_0(\varphi)=0$, implying that, besides the fixed point $c_\varphi^{-1}(0)$, the map φ^p has the 2–periodic orbit $\{c_\varphi^{-1}(-\sqrt{-\alpha_0(\varphi)},0), c_\varphi^{-1}(\sqrt{-\alpha_0(\varphi)},0)\}$ for $\alpha_1(\varphi)<0$ and none for $\alpha_1(\varphi)>0$. Thus, if $s \mapsto \varphi_s$ is a smooth path $(\mathbf{R},0) \to (\mathcal{F},a)$, generic in the sense that $\frac{d}{ds}\alpha_1(\varphi_s)|_{s=0}$ is nonzero, then, according to its sign, one observes for $s=0$ the birth or death at a of a $2p$–periodic point of φ_s. This is the *period doubling bifurcation.*

2.4 Periodic orbits of flows

Assuming $\mathcal{F}=C^\infty(TM)$, for each $X \in \mathcal{F}$, we denote by $g_X=(g_X^t)$ the flow of X, defined by the fact $g_X^t(x)$ is the value at time t of the integral curve of X which passes through x at time 0. If $(X,a) \in \mathcal{F} \times M$ satisfies $g_X^T(a)=a$ for some positive T, the point a is called a *periodic point of period* T of X (or g_X), and $\{g_X^t(a)\}$ is the corresponding *periodic orbit of period* T, or T*–periodic orbit.* If $t \mapsto g_X^t(a)$ is injective on $[0,T)$, then a is called a periodic point of *minimal* period T and, of course, so is every point of the orbit. We assume $n := \dim M > 1$.

Proposition 2.4.1 *The set* $\mathrm{Per}(M)$ *of those* $(\xi,x,\tau) \in \mathcal{F} \times M \times \mathbf{R}_+^*$ *such that* x *is a periodic point of* g_ξ *with* minimal *period* τ *is a smooth submanifold of codimension* n*, endowed with the free smooth action* ρ *of* $\mathbf{T}=\mathbf{R}/\mathbf{Z}$ *given by* $\rho^\theta(\xi,x,\tau) := (\xi, g_\xi^{\tau\theta}(x),\tau)$*. The canonical projection* $\mathcal{F} \times M \times \mathbf{R}_+^* \to \mathcal{F} \times M$*, restricted to* $\mathrm{Per}(M)$*, is an injective immersion whose image* $\mathrm{per}(M)$ *consists of all* $(\xi,x) \in \mathcal{F} \times M \smallsetminus \widetilde{\Sigma}$ *such that* x *is a periodic point of* ξ.[26]

Proof We should prove that, for each $(X,a,T) \in \mathrm{Per}(M)$,

a) the mapping $\rho : (\xi,x,\tau) \mapsto (x, g_\xi^\tau(x))$ is transversal to ΔM at (X,a,T), hence $\rho^{-1}(\Delta M)$ is a smooth n–codimensional submanifold near (X,a,T)

b) there is an open subset $\Omega \ni (X,a,T)$ of $\mathcal{F} \times M \times \mathbf{R}_+^*$ such that $\Omega \cap \rho^{-1}(\Delta M) = \Omega \cap \mathrm{Per}(M)$.

Both assertions are standard [2]: (a) comes from the fact $\partial_\xi g_\xi^T(a)|_{\xi=X} : T_X\mathcal{F} \to T_aM$ is onto, whereas (b) can be proved by contradiction: assuming that every neighbourhood of (X,a,T) contains points (ξ,x,τ) with $g_\xi^{\tau/p}(x)=x$ for some integer $p>1$, either such p are bounded, in which case T is not the minimal period of a for X, or p must be unbounded, in which case $X_a=0$. □

Here is the generalisation of Theorem 2.2.1 to flows:

Theorem 2.4.2 *The submanifold* $\mathrm{Per}(M)$ *is the disjoint union of smooth* ρ*–invariant submanifolds* $\mathrm{Per}(M)_\kappa$ *of codimension* κ^2*,* $0 \le \kappa \le n-1$*, each of which consists of all* $(X,a,T) \in \mathrm{Per}(M)$ *such that* $d(g_X^T)_a - \mathrm{Id}$ *has corank* $\kappa+1$*. The open and dense subset* $\mathrm{Per}(M)_0$ *is the union of those orbits of* ρ *at which the projection* $\mathrm{Per}(M) \ni (\xi,x,\tau) \mapsto \xi \in \mathcal{F}$ *is a local fibration*[27]*, whereas the hypersurface* $\mathrm{Per}(M)_1$ *has the following properties:*

[26] Thus, $\mathrm{per}(M)$ is equipped with the action of the circle defined by $\rho^\theta(\xi,x) := (\xi, g_\xi^{\tau(\xi,x)\theta}(x))$, where $\tau(\xi,x)$ is the minimal period of x for ξ. As $\mathrm{per}(M)$ is definitely *not* a submanifold, what follows is only a first exploration of its very intricate topology. As a rule, it is safer to look for statements about $\mathrm{Per}(M)$ before "going down".

[27] A *principal* fibration, as its fibres are the orbits of ρ.

(i) For each $(X,a,T)\in \mathrm{Per}(M)_1$ *and each smooth hypersurface*[28] $S\ni a$ *of* M *with* $X_a\notin T_aS$, *there is a smooth family* $\big(\mathcal F\times M,(X,a)\big)\ni(\xi,x)\mapsto c_\xi(x)\in(\mathbf R\times\mathbf R^{n-2}\times\mathbf R,0)$ *of local charts such that*

(a) $c_{\xi*}\xi=(0,0,1)$ *and* $c_\xi(S\cap\operatorname{dom}c_\xi)=\operatorname{Im}c_\xi\cap(\mathbf R\times\mathbf R^{n-2}\times\{0\})$

(b) near (X,a,T), *the submanifold* $\mathrm{Per}(M)$ *is the set of those* $(\xi,x,\tau)\in\mathcal F\times M\times\mathbf R_+^*$ *at which* $(y,z,w):=c_\xi(x)$ *and* τ *satisfy the equations* $\tau=\tau_0(\xi,y,z)$, $z=0$ *and* $G(\xi,y)=0$, *where* $\tau_0:\big(\mathcal F\times\mathbf R^{n-1},(X,0)\big)\to(\mathbf R,0)$ *and* $G:\big(\mathcal F\times\mathbf R,(X,0)\big)\to(\mathbf R,0)$ *are smooth functions as follows:*

(*) $\tau_0(\xi,y,z)$ *is the smallest positive* t *such that* $g_\xi^t\big(c_\xi^{-1}(y,z,0)\big)\in S$ ("first return time")

(**) $\partial_yG(X,0)=0$ *and, for each* $k\in\mathbf N$ *such that* $\partial_yG(X,0)=\cdots=\partial_y^{k+1}G(X,0)=0$, *the differentials* $\partial_\xi\partial_y^jG(f,0)$ *with* $0\le j\le k$ *are linearly independent.*

(ii) A decreasing sequence $\big(\mathrm{Per}(M)_{1_{k+1}}\big)_{k\in\mathbf N}$ *of smooth* ρ*–invariant submanifolds of* $\mathrm{Per}(M)_1$ *can be defined inductively from* $\mathrm{Per}(M)_{1_1}:=\mathrm{Per}(M)_1$ *as follows:* $\mathrm{Per}(M)_{1_{k+2}}$ *is the (closed) set of those points of* $\mathrm{Per}(M)_{1_{k+1}}$ *at which the kernel of the differential of the restricted projection* $\mathrm{Per}(M)_{1_{k+1}}\to\mathcal F$ *is not the tangent space of the orbit of* ρ. *Thus,*

(a) near every $(f,a)\in\mathrm{Per}(M)_1$, *with the notation of (i), each* $\mathrm{Per}(M)_{1_{k+1}}$ *is defined by the conditions* $z=0$ *and* $\partial_y^jG(\varphi,y)=0$ *for every integer* $j\le k+1$, *hence* $\mathrm{Per}(M)_{1_{k+1}}$ *has codimension* $n+k+1$ *in* $\mathcal F\times M\times\mathbf R_+^*$ *and, therefore, codimension* k *in* $\mathrm{Per}(M)_1$

(b) in particular, $\mathrm{Per}(M)_{1_{k+1}}$ *is the set of those* $(X,a,T)\in\mathrm{Per}(M)_1$ *such that the function* G *in (i) satisfies* $\partial_y^jG(X,0)=0$ *for* $1\le j\le k+1$

(c) the closed subset $\mathrm{Per}(M)_{1,\infty}=\mathrm{Per}(M)_{1_\infty}:=\bigcap_k\mathrm{Per}(M)_{1_{k+1}}$ *of all* $(X,a,T)\in\mathrm{Per}(M)_1$ *such that the function* G *introduced in (i) satisfies* $\partial_y^jG(X,0)=0$ *for every integer* j, *has infinite codimension.*

(iii) It follows that the submanifold $\mathrm{Per}(M)_1$ *is the disjoint union of* $\mathrm{Per}(M)_{1,\infty}$ *and the submanifolds* $\mathrm{Per}(M)_{1,k}:=\mathrm{Per}(M)_{1_{k+1}}\smallsetminus\mathrm{Per}(M)_{1_{k+2}}$, $k\in\mathbf N$, *each of which has codimension* k *in* $\mathrm{Per}(M)_1$ *and does not intersect the closure* $\mathrm{Per}(M)_{1_{\ell+1}}$ *of* $\mathrm{Per}(M)_{1,\ell}$ *for* $\ell>k$. *Moreover, for* $(X,a,T)\in\mathrm{Per}(M)_{1,k}$, $k<\infty$, *the charts* c_ξ *and the function* G *in (i) can be chosen so that*

$$G(\xi,y)=y^{k+2}+\sum_{j=1}^{k+1}\alpha_j(\xi)\,y^{j-1},$$

where $\alpha_1,\dots,\alpha_{k+1}:(\mathcal F,X)\to(\mathbf R,0)$ *have independent differentials at* X.

Proof By the flow-box theorem, there exists a smooth family (C_ξ) of charts, defined near (X,a) and satisfying (i) (a). The function τ_0 (before further variable changes) is then obtained by applying the implicit function theorem to the equation expressing the vanishing of the last component of $C_{\xi*}g_\xi^\tau(x,0)$. Let h_ξ be the first return map given by $\big(h_\xi(x),0\big):=C_{\xi*}g_\xi^{\tau_0(\xi,x)}(x,0)$. Near (X,a,T), the map $(\xi,x,\tau)\mapsto j^\ell h_\xi\big(C_\xi(x)\big)$ is [2] a submersion of $\mathrm{Per}(M)|_S:=\{(\xi,x,\tau)\in\mathrm{Per}(M):x\in S\}$ into $J^\ell(\mathbf R^{n-1},\mathbf R^{n-1})$ for every integer ℓ. Therefore, our stratification is obtained by taking the inverse images by these submersions of the submanifolds Σ_κ^ℓ and $\Sigma_{1,k}^\ell$ mentioned in the remarks following Theorem 2.1.1 (with $M:=\mathbf R^{n-1}$), the function G being deduced from the function F of Theorem 2.1.1 by the formula $G(\xi,y)=F(h_\xi,y)$. □

[28] "Poincaré section".

Remarks This makes sense, as $F(\varphi, y)$ is determined by the restriction of φ to a neighbourhood of 0 and the submanifolds $\Sigma_{\kappa}^{\ell}, \Sigma_{1,k}^{\ell}$ are invariant by smooth variable changes.

Of course, what should be visualised is the *local* image of $\mathrm{Per}(M)$ by the projection into $\mathcal{F} \times M$: locally, the globally dreadful injective immersion of $\mathrm{Per}(M)$ as $\mathrm{per}(M)$ is a harmless embedding.

If $k = 0$, one gets the *fold catastrophe for periodic orbits*: two periodic orbits collide for $\alpha_0(\xi) = 0$ and disappear for $\alpha_0(\xi) > 0$.

As the action ρ is free, the whole situation (including the strata $\mathrm{Per}(M)_{\kappa}$ with $\kappa > 1$) can be handled by the ordinary theory of singularities of smooth maps.

The story in paragraph 2.2 could be told similarly, introducing $\mathrm{Per}_{\mathbf{Z}}(M) := \bigcup_{p \geq 1} \widetilde{\Sigma}^{(p)} \times \{p\}$, whose p–th slice is endowed with an action of $\mathbf{Z}/p\mathbf{Z}$.

2.5 The period doubling bifurcation for flows and such

If $(X, a, T) \in \mathcal{F} \times M \times \mathbf{R}_{+}^{*}$ lies in the closure $\mathrm{cl}\,\big(\mathrm{Per}(M)\big)$ of $\mathrm{Per}(M)$, then $g_X^T(a) = a$ and therefore either $X_a = 0$, or $(X, a, T/p) \in \mathrm{Per}(M)$ for some positive integer p. Hence, for each integer $p > 1$, we introduce the closed subset $\mathrm{Per}^{*p}(M)$ of $\mathrm{Per}(M)$ which consists of those (X, a, T) with $(X, a, pT) \in \mathrm{cl}\,\big(\mathrm{Per}(M)\big)$. Here, we are interested in the case $p = 2$; the following result is proved in the same way as Proposition 2.3.1:

Proposition 2.5.1 *The subset* $\mathrm{Per}^{*2}(M)$ *is the set of those* $(X, a, T) \in \mathrm{Per}(M)$ *such that* -1 *is an eigenvalue of* $d(g_X^T)_a$. *It is the disjoint union of the submanifolds* $\mathrm{Per}^{*2}(M)_1, \ldots, \mathrm{Per}^{*2}(M)_{n-1}$ *defined as follows:* $\mathrm{Per}^{*2}(M)_{\kappa}$ *is the set of those* $(X, a, T) \in \mathrm{Per}(M)$ *such that* $d(g_X^T) + \mathrm{Id}$ *has corank* κ. *In particular,* $\mathrm{Per}^{*2}(M)_1$ *is a hypersurface of* $\mathrm{Per}(M)$. □

Remark If M is an orientable surface, $\mathrm{Per}^{*2}(M)$ must be empty. It is a nice fact of Nature that Möbius strips should appear so naturally in period doubling bifurcations.

Theorem 2.5.2 *Denoting by* $\mathrm{Per}_2(M)$ *(resp.* $\mathrm{Per}_2^{*2}(M)$*) the set of all* $(\xi, x, 2\tau) \in \mathcal{F} \times M \times \mathbf{R}_{+}^{*}$ *with* $(\xi, x, \tau) \in \mathrm{Per}(M)$ *(resp.* $(\xi, x, \tau) \in \mathrm{Per}^{*2}(M)$*), the intersection of* $\mathrm{Per}^{*2}(M)_1$ *with the "big stratum"* $\mathrm{Per}(M)_0$ *of Theorem 2.4.2 has the following properties:*

(i) For every $(X, a, T) \in \mathrm{Per}^{*2}(M)_1 \cap \mathrm{Per}(M)_0$, *the subset* $\mathrm{cl}\,\big(\mathrm{Per}(M)\big)$ *is a smooth* n*–codimensional submanifold of* $\mathcal{F} \times M \times \mathbf{R}_{+}^{*}$ *in the neighbourhood of* $(X, a, 2T)$, *intersecting* $\mathrm{Per}_2(M)$ *along* $\mathrm{Per}_2^{*2}(M)$. *More precisely, for each Poincaré section* S *of* X *at* a, *there is a smooth family* $\big(\mathcal{F} \times M, (X, a)\big) \ni (\xi, x) \mapsto c_{\xi}(x) \in (\mathbf{R} \times \mathbf{R}^{n-2} \times \mathbf{R}, 0)$ *of local charts such that, near* $(X, a, 2T)$,

  - *(a)* $c_{\xi *}\xi = (0, 0, 1)$ *and* $c_{\xi}(S \cap \mathrm{dom}\, c_{\xi}) = \mathrm{Im}\, c_{\xi} \cap (\mathbf{R}^{n-1} \times \{0\})$
  - *(b)* $\mathrm{Per}_2(M)$ *is the set of those* $(\xi, x, 2\tau)$ *such that* $\tau = \tau_0(\xi, x)$ *and* $c_{\xi}(x) \in \{0\} \times \mathbf{R}$
  - *(c)* $\mathrm{cl}\,\big(\mathrm{Per}(M)\big)$ *consists of those* $(\xi, x, \tau) \in \mathrm{Per}(M) \cup \mathrm{Per}_2^{*2}(M)$ *such that* $(y, z, w) := c_{\xi}(x)$ *satisfies the equations* $z = 0$ *and* $F(\xi, y) = 0$, *where* $F : \big(\mathcal{F} \times \mathbf{R}, (X, 0)\big) \to (\mathbf{R}, 0)$ *is a smooth local function satisfying* $\partial_{\xi} F(X, 0) \neq 0$ *and* $F(\xi, -y) = F(\xi, y)$
  - *(d) for each integer* k *such that* $\partial_y F(X, 0) = \cdots = \partial_y^{2k+1} F(X, 0) = 0$, *the differentials* $\partial_{\xi} \partial_y^{2j} F(X, 0)$ *with* $0 \leq j \leq k$ *are linearly independent.*

(ii) One can define a decreasing sequence $\big(\mathrm{Per}^{*2}(M)_{1_{k+1}}\big)_{k\in\mathbf{N}}$ *of smooth submanifolds of the smooth manifold* $\mathrm{Per}^{*2}(M)_1 \cap \mathrm{Per}(M)_0 =: \mathrm{Per}^{*2}(M)_{1_1}$ *as follows:* $\mathrm{Per}^{*2}(M)_{1_{k+2}}$ *is the (closed) set of those* $(\xi, x, \tau) \in \mathrm{Per}^{*2}(M)$ *such that* $(\xi, x, 2\tau)$ *lies in the closure of the subset* $\mathrm{Per}(M)_{1_{k+1}}$ *of Theorem 2.4.2. Thus,*

(a) for every $(X, a, T) \in \mathrm{Per}^{*2}(M)_1$*, with the notation of (i),*

– *each* $\mathrm{Per}(M)_{1_{k+1}}$ *is defined near* $(X, a, 2T)$ *by the conditions* $z = 0$*,* $y \neq 0$ *and* $\partial_y^j F(\xi, y) = 0$ *for every integer* $j \leq k+1$

– *each* $\mathrm{Per}^{*2}(M)_{1_{k+1}}$ *is defined near* (X, a, T) *by the conditions* $c_\xi(x) \in \{0\} \times \mathbf{R}$ *and* $\partial_y^{2j} F_p(\xi, y) = 0$ *for every integer* $j \leq k$*, hence* $\mathrm{Per}^{*2}(M)_{1_{k+1}}$ *has codimension* $n+k+1$ *in* $\mathcal{F} \times M \times \mathbf{R}_+^*$ *and, therefore, codimension* k *in* $\mathrm{Per}^{*2}(M)$

(b) in particular, $\mathrm{Per}^{*2}(M)_{1_{k+1}}$ *is the set of those* $(X, a, T) \in \mathrm{Per}^{*2}(M)_1$ *such that the function* F *in (i) satisfies* $\partial_y^{2j} F(X, 0) = 0$ *for* $1 \leq j \leq k$

(c) the closed subset $\mathrm{Per}^{*2}(M)_{1,\infty} = \mathrm{Per}^{*2}(M)_{1_\infty} := \bigcap_k \mathrm{Per}^{*2}(M)_{1_{k+1}}$*, consisting of all* $(X, a, T) \in \mathrm{Per}^{*2}(M)_1$ *such that the function* F *introduced in (i) satisfies* $\partial_y^j F(X, 0) = 0$ *for every integer* j*, has infinite codimension.*

(iii) It follows that the submanifold $\mathrm{Per}^{*2}(M)_1$ *is the disjoint union of the subset* $\mathrm{Per}^{*2}(M)_{1,\infty}$ *and the submanifolds* $\mathrm{Per}^{*2}(M)_{1,k} := \mathrm{Per}^{*2}(M)_{1_{k+1}} \smallsetminus \mathrm{Per}^{*2}(M)_{1_{k+2}}$*,* $k \in \mathbf{N}$*, each of which has codimension* k *in* $\mathrm{Per}^{*2}(M)_1$ *and does not intersect the closure* $\mathrm{Per}^{*2}(M)_{1_{\ell+1}}$ *of* $\mathrm{Per}^{*2}(M)_{1,\ell}$ *for* $\ell > k$*. Moreover, for* $(X, a) \in \mathrm{Per}^{*2}(M)_{1,k}$*,* $k < \infty$*, the function* F *in (i) can be chosen of the form*

$$F(\xi, y) = y^{2k+2} + \sum_{j=0}^{k} \alpha_j(\xi)\, y^{2j},$$

where $\alpha_0, \ldots, \alpha_k : (\mathcal{F}, X) \to (\mathbf{R}, 0)$ *are smooth local functions whose differentials at* X *are independent.*

Proof As in the proof of Theorem 2.4.2, this is obtained by applying Theorem 2.3.3 to the Poincaré first return maps h_ξ. □

Remarks What should be visualised is the normal crossing at (X, a) of the two local embeddings $\big(\mathrm{Per}(M), (X, a, T)\big) \to \mathcal{F} \times M$ and $\big(\mathrm{cl}\,\mathrm{Per}(M), (X, a, 2T)\big) \to \mathcal{F} \times M$ obtained by restricting the projection $\mathcal{F} \times M \times \mathbf{R}_+^* \to \mathcal{F} \times M$ (a first reason why the injectively immersed manifold $\mathrm{per}(M)$ is not a manifold).

If $k = 0$, one gets the *period doubling bifurcation for periodic orbits*: a periodic orbit of ξ whose period is twice that of the (robust) periodic orbit at which one is localized appears for $\alpha_0(\xi) = 0$ and develops for $\alpha_0(\xi) < 0$. When the eigenvalue -1 of the Poincaré map h_X is simple and the others lie off the unit circle, the phenomenon takes place in a family of 2–dimensional Möbius strips (the sections $\xi =$ constant of a central manifold) centred at the robust orbit from which the bifurcation occurs.

As before, I do not know if the structure of $\mathrm{cl}\,\big(\mathrm{Per}(M)\big)$ near $\mathrm{Per}^{*2}(M)_\kappa$ is understood for $\kappa > 1$, even though this looks like a *bona fide* problem in the theory of singularities of smooth maps.

2.6 The AndronHopf bifurcation and such

We now consider the set $\mathrm{Hopf}(M)$ of those $(X,a,T)\in \mathrm{cl}\big(\mathrm{Per}(M)\big)$ such that a is a *non-degenerate* zero of X (which writes $(X,a)\in\widetilde{\Sigma}_0$ with the notation of Proposition 1.2.1 for vector fields).

Proposition 2.6.1 *(i) The set* $\mathrm{Hopf}(M)$ *consists of points* $(X,a,T)\in\widetilde{\Sigma}_0\times\mathbf{R}_+^*$ *such that a is a degenerate fixed point of* g_X^T.

(ii) Given $(X,a)\in\widetilde{\Sigma}_0$*, if the spectrum of* d_aX *contains* $\frac{2\pi}{T}i$ *and has no further intersection with* $\frac{2\pi}{T}i\mathbf{N}^*$*, then* (X,a,T) *lies in* $\mathrm{Hopf}(M)$*.*

(iii) In particular, the smooth submanifold $\mathrm{Hopf}(M)_1$ *of codimension 2 of* $\widetilde{\Sigma}\times\mathbf{R}_+^*$ *consisting of those* $(X,a,T)\in\widetilde{\Sigma}_0\times\mathbf{R}_+^*$ *such that the spectrum of* d_aX *contains* $\frac{2\pi}{T}i$ *and that* $e^{T\,d_aX}-\mathrm{Id}$ *has corank 2 is contained in* $\mathrm{Hopf}(M)$*.*

Proof By Proposition 1.2.1, $\widetilde{\Sigma}$ is the graph of a smooth function Φ near each $(X,a)\in\widetilde{\Sigma}_0$. Given a chart $c:(M,a)\to(\mathbf{R}^n,0)$, composing $(\xi,x)\mapsto\big(\xi,c(x)\big)$ with $(\xi,x)\mapsto\big(\xi,x-c\circ\Phi(\xi)\big)$, we get a smooth family (c_ξ) of local charts such that the image of $\widetilde{\Sigma}$ by $\tilde{c}:(\xi,x)\mapsto\big(\xi,c_\xi(x)\big)$ is $\mathrm{Im}\,\tilde{c}\cap(\mathcal{F}\times\{0\})$. Thus, if (ξ_k,x_k,τ_k) is a sequence in $\mathrm{Per}(M)$ tending to $(X,a,T)\in\mathrm{Hopf}(M)$, then $\tilde{c}(\xi_k,x_k)=(\xi_k,r_ku_k)$ with $r_k>0$ tending to 0 and $|u_k|=1$; the equation $g_{\xi_k}^{\tau_k}(x_k)=x_k$ writes $G(\xi_k,r_k,u_k,\tau_k)=0$, where

$$G(\xi,r,u,\tau):=\begin{cases} r^{-1}g^{\tau}_{c_{\xi*}\xi}(ru)-u & \text{for } r>0,\\ \big(Dg^{\tau}_{c_{\xi*}\xi}(0)-\mathrm{Id}\big)u & \text{if } r=0\end{cases}$$

is smooth since $c_{\xi*}\xi(0)=0$. Replacing (ξ_k,x_k,τ_k) by a subsequence, we may assume that (u_k) converges to u_∞ in the unit sphere, hence $G(X,0,u_\infty,T)=0$. It follows that u_∞ is a fixed point of $Dg^T_{c_{X*}X}(0)=(dc_X)_{a*}e^{T\,d_aX}$; in other words, $(dc_X)_a^{-1}u_\infty$ is a fixed point of $Dg^T_{c_{X*}X}(0)=(dc_X)_{a*}e^{T\,d_aX}$ hence (i). Under the hypothesis of (ii), if $u_\infty\in\mathbf{S}^{n-1}$ lies in an invariant 2–plane associated to the eigenvalue $\frac{2\pi}{T}i$ of d_aX, then $\xi\mapsto\big(Dg^T_{c_{\xi*}\xi}(0)-\mathrm{Id}\big)u_\infty$ is a submersion at X; therefore, the equation $G(\xi,r,u,\tau)=0$ has solutions with $r>0$ arbitrarily close to $(X,0,u_\infty,T)$, proving (ii). Finally, as $\mathrm{Hopf}(M)_1$ consists of those $(X,a,T)\in\widetilde{\Sigma}_0\times\mathbf{R}_+^*$ such that (i) is satisfied and $\frac{2\pi}{T}i$ is a geometrically simple eigenvalue of d_aX, it is contained in $\mathrm{Hopf}(M)$ and is a submanifold of codimension 2 by transversality in jet spaces. □

Theorem 2.6.2 *The submanifold* $\mathrm{Hopf}(M)_1$ *has the following properties:*

(i) Near every $(X,a,T)\in\mathrm{Hopf}(M)_1$*, the subset* $\mathrm{cl}\big(\mathrm{Per}(M)\big)$ *is a smooth n–codimensional submanifold of* $\mathcal{F}\times M\times\mathbf{R}_+^*$ *intersecting* $\widetilde{\Sigma}\times\mathbf{R}_+^*$ *at* $\mathrm{Hopf}(M)_1$*. More precisely, there exists a smooth family* $\big(\mathcal{F}\times M,(X,a)\big)\ni(\xi,x)\mapsto c_\xi(x)\in(\mathbf{C}\times\mathbf{R}^{n-2},0)$ *of local charts such that, near* (X,a,T)*,*

(a) $\widetilde{\Sigma}\times\mathbf{R}_+^*$ *is the smooth submanifold of those* (ξ,x,τ) *such that* $c_\xi(x)=0$

(b) $\mathrm{cl}\big(\mathrm{Per}(M)\big)$ *is the smooth submanifold of those* (ξ,x,τ) *such that* $(w,z):=c_\xi(x)$ *satisfies the equations* $z=0$*,* $\tau=\tau_0(\xi,x)$ *and* $F(\xi,w)=0$*, where*

(*) $F:\big(\mathcal{F}\times\mathbf{C},(X,0)\big)\to(\mathbf{R},0)$ *is a smooth local function satisfying* $\partial_\varphi F(f,0)\neq 0$ *and* $F(\xi,e^{i\theta}w)=F(\xi,w)$ *for all* $\theta\in\mathbf{R}$

(**) $\tau_0 : \big(\mathcal{F}\times M,(X,a)\big)\to(\mathbf{R},T)$ *is the (smooth) continuous extension of the function defined in Theorem 2.4.2.*

(c) *For* $(\xi,x,T)\in \operatorname{cl}\big(\operatorname{Per}(M)\big)$ *and* $x=c_\xi^{-1}(w,0)$*, we have that* $c_{\xi *}\xi(w,0)=(\frac{2\pi}{T}iw,0)$*; thus,* ρ *extends to the smooth local* $\mathbf{T}$*–action on* $\operatorname{cl}\big(\operatorname{Per}(M)\big)$ *which reads* $(\theta,w,0)\mapsto(e^{2\pi i\theta}w,0)$ *in the charts* c_ξ

(d) *for each integer* k *satisfying* $\partial_w F(f,0)=\cdots=\partial_w^{2k+1}F(f,0)=0$*, the differentials* $\partial_\varphi\partial_w^{2j}F(f,0)$ *with* $0\le j\le k$ *are linearly independent.*

In particular, the restricted projection $\operatorname{cl}\big(\operatorname{Per}(M)\big)\to\mathcal{F}\times M$ *is a smooth embedding near* (X,a,T)*; its image consists of those* (ξ,x) *such that* $(w,z):=c_\xi(x)$ *satisfies the equations* $z=0$ *and* $F(\xi,w)=0$*, and it is endowed with a smooth* $\mathbf{T}$*–action.*

(ii) *A decreasing sequence* $\big(\operatorname{Hopf}(M)_{1_{k+1}}\big)_{k\in\mathbf{N}}$ *of smooth submanifolds of* $\operatorname{Hopf}(M)_1=:\operatorname{Hopf}(M)_{1_1}$ *can be defined as follows:* $\operatorname{Hopf}(M)_{1_{k+2}}$ *is the (closed) set of those points of* $\operatorname{Hopf}(M)$ *which lie in the closure of the subset* $\operatorname{Per}(M)_{1_{k+1}}$ *of Theorem 2.4.2. Thus,*

(a) *near every* $(X,a,T)\in\operatorname{Hopf}(M)_1$*, with the notation of (i),*

- *each* $\operatorname{Per}(M)_{1_{k+1}}$ *is defined by the conditions* $z=0$*,* $w\neq 0$ *and* $\partial_w^j F(\varphi,w)=0$ *for every integer* $j\le k+1$
- *each* $\operatorname{Hopf}(M)_{1_{k+1}}$ *is defined by the conditions* $c_\xi(x)=0$*,* $\tau=\tau_0(\xi,x)$ *and* $\partial_w^{2j}F_p(\xi,0)=0$ *for every integer* $j\le k$*, hence* $\operatorname{Hopf}(M)_{1_{k+1}}$ *has codimension* $n+k+2$ *in* $\mathcal{F}\times M\times\mathbf{R}_+^*$ *and, therefore, codimension* k *in* $\operatorname{Hopf}(M)$

(b) *in particular,* $\operatorname{Hopf}(M)_{1_{k+1}}$ *is the set of those* $(X,a,T)\in\operatorname{Hopf}(M)_1$ *such that the function* F *in (i) satisfies* $\partial_w^{2j}F(X,0)=0$ *for* $1\le j\le k$

(c) *the closed subset* $\operatorname{Hopf}(M)_{1,\infty}=\operatorname{Hopf}(M)_{1_\infty}:=\bigcap_k\operatorname{Hopf}(M)_{1_{k+1}}$*, consisting of all* $(f,a)\in\operatorname{Hopf}(M)_1$ *such that the function* F *introduced in (i) satisfies* $\partial_w^j F(f,0)=0$ *for every integer* j*, has infinite codimension.*

(iii) *It follows that the submanifold* $\operatorname{Hopf}(M)_1$ *is the disjoint union of the subset* $\operatorname{Hopf}(M)_{1,\infty}$ *and the submanifolds* $\operatorname{Hopf}(M)_{1,k}:=\operatorname{Hopf}(M)_{1_{k+1}}\smallsetminus\operatorname{Hopf}(M)_{1_{k+2}}$*,* $k\in\mathbf{N}$*, each of which has codimension* k *in* $\operatorname{Hopf}(M)_1$ *and does not intersect the closure* $\operatorname{Hopf}(M)_{1_{\ell+1}}$ *of* $\operatorname{Hopf}(M)_{1,\ell}$ *for* $\ell>k$*. Moreover, for* $(X,a,T)\in\operatorname{Hopf}(M)_{1,k}$*,* $k<\infty$*, the function* F *in (i) can be chosen of the form*

$$F(\xi,w)=|w|^{2k+2}+\sum_{j=0}^{k}\alpha_j(\xi)\,|w|^{2j},$$

where $\alpha_0,\dots,\alpha_k:(\mathcal{F},X)\to(\mathbf{R},0)$ *are smooth local functions whose differentials at* X *are independent.*

Proof Composing a smooth family of local charts satisfying (i) (a) with a smooth family of linear isomorphisms, we get a smooth family C_ξ of local charts such that $C_{\xi *}\xi(0)=0$ and $D(C_{\xi *}\xi)(0)(w,0)=(\lambda(\xi)w,0)$ for ξ close enough to X, where the smooth complex function λ satisfies $\lambda(X)=\frac{2\pi}{T}i$. If $L\subset\mathbf{C}$ is any real line through the origin, it follows that the first return map $h_{\xi,L}:(L\times\mathbf{R}^{n-2},0):(L\times\mathbf{R}^{n-2},0)$ for the flow of $C_{\xi *}\xi$ is well-defined and smooth (as a function of (ξ,x) near $\xi=x$). Our result is essentially Theorem 2.3.3 applied to these first return maps, but smoothness is not so obvious, at least to the author: it can be proved using a nice trick due to S. López de Medrano [11]. □

Remarks What should be visualised is the transversal intersection at (X,a) of $\widetilde{\Sigma}$ with the image of the local embedding $\big(\operatorname{cl}\operatorname{Per}(M),(X,a,T)\big)\to\mathcal{F}\times M$ obtained by restricting the projection.

If $k=0$, one gets the *Andronov-Hopf bifurcation*: a periodic orbit of ξ appears for $\alpha_0(\xi)=0$ and develops for $\alpha_0(\xi)<0$. When the eigenvalue $\frac{2\pi}{T}i$ is simple and the others lie off the imaginary line, the phenomenon takes place in a family of 2–dimensional orientable surfaces (the sections $\xi=$ constant of a central manifold).

As before, I do not know if the structure of $\operatorname{cl}\big(\operatorname{Per}(M)\big)$ near $\operatorname{Hopf}(M)_\kappa$ is understood for $\kappa>1$.

Interesting examples where (X,a,T) lies in $\operatorname{Hopf}(M)_{1,\infty}$ are provided by Hamiltonian vector fields X. In that case [11], near (X,a,T), the intersection of $\operatorname{cl}\big(\operatorname{Per}(M)\big)$ with $\{\xi=X\}$ is a smooth surface containing (X,a,T), whose projection into M is smooth (Lyapunov's theorem). Note that a may be an elliptic rest point of X, in which case there are generically $\frac{1}{2}n-1$ other values of T with the same property.

2.7 p–upling bifurcations, Arnol'd tongues and such

If $\mathcal{F}=C^\infty(M,M)$, given an integer $p>2$, the closure of the smooth submanifold $\widetilde{\Sigma}^{(p)}$, clearly consists of points (f,a) with $f^p(a)=a$. Hence, for $(f,a)\in\operatorname{cl}(\widetilde{\Sigma}^{(p)})\smallsetminus\widetilde{\Sigma}^{(p)}$ the primitive period of a for f is less than p and divides p; in particular, if p is prime, (f,a) lies in $\widetilde{\Sigma}$. For brevity, we restrict to that case. The following result is proved like Proposition 2.6.1:

Proposition 2.7.1 *Given a prime integer $p>2$, the subset $\widetilde{\Sigma}_0\cap\operatorname{cl}(\widetilde{\Sigma}^{(p)})$ is the set $\widetilde{\Sigma}^{1,p}$ of those $(f,a)\in\widetilde{\Sigma}_0$ such that the spectrum of d_af contains a primitive p–th root of unity. Conversely, for any integer $p>2$, the subset $\widetilde{\Sigma}^{1,p}$ lies in $\operatorname{cl}(\widetilde{\Sigma}^{(p)})$. It contains as a dense subset the smooth 2–codimensional manifold $\widetilde{\Sigma}^{1,p}_1$ of those $(f,a)\in\widetilde{\Sigma}^{1,p}$ such that $d_af^p-\operatorname{Id}$ has corank 2.* □

At points $(f,a)\in\widetilde{\Sigma}^{1,p}_1$, the subset $\operatorname{cl}(\widetilde{\Sigma}^{(p)})$ is a submanifold of class only C^{p-3} in general [10] and its projection into $\mathcal{F}$ is a thinner and thinner "tongue" at f when p increases (at generic points, its intersection with a generic surface is the celebrated picture known as an Arnol'd tongue). As the set of primitive roots of unity is dense in the unit circle, these local submanifolds form an almost unavoidable maze near the large subset of $\widetilde{\Sigma}_0\smallsetminus\widetilde{\Sigma}_h$ consisting of those (f,a) such that d_af has a pair of eigenvalues on the unit circle, which will be considered in more detail in the next two paragraphs.

Stratifications and normal forms for $\operatorname{cl}(\widetilde{\Sigma}^{(p)})$ near $\widetilde{\Sigma}^{1,p}_1$ may not be quite as obvious as previously, even though the normal form theorem 1.6.5 can help.

As in the case $p=2$, this p–upling bifurcation extends from fixed points of maps to periodic orbits of maps or vector fields.

2.8 The Naimark-Sacker bifurcation

Despite superficial analogy (and its popularisation as the "Hopf bifurcation for maps"), it is essentially different from the Andronov-Hopf bifurcation and occurs at far less general points. We assume that $\mathcal{F}=C^\infty(M,M)$ and denote by $\operatorname{NS}(M)$ the set of those $(f,a)\in\widetilde{\Sigma}_0\smallsetminus\widetilde{\Sigma}_h$ such that

(a) d_af has a pair of simple eigenvalues $e^{\pm i\alpha_f}$, $0<\alpha_f<\pi$, on the unit circle, with $\alpha_f\neq\frac{2\pi}{p}$ for[29] $3\le p\le 5$

[29] This can be improved a bit, at the expense of comfort (the value $p=3$ is forbidden anyway).

(b) its other eigenvalues lie off the unit circle, hence f has a unique formal central manifold of dimension 2 and the (formal) restriction of f to this central manifold is the time 1 of the flow of a (formal) vector field X such that, in a suitable smooth complex coordinate vanishing at a, the jet $j_a^4 X$ identifies to a polynomial vector field $w \mapsto w\big(i\alpha_f + (\beta_f + i\gamma_f)|w|^2\big)$ with $\beta_f \in \{-1, 0, 1\}$ and $\gamma_f \in \mathbf{R}$

(c) the "Birkhoff invariant" β_f is nonzero.

Theorem 2.8.1 *The subset* $\mathrm{NS}(M)$ *is a smooth hypersurface of* $\widetilde{\Sigma}$ *and* $\mathrm{NS}(M) \cup \widetilde{\Sigma}^{1,2}$ *is dense in* $\widetilde{\Sigma}_0 \smallsetminus \widetilde{\Sigma}_h$*, of which* $\mathrm{NS}(M) \cup \widetilde{\Sigma}^{1,2}_{1_1}$ *therefore is the "big stratum" for a hypothetic stratification. Every* $(f, a) \in \mathrm{NS}(M)$ *lies in the closure of the set* $\mathrm{Circ}(M)$ *of those* $(\varphi, x) \in \mathcal{F} \times M$ *such that* x *belongs to a* φ*–invariant closed curve* $\mathrm{Circ}_\varphi(M)$*. More precisely,*

(i) there exist a fairly smooth family $\big(\mathcal{F} \times M, (f, a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{C} \times \mathbf{R}^{n-2}, 0)$ *of local charts and a smooth submersion* $\lambda : (\mathcal{F}, a) \to (\mathbf{R}, 0)$ *such that, near* (f, a)*, the closure* $\mathrm{cl}\big(\mathrm{Circ}(M)\big)$ *is the* $(n-1)$*–codimensional submanifold of those* $(\varphi, x) \in \mathcal{F} \times M$ *such that* $(w, y) := c_\varphi(x)$ *satisfies* $y = 0$ *and* $|w|^2 = \lambda(\varphi)$

(ii) this submanifold intersects $\widetilde{\Sigma}$ *along* $\mathrm{NS}(M)$*.*

Proof The first assertion is easy. To prove the rest, one can work in a central manifold W^c of the tautological unfolding $(\varphi, x) \mapsto \big(\varphi, \varphi(x)\big)$ at (f, a), which can be taken of arbitrarily high finite smoothness; there exists a very smooth family $\big(\mathcal{F} \times M, (f, a)\big) \ni (\varphi, x) \mapsto C_\varphi(x) \in (\mathbf{C} \times \mathbf{R}^{n-2}, 0)$ of local charts such that $C_{\varphi *}\varphi(0) = 0$ and $C_\varphi(W^c_\varphi) = (\mathbf{C} \times \{0\}) \cap \mathrm{Im}\, C_\varphi$ for every φ. This reduces our problem to the study of the maps $C_{\varphi *}\varphi|_{\mathbf{C} \times \{0\}} : (\mathbf{C} \times \{0\}, 0) \to (\mathbf{C} \times \{0\}, 0)$. Up to a polynomial change of variables depending smoothly on φ, each one of them has contact of order 4 at 0 with the time 1 of the flow of a polynomial normal form $\xi_\varphi : w \mapsto w\big(-\beta_f \lambda(\varphi) + i\alpha(\varphi) + \big(\beta_f + i\gamma(\varphi)\big)|w|^2\big)$, where $\lambda : (\mathcal{F}, f) \to (\mathbf{R}, 0)$, $\alpha : (\mathcal{F}, f) \to (\mathbf{R}, \alpha_f)$ and $\gamma : (\mathcal{F}, f) \to (\mathbf{R}, \gamma_f)$ are smooth and λ is a submersion. The circle $|w|^2 = \lambda(\varphi)$ is invariant by this normal form and attracting or repulsing according to the sign of β_f.

Let $\varphi_1 : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ be the map obtained fom $\varphi|_{W_\varphi}$ after these variable changes. A nice way to prove that φ itself admits an invariant closed curve near this circle consists [9, 13] in making the change of variable $w = \lambda(\varphi)^{\frac{1}{2}} W$ and considering the "accelerated" map $\varphi_1^{[\lambda(\varphi)^{-1}]}$ (integer part) for $\lambda(\varphi) > 0$. The change of variable transforms the normal form ξ_φ into the vector field $\lambda(\varphi)\Xi_\varphi$, where $\Xi_\varphi(W) := W\big(-\beta_f + i\lambda(\varphi)^{-1}\alpha(\varphi) + \big(\beta_f + i\gamma(\varphi)\big)|W|^2\big)$. When $\lambda(\varphi)$ tends to 0, the transformation $g_{\lambda(\varphi)\Xi_\varphi}^{[\lambda(\varphi)^{-1}]}$ does not tend to anything, but $e^{-i[\lambda(\varphi)^{-1}]\alpha(\varphi)} g_{\lambda(\varphi)\Xi_\varphi}^{[\lambda(\varphi)^{-1}]}$ tends in the C^1 sense to the time 1 of the flow of $W \mapsto W\big(-\beta_f + \big(\beta_f + i\gamma_f\big)|W|^2\big)$ and so does $W \mapsto e^{-i[\lambda(\varphi)^{-1}]\alpha(\varphi)} \lambda(\varphi)^{-\frac{1}{2}} \varphi_1^{[\lambda(\varphi)^{-1}]}\big(\lambda(\varphi)^{\frac{1}{2}} W\big)$. As the limit admits the unit circle as a normally hyperbolic invariant manifold and each $e^{-i[\lambda(\varphi)^{-1}]\alpha(\varphi)}$ lies in the compact group $\mathbf{U}(1)$, which leaves the unit circle and $W \mapsto \big(-\beta_f + \big(\beta_f + i\gamma_f\big)|W|^2\big)W$ invariant, it is easy to conclude. □

Remarks Inside each "tongue" considered in paragraph 2.7, near $\mathrm{NS}(M)$, the periodic orbit of φ must lie on the invariant circle $\mathrm{Circ}_\varphi(M)$; hence, the (rational) rotation number of $\varphi|_{\mathrm{Circ}_\varphi(M)}$ is locally constant in such zones; this makes a difference with the far more regular rotation number of the normal form on $|w|^2 = \lambda(\varphi)$.

Using first return maps, one gets the periodic/quasiperiodic bifurcation, in which a persistent periodic orbit of a vector field gives birth to an invariant 2–torus.

If $\beta_f = 0$, there may be no invariant circles around (what makes the Naimark-Sacker invariant circles $\mathrm{Circ}_\varphi(M)$ stable is normal hyperbolicity and not the fact that φ is topologically transitive on them, which it is not in general—in contrast with the genuine Hopf bifurcation). Generic cases of codimension at least 2 become very complex almost at once: see [14], where the analogue of the situation considered in Theorem 2.6.2 (iii) with $k = 1$ gives rise to a wealth of dynamical phenomena.

2.9 Generalisations

Since much of biology, chemistry, psychology, etc. consists in coupling oscillators, one may show some interest in points of $\widetilde{\Sigma}_0 \smallsetminus \widetilde{\Sigma}_h$ where the differential $d_a f$ (resp. $d_a X$) has several pairs of eigenvalues on the unit circle (resp. the imaginary axis)[30]. For brevity, as in the first paragraphs of the paper, we denote both vector fields and maps by f, φ.

2.9.1 Hypotheses and notation: the case of vector fields

Assuming $\mathcal{F} = C^\infty(TM)$, we consider points $(f, a) \in \widetilde{\Sigma}_0$ with the following properties:

(H_1) The eigenvalues $\pm i\alpha_1(f), \ldots, \pm i\alpha_\nu(f)$ ($\alpha_j(f) > 0$) of $d_a f$ which lie on the imaginary axis are nonzero, simple and satisfy no "resonance" condition

$$i\alpha_k(f) = \sum_{j=1}^{\nu} i\alpha_j(f)(p_j - q_j) \quad \text{with} \quad (p,q) \in \mathbf{N}^\nu \times \mathbf{N}^\nu \quad \text{and} \quad 2 \le \sum_{j=1}^{\nu}(p_j + q_j) \le 4$$

except the unavoidable $p_k = q_k + 1$ and $p_j = q_j$ for $j \neq k$, in which case the constraint $2 \le \sum(p_j + q_j) = 1 + 2\sum q_j \le 4$ imposes that one of the integers $q_1, \ldots, q_\nu$ equals 1 and that the others vanish.

It follows that f has a unique formal central manifold of dimension 2ν, the (formal) restriction of f to which, in suitable smooth complex coordinates $w_1, \ldots, w_\nu$ vanishing at a, has fourth order contact with a polynomial normal form

$$N_f : (w_1, \ldots, w_\nu) \mapsto \left(w_j \Big(i\alpha_j(f) - \sum_{\ell=1}^{\nu} \left(\beta_{j\ell}(f) + i\gamma_{j\ell}(f)\right) |w_\ell|^2 \Big) \right)_{1 \le j \le \nu}$$

with $\beta_{j\ell}(f), \gamma_{j\ell}(f) \in \mathbf{R}$.

As in 2.8, one can work in a central manifold W^c of the tautological unfolding $(\varphi, x) \mapsto \big(\varphi, \varphi(x)\big)$ at (f, a), which can be taken of arbitrarily high finite smoothness; by the theory of normal forms, there exists a fairly smooth family $\big(\mathcal{F} \times M, (f,a)\big) \ni (\varphi, x) \mapsto c_\varphi(x) \in (\mathbf{C}^\nu \times \mathbf{R}^{n-2\nu}, 0)$ of local charts such that $c_{\varphi *}\varphi(0) = 0$, that $c_\varphi(W^c_\varphi) = (\mathbf{C}^\nu \times \{0\}) \cap \mathrm{Im}\, c_\varphi$ and that $c_{\varphi *}\varphi|_{\mathbf{C}^\nu \times \{0\}}$ has contact of order 4 at 0 with a polynomial normal form

$$N_\varphi(w) = \left(w_j \Big(\lambda_j(\varphi) + i\alpha_j(\varphi) - \sum_{\ell=1}^{\nu} \left(\beta_{j\ell}(\varphi) + i\gamma_{j\ell}(\varphi)\right) |z_\ell|^2 \Big) \right)_{1 \le j \le \nu} \tag{7}$$

[30] This paragraph was designed [9] as a perhaps embarrassing birthday present to V.I. Arnol'd.

where $w = (w_1, \ldots, w_\nu) \in \mathbf{C}^\nu$ and the $\lambda_j, \alpha_j, \beta_{j\ell}, \gamma_{j\ell}$ are smooth real functions with $\lambda_j(f) = 0$ and $\alpha_j(f), \beta_{j\ell}(f), \gamma_{j\ell}(f)$ as before.

The normal form N_φ is invariant under the natural action $(e^{i\theta_1}, \ldots, e^{i\theta_n}; w) \mapsto (e^{i\theta_1}w_1, \ldots, e^{i\theta_\nu}w_\nu)$ of $\mathbf{U}(1)^\nu$ on $\mathbf{C}^\nu$. In particular, the mapping $\rho : \mathbf{C}^\nu \to \mathbf{R}^\nu_+$ defined by $\rho(w) = \big(|w_1|, \ldots, |w_n|\big)$ sends N_φ onto the smooth vector field $V_\varphi = \sum_j \Big(\lambda_j(\varphi) - \sum_\ell \beta_{j\ell}(\varphi) r_\ell^2\Big) r_j \frac{\partial}{\partial r_j}$. To each $\eta \in T_f\mathcal{F} = \mathcal{F}$ we associate the vector field

$$W_\eta(x) = \sum_j x_j \Big(D\lambda_j(f)\eta - \sum_\ell \beta_{j\ell}(f)\, x_\ell^2\Big) \frac{\partial}{\partial x_j}$$

on $\mathbf{R}^\nu$. It is invariant under the action of $\mathbf{O}(1)^\nu$ generated by the symmetries with respect to coordinate hyperplanes. Here is our second hypothesis:

(H_2) For some $\eta_0 \in T_f\mathcal{F} \smallsetminus \{0\}$, the vector field W_{η_0} admits a normally hyperbolic compact invariant manifold Σ_{η_0}, invariant under the action of $\mathbf{O}(1)^\nu$ and whose intersection with the non-negative orthant $\mathbf{R}^\nu_+$ is connected, hence:

(P) If $\mathbf{R}^J$, $J \subset \{1, \ldots, \nu\}$, is the smallest coordinate subspace which contains Σ_{η_0}, then Σ_{η_0} is transversal *in* $\mathbf{R}^J$ to the coordinate subspaces lying in $\mathbf{R}^J$.

Since normal hyperbolicity is open, there is an open neighbourhood $\mathcal{V}_{\eta_0}$ of η_0 in $T_f\mathcal{F}$ such that every W_η with $\eta \in \mathcal{V}_{\eta_0}$ has a normally hyperbolic invariant manifold Σ_η diffeomorphic to Σ_{η_0} and close to it, unique and therefore $\mathbf{O}(1)^\nu$–invariant, such that Σ_η satisfies (P) with the same J. It follows that $S_{0,\eta} := \rho^{-1}(\Sigma_\eta)$ is a compact $\mathbf{U}(1)^J$–invariant submanifold of $\mathbf{C}^J$ with the same smoothness and codimension as Σ_v.

Remarks Each submanifold $\varepsilon^{\frac{1}{2}}\Sigma_{\eta_0}$, $\varepsilon > 0$, is a normally hyperbolic invariant manifold of $W_{\varepsilon\eta_0}$.

As the mapping $\varphi \mapsto \big(\lambda_1(\varphi), \ldots, \lambda_n(\varphi)\big)$ is a submersion at f, the existence of η_0 is equivalent to that of $v \in \mathbf{R}^\nu$ such that the vector field $\sum_j x_j \Big(v_j - \sum_\ell \beta_{j\ell}(f)\, x_\ell^2\Big) \frac{\partial}{\partial x_j}$ has a compact normally hyperbolic invariant manifold as before.

Using diagonal changes of coordinates in $\mathbf{C}^\nu$, one can see that, for each ℓ, the functions $\beta_{j\ell} + i\,\gamma_{j\ell}$ are defined up to multiplication by the same positive function of φ, which enables one to assume $\beta_{jj}(f) \in \{-1, 01\}$ for every j and $\beta_{jj}(\varphi) = \pm 1$ for all φ if $\beta_{jj}(f) = \pm 1$.

2.9.2 Hypotheses and notation in the case of maps

Assuming $\mathcal{F} = C^\infty(M, M)$, we consider points $(f, a) \in \widetilde{\Sigma}_0$ satisfying:

(H_1) The eigenvalues $e^{\pm i\alpha_1(f)}, \ldots, e^{\pm i\alpha_\nu(f)}$, $0 < \alpha_1(f) < \cdots < \alpha_\nu(f) \le \pi$, of d_af which have modulus 1 are simple[31] and for $1 \le j \le \nu$, the only solutions $(p, q) \in (\mathbf{N}^\nu)^2$ with $\sum(p_\ell + q_\ell) \le 4$ of the equation $e^{i\alpha_j(f)} = e^{i\sum(p_\ell - q_\ell)\alpha_\ell(f)}$ are the obvious ones: $p_j = q_j + 1$ and $p_\ell = q_\ell$ for $\ell \neq j$.

Set $\mathbf{K} := \begin{cases} \mathbf{R} & \text{if } \alpha_\nu(f) = \pi, \\ \mathbf{C} & \text{otherwise} \end{cases}$ and $h := \begin{cases} n - 2\nu + 1 & \text{if } \alpha_\nu(f) = \pi, \\ n - 2\nu & \text{otherwise.} \end{cases}$

[31] Including the eigenvalue -1 if $\alpha_\nu(f) = \pi$.

Given a central manifold W^c of the tautological unfolding $(\varphi,x)\mapsto(\varphi,\varphi(x))$ at (f,a) (whose "slices" W^c_φ have codimension h) the theory of normal forms implies that there exists a fairly smooth family of local charts $(\mathcal{F}\times M,(f,a))\ni(\varphi,x)\mapsto c_\varphi(x)\in(\mathbf{C}^{\nu-1}\times\mathbf{K}\times\mathbf{R}^h,0)$ such that $c_{\varphi*}\varphi(0)=0$, that $c_\varphi(W^c_\varphi)=(\mathbf{C}^{\nu-1}\times\mathbf{K}\times\{0\})\cap\operatorname{Im}c_\varphi$ and that $c_{\varphi*}\varphi|_{\mathbf{C}^{\nu-1}\times\mathbf{K}\times\{0\}}$ has contact of order 4 at 0 with $\sigma\circ g^1_{N_\varphi}=g^1_{N_\varphi}\circ\sigma$, where N_φ is of the form (7) with $w=(w_1,\dots,w_\nu)\in\mathbf{C}^{\nu-1}\times\mathbf{K}$,

$$\sigma(w)=\begin{cases}w & \text{if } \mathbf{K}=\mathbf{C},\\ (w_1,\dots,w_{n-1},-w_n) & \text{otherwise,}\end{cases}$$

and $\lambda_j,\alpha_j,\beta_{j\ell},\gamma_{j\ell}$ are smooth real functions with $\lambda_j(f)=0$, $\alpha_j(f)$ is as before for $j<\nu$ and, if $\mathbf{K}=\mathbf{C}$, for $j=\nu$ and, of course, $\alpha_\nu(\varphi)=\gamma_{\nu\ell}(\varphi)=0$ if $\mathbf{K}=\mathbf{R}$.

We assume that N_φ satisfies (H_2) and, if $\mathbf{K}=\mathbf{R}$, we denote again by $S_{0,\eta}$ the intersection of $S_{0,\eta}\subset\mathbf{C}^\nu$ with $\mathbf{C}^{\nu-1}\times\mathbf{R}$.

2.9.3 The birth(day) lemma and some consequences

The proof of the following result follows the same lines as the proof sketched for Theorem 2.8.1.

Theorem 2.9.1 (birth lemma) *Under the previous hypotheses (H_1) and (H_2), there is an open subset $\mathcal{U}_{\eta_0}$ of $\mathcal{F}$, whose closure contains f and whose tangent cone*[32] *at f is an open cone with vertex 0 containing $\mathbf{R}^*_+\mathcal{V}_{\eta_0}$, such that every $\varphi\in\mathcal{U}_{v_0}$ has a compact normally hyperbolic invariant manifold S_φ diffeomorphic to S_{0,η_0}, with the same index and co-index as the invariant manifold Σ_{η_0} of W_{η_0}, depending at least $C^{1+\alpha}$ on φ and tending to $\{a\}$ when $\varphi\to f$. More precisely, every smooth path $\gamma:(\mathbf{R}_+,0)\to(\mathcal{F},f)$ with $\dot\gamma(0)\in\mathcal{V}_{\eta_0}$ satisfies $\gamma(\varepsilon)\in\mathcal{U}_{\eta_0}$ for small enough positive ε and $\lim_{\varepsilon\to0}\varepsilon^{-\frac12}c_{\gamma(\varepsilon)}(S_{\gamma(\varepsilon)})=S_{0,\dot\gamma(0)}$ in the at least C^1 sense.*

Remarks Of course, the set of such points (f,a) has codimension ν in $\widetilde\Sigma$ (it is a smooth submanifold). On the other hand, in contrast with the "tongues" of paragraph 2.7 for $p>3$, the open subsets $\mathcal{U}_{\eta_0}$ are not thin at all, as their tangent cone at (f,a) has nonempty interior.

The advantage of maps over vector fields is that one can draw the curves or surfaces S_ξ in three-space if $\nu=2$ and $\mathbf{K}=\mathbf{R}$.

To stick to the viewpoint of this article, it would probably be better to consider the submanifold of $\mathcal{F}\times M$ consisting of all (φ,x) with $x\in S_\varphi$ and its closure near (f,a). However, in contrast with paragraph 2.8, for $\nu>1$, the boundary of $\mathcal{U}_{\eta_0}$ is unknown and certainly not clean at all, marking the beginning of a *terra incognita* where terrible things (much worse than in [14]) must happen.

Example 1 (period doubling, Hopf and Naimark-Sacker bifurcations) If $J=\{\ell\}$ with $1\le\ell\le\nu$, hypothesis (H_2) means that $\beta_{\ell\ell}(f)\,D\lambda_\ell(f)\eta_0>0$, $D\lambda_j(f)\eta_0\neq0$ for $j\neq\ell$ and $\Sigma_{\eta_0}=\left\{x\in\mathbf{R}^J:|x_\ell|=\sqrt{\frac{D\lambda_\ell(f)\eta_0}{\beta_{\ell\ell}(\eta_0)}}\right\}$. Therefore,

– in the case of vector fields, the S_φ's are periodic orbits: if $\nu=1$, we get a somewhat weak version of the Hopf bifurcation

– in the case of maps with $\ell=\nu$ and $\mathbf{K}=\mathbf{R}$, the S_φ's are 2–periodic orbits: if $\nu=1$, we get a somewhat weak version of the period doubling bifurcation

[32] Set of velocities $\dot\gamma(0)$ of differentiable paths $\gamma:(\mathbf{R},0)\to(\mathcal{F},f)$ satisfying $\gamma(\varepsilon)\in\mathcal{U}_{\eta_0}$ for every small positive ε.

– in the case of maps with $\ell < \nu$ or $\mathbf{K} = \mathbf{C}$, the S_φ's are invariant circles: if $\nu = 1$, we get the Naimark-Sacker bifurcation.

Example 2 (birth of invariant tori of dimension d, $1 < d \leq \nu$) For $\#J = d > 1$, hypothesis (H_2) is satisfied with $\dim \Sigma_{v_0} = 0$ if and only if the following three conditions hold: the matrix $A = (\beta_{j\ell})_{j,\ell \in J}$ is invertible, the point $y_{\eta_0} := A^{-1}\big(D\lambda_j(f)\eta_0\big)_{j\in J}$ of $\mathbf{R}^J$ has all its coordinates positive and the zero $x_{\eta_0} \in \mathbf{R}^J$ of X_{η_0} defined by $x_{\eta_0 j} = \sqrt{y_{\eta_0 j}}$, $j \in J$, is hyperbolic. Then, $\Sigma_{\eta_0} = \mathbf{O}(1)^\nu x_{v_0}$ and S_{0,η_0} is

– a d–dimensional torus in the case of vector fields or, in the case of maps, if $\mathbf{K} = \mathbf{C}$ or $\nu \notin J$

– a pair $(d-1)$–dimensional tori in the case of maps for $\mathbf{K} = \mathbf{R}$ and $\nu \in J$,

hence so are the S_φ's.

Trivial *modulo* the birth lemma, those examples are the only ones for which Σ_{η_0} has dimension 0. They occur near most of the points (f, a) satisfying (H_1). It is not surprising that the coupling of d oscillators should give rise to invariant d–tori. The reason for this work is that other invariant manifolds seem to appear as well in Nature, though probably in a less naive setting than ours.

Example 3 (birth of invariant spheres of dimension $2d-1$, $2 \leq d \leq \nu$) For $\#J = d > 1$, the submanifold Σ_{η_0} can be [19, 20, 13] an embedded sphere $\mathbf{S}^{d-1}$ around the origin in $\mathbf{R}^J$, a situation *which occurs in a nonempty open subset of the ν–codimensional submanifold of $\widetilde{\Sigma}$ defined by* (H_1) [13]. Then, the S_φ's are embedded spheres $\mathbf{S}^{2d-1}$ ($\mathbf{S}^{2d-2}$ in the case of maps with $\mathbf{K} = \mathbf{R}$ and $\nu \in J$). If $d = \nu$, they lie around the origin in $\mathbf{C}^{\nu-1} \times \mathbf{K} \times \{0\}$ and the result is much more reminiscent of the Hopf or Naimark-Sacker bifurcation than the birth of invariant tori.

In contrast with the latter case, the existence of such a sphere Σ_{η_0} is nontrivial. To introduce more familiar objects, one can remark that the change of variables $y = \pi(x) := (x_1^2, \ldots, x_n^2)$ sends X_{η_0} onto the Lotka-Volterra field $Y_{\eta_0}(y) = 2\sum_j y_j \Big(D\lambda_j(f)\eta_0 - \sum_\ell \beta_{j\ell}(f)\, y_\ell\Big) \frac{\partial}{\partial y_j}$ on $\mathbf{R}^n_+$. The sphere Σ_{η_0} is simply the inverse image by π of the carrying simplex of Y_{v_0}, whose existence can follow from Hirsch's theorem[33] [16]. Details will be given in [13].

Example 4 (birth of many other kinds of compact invariant manifolds) This answers a theoretical question: what can be the topology of higher dimensional analogues of the Hopf or Naimark-Sacker circles? The birth lemma yields many topologically different such invariant manifolds at high (but finite) codimensional points of the submanifold defined by (H_1). More precisely, these higher dimensional invariant manifolds include all *moment-angle manifolds*, i.e. nonempty real algebraic submanifolds S of $\mathbf{C}^n$ of the form $S = Q^{-1}(b)$, where $b \in \mathbf{R}^c$ is a regular value of a quadratic map $Q(z) = \Lambda_1|z_1|^2 + \cdots + \Lambda_n|z_n|^2$ such that the convex hull of the points $\Lambda_j \in \mathbf{R}^c$ does not contain the origin. The proof, thanks to Santiago López de Medrano, is quite easy [12]. Such manifolds can have extremely diverse topologies [21, 4]

Remarks The invariant manifolds provided by the birth lemma can form a series of matrioshkas. However, a remarkable feature of the last examples is that, the dynamics on the invariant manifold being extremely unstable (the invariant manifold Σ_{η_0} of our construction is *pointwise* invariant by the flow of W_{η_0}), no obvious smaller invariant manifold for ξ is contained in S_ξ.

[33] If the $\beta_{j\ell}(f)$ are all positive, it is quite easy to show the birth of a compact invariant"homology sphere", but additional conditions are needed to ensure that it is a regular normally hyperbolic manifold.

It is not difficult to construct stable situations where an invariant (attracting) 3–sphere of a flow persists whereas periodic/quasiperiodic bifurcations, for example, occur inside that big sphere. This can model interesting simple facts of Nature, such as the transition between two periodic regimes.

From a theoretical viewpoint, the invariant submanifolds of a dynamical system can be viewed as fixed points of the dynamical system induced by the given one on the space of submanifolds. Normal hyperbolicity corresponds to hyperbolic fixed points and one can dream of extending the theory beyond normal hyperbolicity. KAM theory might be the beginning of such an extension.

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References

[1] B. Abbaci. On a theorem of Philip Hartman. *C. R. Acad. Sc. Paris* Sér. I **339** (2004) 781–786

[2] Abraham, R., Robbin, J.: Transversal mappings and flows. Benjamin, New York (1967).

[3] Arnold, V.I.: Geometric methods in the theory of ordinary differential equations, Springer, Berlin Heidelberg New York (1983)

[4] F. Bosio, L. Meersseman. Real quadrics in $\mathbf{C}^n$, complex manifolds and convex polytopes. Acta Math. 197 n^o 1 (2006), 53–127

[5] Chaperon, M.: Linéarisation des germes hyperboliques d'actions différentiables de $\mathbf{R}^k \times \mathbf{Z}^m$: le domaine de Poincaré. C. R. Acad. Sc. Paris **289**, série A (1979), 325–328

[6] Chaperon, M.: Géométrie différentielle et singularités de systèmes dynamiques. Astérisque **138**–**139** (1986)

[7] Chaperon, M.: C^k–conjugacy of holomorphic flows near a singularity. Publ. Math. I.H.E.S. **64** (1987), 143–183

[8] Chaperon, M.: Invariant manifolds revisited. Proceedings of the Steklov Institute **235** (2002), 415–433

[9] Chaperon, M.: Birth control in generalized Hopf bifurcations, C. R. Acad. Sci. Paris, Ser. I 345 (2007), 453–458

[10] Chaperon, M., López de Medrano, S., Samaniego, J.L.: On sub-harmonic bifurcations, C. R. Acad. Sci., Paris, Sér. I, **340** (2005), 827–832

[11] Chaperon, M., López de Medrano, S.: On the Hopf bifurcation for flows, C. R. Acad. Sci., Paris, Sér. I, **340** (2005), 833–838

[12] Chaperon, M., López de Medrano, S.: Birth of attracting compact invariant submanifolds diffeomorphic to moment-angle manifolds in generic families of dynamics. C. R. Acad. Sci. Paris, Sér. I, **346** (2008), 1099–1102

[13] Chaperon, M., López de Medrano, S.: Generalized Hopf bifurcations, to appear

[14] Chenciner, A.: Bifurcations de points fixes elliptiques,

I. Courbes invariantes. Publ. Math. I.H.E.S. **61**, 67–127 (1985)

II. Orbites périodiques et ensembles de Cantor invariants. Invent. Math. **80**, 81–106 (1985)

III. Orbites périodiques de "petites" périodes et élimination résonnante des couples de courbes invariantes. Publ. Math. I.H.E.S. **66**, 5–91 (1988)

[15] P. Hartman, *On local homeomorphisms of Euclidean spaces*, Bol. Soc. Mat. Mex. **5** (1960), 220–241

[16] M. W. Hirsch. Systems of differential equations that are competitive or cooperative. III: Competing species, *Nonlinearity* 1 (1988), 51–71

[17] Hunt, B.R., Sauer, T., Yorke, J.A.: Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces. Bull. Amer. Math. Soc. (N.S.) **27**, no. 2, 217–238 (1992)

[18] Iglesias, P.: Diffeology. To appear.

[19] Kammerer-Colin de Verdière, M.: Stable products of spheres in the non-linear coupling of oscillators or quasi-periodic motions, C. R. Acad. Sci. Paris, Série I **335** (2004), 625–629

[20] Kammerer-Colin de Verdière, M.: Bifurcations de variétés invariantes. Thèse, Université de Bourgogne, décembre 2006

[21] S. López de Medrano. The space of Siegel leaves of a holomorphic vector field. Dynamical Systems, Lecture Notes in Mathematics, Vol. 1345 (1988), Springer-Verlag, 233–245

[22] Nelson, E.: Topics in Dynamics—I: flows. Princeton University Press (1969)

[23] Rodrigues, H. M., Solà-Morales, J.: An Invertible Contraction that is not C^1-linearizable, C. R. Acad. Sci. Paris, Ser. I 340 (2005), 847–850

[24] Sard, A.: The measure of the critical points of differentiable maps. Bull. Amer. Math. Soc. **48**, 883–890 (1942)

[25] Smale, S.: An infinite dimensional version of Sard's theorem. Am. J. Math. **87**, 861–866 (1965)

[26] Sternberg, S.: On the structure of local homeomorphisms of euclidean n–space II. Amer. J. of Math. **80**, 623–631 (1958)

[27] Thom, R.: Un lemme sur les applications différentiables, Bol. Soc. Mat. Mexicana, 59–71 (1956)