Derived Canonical Algebras as One-Point Extensions

Michael Barot and Helmut Lenzing

Abstract. Canonical algebras have been intensively studied, see for example [12], [3] and [11] among many others. We are interested in the question when a one-point extension of a finite-dimensional algebra $\Sigma$ by a $\Sigma$-module $M$ is derived canonical, i.e. derived equivalent to a canonical algebra. We give necessary conditions on the algebra $\Sigma$ and the module $M$. If the canonical algebra associated with $\Sigma$ is tame the conditions are even sufficient. As a further result we obtain that, if $\Sigma$ is derived canonical then the one-point extension of $\Sigma$ by $M$ is derived canonical again if and only if $M$ is derived simple, i.e. $M$ is indecomposable and belongs to the mouth of a tube in the bounded derived category of $\Sigma$-modules.

1. Prerequisites

We work over an algebraically closed base field $k$. By algebra we mean always a basic, finite dimensional $k$-algebra and by module we mean finitely generated right module. For an algebra $\Sigma$ we denote by $\text{mod} \Sigma$ the category of $\Sigma$-modules.

An algebra $\Sigma$ which is derived equivalent to a canonical algebra will be called derived canonical. This terminology replaces the less suggestive term quasi-canonical used in [10]. For each derived canonical algebra there exists a weighted projective line $X = X(p, \lambda)$ such that the (bounded) derived category $D^b(\Sigma)$ of $\text{mod} \Sigma$ is equivalent to the (bounded) derived category of $\text{coh} X$, the category of coherent sheaves over $X$, see [3]. Two derived canonical algebras are derived equivalent if and only if their associated weighted projective lines are isomorphic, see [4] and [10] for further details. Quite important information on $X$, the weight type $p = (p_1, \ldots, p_t)$, can already be read from the Coxeter polynomial of $\Sigma$ which has the shape

$$(T - 1)^2 \prod_{i=1}^{t} \frac{T^{p_i} - 1}{T - 1}.$$  

Note that, if $t \leq 3$, then $X$ is completely determined by its weight type. The genus of $X$, hence of $\Sigma$, defined by

$$g_X = g_\Sigma := 1 + \frac{1}{2} \left( (t-2)p - \frac{p}{p_1} - \cdots - \frac{p}{p_t} \right),$$

where $p = \text{l.c.m.}(p_1, \ldots, p_t)$.

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determines the representation type of coh X and of the canonical algebra \( \Lambda \) associated with \( X \), see [3]: for \( g_X \leq 1 \) both coh \( X \) and \( \Lambda \) are tame, whereas coh \( X \) and \( \Lambda \) are wild if \( g_X > 1 \). If \( g_X = 1 \) the algebra \( \Lambda \) is tubular, see [12, 10]. This happens if and only if the weight type is — up to permutation — one of \((2,2,2,2),(3,3,3),(2,4,4)\) or \((2,3,6)\). Note, in this context, that passing from the canonical algebra \( \Lambda \) to a derived equivalent algebra \( \Sigma \) may simplify the representation type.

For a hereditary category \( \mathcal{C} \) like coh \( X \) or mod \( H \), \( H \) hereditary, the derived category \( D^b(\mathcal{C}) = \bigcup_{n \in \mathbb{Z}} \mathcal{C}[n] \), the additive closure of the union of all \( \mathcal{C}[n] \)'s, is known as well as \( \mathcal{C} \). Here, each \( \mathcal{C}[n] \) is a copy of \( \mathcal{C} \) with objects written \( X[n] \), \( X \in \mathcal{C} \), and morphisms are given by

\[
\text{Hom}_{D^b(\mathcal{C})}(X[m], Y[n]) = \text{Ext}^n_{\mathcal{C}}(X, Y).
\]

Note that we have natural identifications for the Grothendieck-groups

\[
K_0(\mathcal{C}) = K_0(\Sigma) = K_0(D^b(\Sigma)),
\]

and that the identifications preserve the Euler forms given on classes of \( \Sigma \)-modules (coherent sheaves, objects from the derived category, respectively) by the formula

\[
\langle [X], [Y] \rangle = \sum_{i = -\infty}^{\infty} (-1)^i \dim_k \text{Hom}(X, Y[i]).
\]

Let \( A \) be an algebra of finite global dimension. We recall from [5] that the bounded derived category \( D^b(A) \) of finite dimensional \( A \)-modules has Auslander-Reiten triangles. We say that an indecomposable \( A \)-module \( M \) is derived peripheral if the “middle term” \( E \) of the Auslander-Reiten triangle \( \tau M \to E \to M \to \tau M[1] \) is indecomposable. An indecomposable \( A \)-module \( M \) is further called derived simple or also derived simple regular, if \( M \) is derived peripheral and \( \tau \) is periodic for the Auslander-Reiten translation \( \tau \) of \( D^b(A) \). Moreover, still assuming that \( M \) is indecomposable, we call \( M \) derived quasi-simple if \( M \) is derived peripheral and lies in a component of the form \( \mathbb{Z} \Delta \) in \( D^b(A) \).

Assuming that \( \Sigma \) is derived canonical, it follows from [3, 11] and [10] that \( M \) is derived simple if and only if there exists a self-equivalence \( \varphi \) of \( D^b(\Sigma) = D^b(\text{coh } X) \) and a simple sheaf \( S \) on \( X \) such that \( M = \varphi(S) \). For \( g_X \neq 1 \) the situation simplifies, and we may choose \( \varphi \) to be a translation functor \( X \mapsto X[n] \).

Assuming that the algebra \( \Sigma \) is derived equivalent to a wild hereditary algebra \( H \), a \( \Sigma \)-module \( M \) is derived quasi-simple if and only if there exists an integer \( n \) such that \( M[n] \) is a regular \( H \)-module of quasi-length one.

For a representation-finite connected hereditary algebra \( H \) we fix an identification of the Auslander-Reiten quiver of \( D^b(H) \) with the translation quiver \( \mathbb{Z} \Delta \) of the Dynkin diagram \( \Delta \) attached to \( H \). Relative to such an identification we define the derived type of an indecomposable \( \Sigma \)-module, with \( \Sigma \) derived equivalent to \( H \), as the vertex \( v \) of \( \Delta \) such that \( M \) belongs to the \( \tau \)-orbit of \( v \) in \( \mathbb{Z} \Delta \).

2. Derived canonical one-point extensions

Given an algebra \( \Sigma \) and a \( \Sigma \)-module \( M \), we will consider the one-point sink extension or just sink extension \([M] \Sigma \) which is given as

\[
[M] \Sigma = \begin{bmatrix}
\Sigma & 0 \\
M & k
\end{bmatrix}
\]
with the corresponding matrix operations. Dually we define the one-point source extension or just source extension $\Sigma[M]$ (they are also called one-point coextension and one-point extension in the literature).

We are now going to investigate when a one-point extension is derived canonical. Since an algebra is derived canonical if and only if its opposite algebra is it suffices to investigate under which conditions a sink extension $[M]$ is derived canonical.

We need to introduce some notation for the special case where $\Sigma$ is derived equivalent to $k[\Lambda_n]$, i.e. $\Sigma$ is a branch with $n$ points, see [8, 1]. In this case, the indecomposable objects in the derived category $D^b(\Sigma)$ form a single Auslander-Reiten component of type $\mathbb{Z}\Lambda_n$. A slice $S$ in $D^b(\Sigma)$ is called a $(p; q)$-slice if in the quiver of $S$, there are $p$ arrows pointing upwards and the remaining $n - p - 1 = q$ arrows point downwards.

In the above figure, we have marked a $(1, 4)$-slice, and the two peripheral objects belonging to that slice.

Back in the general situation, where we consider the sink extension $[M]\Sigma$, we denote by $\overline{M}$ the indecomposable projective module corresponding to the sink vertex of $\overline{\Sigma} = [M]\Sigma$, thus the radical of $\overline{M}$ is just $M$. We view mod $\Sigma$ as the full exact subcategory of mod $\overline{\Sigma}$ consisting of all $\overline{\Sigma}$-modules $X$ with $\text{Hom}_{\overline{\Sigma}}(\overline{M}, X) = 0$. The inclusion mod $\Sigma \subset$ mod $\overline{\Sigma}$ induces an inclusion $D^b(\Sigma) \subset D^b(\overline{\Sigma})$. In particular, identifying modules with stalk complexes in the corresponding derived category, mod $\Sigma$ and mod $\overline{\Sigma}$ become full subcategories of $D^b(\overline{\Sigma})$.

**Theorem 1.** Let $\Sigma$ be an algebra and $M$ a $\Sigma$-module. Assume that $\overline{\Sigma} = [M]\Sigma$ is derived canonical. Then $M$ is the middle term of the Auslander-Reiten triangle $\tau\overline{M} \to M \to \overline{M} \to \tau\overline{M}[1]$ in $D^b(\overline{\Sigma})$ associated with the projective module $\overline{M}$ attached to the sink vertex of $\overline{\Sigma}$.

If further $\Sigma = \Sigma_1 \times \cdots \times \Sigma_s$ is the decomposition of $\Sigma$ into connected algebras, and $M = M_1 \times \cdots \times M_s$ is the corresponding decomposition of $M$ into $\Sigma_i$-modules $M_i$, then $s \leq 4$ and exactly one of the following cases happens:

1. a. $\Sigma$ is derived equivalent to $k[\Lambda_4]$ and $M = M' \oplus M''$ is the direct sum of two indecomposable modules $M'$ and $M''$ forming the periphery of a $(p - 1, q - 1)$-slice, $\ell = p + q - 1$, of the component $\mathbb{Z}\Lambda_4$ of $D^b(\Sigma)$.

Conversely, for each such choice for $\Sigma$ and $M$ the sink extension $[M]\Sigma$ is derived canonical of weight type $(p, q)$. 
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Table 1: Choices for $(\Sigma_i, M_i)$
b. Each $\Sigma_i$ is derived equivalent to a representation-finite connected hereditary algebra $H_i$, and each $M_i$ is an indecomposable $\Sigma_i$-module, where the Dynkin types for the $\Sigma_i$'s and the derived types for the $M_i$'s are listed in the table below.

Conversely, for each choice of $(\Sigma, M)$ conforming to the table, the sink extension $[M]\Sigma$ is derived canonical with the type given by the first column of the table.

2. $\Sigma = \Sigma_1 \times \Sigma_2$, where $\Sigma_1 \neq 0$ is derived canonical and $\Sigma_2$ is derived equivalent to a hereditary algebra of type $A_\ell$, $0 \leq \ell$. Moreover, $M_1$ is derived simple over $\Sigma_1$, and — if $\Sigma_2$ is non-zero — the $\Sigma_2$-module $M_2$ is derived peripheral.

Conversely, for each such choice of $(\Sigma, M)$, the sink extension $[M]\Sigma$ is derived canonical of weight type $(p_1, \ldots, p_{\ell-1}, p_\ell + \ell)$, if $\Sigma_1$ has weight type $(p_1, \ldots, p_\ell)$ and $M_1$ has $\tau$-period $p_\ell$ in the derived category.

3. $\Sigma = \Sigma_1 \times \Sigma_2$, where $\Sigma_1 \neq 0$ is derived equivalent to a connected wild hereditary algebra $H$, and $\Sigma_2$ is derived equivalent to a hereditary algebra of type $A_\ell$, $0 \leq \ell \leq 5$. Moreover, the $\Sigma_1$-module $M_1$ is derived quasi-simple, and — if $\Sigma_2$ is non-zero — the $\Sigma_2$-module $M_2$ is derived peripheral.

Concerning statement 3 it is an interesting open question when the sink extension $[M]\Sigma$ of a wild hereditary algebra $\Sigma$ by a regular quasi-simple module $M$ is derived canonical. The problem is related to the question when sink and source extension algebras are quasi-tilted [6].

Proof. The first assertion is proved in [2]. Next we show that the conditions listed in 1, 2 and 3 are necessary, and exhaust all possible cases. Since $\Sigma$ is derived canonical, there is a weighted projective line $\overline{\Sigma}$ such that $\Sigma$ can be realized as a tilting complex in the derived category $D^b(\overline{\Sigma})$, where $\overline{\Sigma} = \text{coh} \overline{\Sigma}$, i.e. $\Sigma$ is a full subcategory of $D^b(\overline{\Sigma})$ consisting of indecomposable objects $\overline{\Sigma}_j$, $j = 1, \ldots, n + 1$, which satisfy the condition

$$\text{Hom}_{D^b(\overline{\Sigma})}(\overline{\Sigma}_i, \overline{\Sigma}_j[m]) = 0$$

for all $m \in \mathbb{Z} \setminus \{0\}$ and all $i, j = 1, \ldots, n + 1$ and generate $D^b(\overline{\Sigma})$ as a triangulated category. In the present setting this latter condition is satisfied if and only if $n + 1$ equals the rank of $K_0(\overline{\Sigma})$.

We denote by $M$ the indecomposable object $\overline{\Sigma}_{n+1}$ corresponding to the coextension vertex, and by $\Sigma$ the full subcategory consisting of $\overline{\Sigma}_1, \ldots, \overline{\Sigma}_n$. By translation we may moreover assume that $M$ lies in $\overline{\Sigma}$. Notice that $M$ is an exceptional object of $\overline{\Sigma}$, i.e. has trivial endomorphism ring and no self-extensions. Let $H = M^\perp_{\overline{\Sigma}}$ denote the perpendicular category

$$H = \{ X \in \overline{\Sigma} \mid \text{Hom}_{\overline{\Sigma}}(M, X) = 0 = \text{Ext}^1_{\overline{\Sigma}}(M, X) \}$$

of $M$ formed in $\overline{\Sigma}$, and let $D = M^\perp_{D^b(\overline{\Sigma})}$ denote the perpendicular category

$$D = \{ X \in D^b(\overline{\Sigma}) \mid \text{Hom}_{D^b(\overline{\Sigma})}(M, X[m]) = 0 \text{ for all } m \in \mathbb{Z} \}$$

of $M$ formed in the derived category $D^b(\overline{\Sigma})$. It is easily checked that $H$ is an abelian hereditary category, and that $D = \bigvee_{n \in \mathbb{Z}} H[n]$. Moreover, $\Sigma$ is a tilting complex in $D$.

The structure of $H$ is given as follows:

Case (i): If $M$ has finite length $n$, then $M$ lies in an exceptional tube and $H \cong \overline{\Sigma} \times \text{mod} \overline{H}$, where $\overline{\Sigma} = \text{coh} \overline{X}$ for a weighted projective line of a weight type
dominated by the weight type of \( \Delta \) and where \( H = k[1 \to 2 \to \cdots \to n - 1] \). This follows from [4, Thm. 9.5] invoking an argument of Strauß [13]. The figure below shows the relevant part of the component of \( \mathcal{D}^b(\mathcal{C}) \) containing \( \mathcal{M} \):

Here, the indecomposable \( H \)-modules form the subwing with “top” \( M_2 \). According to the decomposition \( \mathcal{D}^b(\mathcal{H}) = \mathcal{D}^b(\mathcal{C}) \times \mathcal{D}^b(\text{mod } H) \), the algebra \( \Sigma \) decomposes into two connected algebras \( \Sigma_1 \) and \( \Sigma_2 \), where \( \Sigma_1 \) is a tilting complex in \( \mathcal{D}^b(\mathcal{C}) \) and \( \Sigma_2 \) is a tilting complex in \( \mathcal{D}^b(\text{mod } H) \). Note that \( M_2 \) is derived peripheral over \( H \), hence over \( \Sigma_2 \).

Further the object \( M_1 \) at the top of the figure is a simple object in \( \mathcal{C} \), and therefore \( M_1 \) becomes a derived simple \( \Sigma_1 \)-module. This proves the first part of the statement 2.

Case (ii): \( \mathcal{M} \) is an exceptional vector bundle in \( \text{coh } \mathcal{X} \). Here, it follows from [7] that \( \mathcal{H} \) is equivalent to a module category over a (not necessarily connected) hereditary algebra. For a more complete analysis, we need to distinguish the various representation types for \( \mathcal{C} = \text{coh } \mathcal{X} \):

1. \( \mathcal{X} \) has genus \( < 1 \), i.e. the weight type \( \Delta = (p, q, r) \) is of Dynkin type. Here the vector bundles form one component of type \( \Xi \), where \( \Xi \) is the extended Dynkin diagram corresponding to \( \Delta \). To calculate the perpendicular category of \( \mathcal{M} \) in \( \mathcal{D}^b(\Sigma) \) we choose a slice \( \overline{\Pi} \) of \( \Xi \) such that \( \mathcal{M} \) becomes a sink in \( \overline{\Pi} \), which is a tilting object of \( \overline{\mathcal{C}} \) whose endomorphism ring, here identified with \( \overline{\Pi} \), is a tame hereditary algebra. Since \( \mathcal{D}^b(\overline{\mathcal{C}}) = \mathcal{D}^b(\text{mod } \overline{\Pi}) \), the perpendicular category of \( \mathcal{M} \) in \( \mathcal{D}^b(\overline{\mathcal{C}}) \) equals the derived category of \( \mathcal{M} \text{mod } \overline{\Pi} \). This category is equivalent to the module category of a not necessarily connected hereditary algebra \( H \), whose indecomposable objects consist of the objects of the slice \( \overline{\Pi} \) different from \( \mathcal{M} \). The arising cases for \( H \) are listed in the table. Moreover, the almost-split sequence \( 0 \to \tau \mathcal{M} \to M \to \mathcal{M} \to 0 \) in \( \overline{\mathcal{C}} \), obtained from the first assertion, yields the
derived types of the $M_i$, $M = \bigoplus_{i=1}^s M_i$, as marked in the table. Since

$$M^\mathcal{L}_\mathcal{M} = \text{D}^b(\mathcal{M}^\mathcal{L}_{\text{mod} \mathcal{M}}) = \text{D}^b(\text{mod} \ H) = \prod_{i=1}^s \text{D}^b(\text{mod} \ H_i)$$

the tilting complex $\Sigma$ decomposes into $s$ connected pieces $\Sigma_i$, where each $\Sigma_i$ as a tilting complex in $\text{D}^b(\text{mod} \ H_i)$ is derived-equivalent to $H_i$. This shows the first part of statement 1.

2. $\mathcal{X}$ has genus one. By an automorphism of the derived category we can in this case achieve that $\mathcal{M}$ has finite length, see [10]. The assertion thus reduces to case (i).

3. $\mathcal{X}$ has genus $> 1$. In this case, $\mathcal{M}$ belongs to a component of $\mathcal{C}$ having type $\mathcal{Z}A_{\infty}$ [11], and it is known that the quasi-length $\ell$ of $\mathcal{M}$ is at most 5 [loc. cit.]. Invoking arguments of [13] it further follows from [7] that $\mathcal{M}_{\mathcal{P}}$ is equivalent to the product of mod $A_{\ell-1}$, where $A_{\ell-1} = k[1 \to \cdots \to \ell - 1]$, with the module category mod $H$ over a connected wild hereditary algebra $H$. Accordingly $\Sigma$ decomposes into a product $\Sigma_1 \times \Sigma_2$, where $\Sigma_1$ is connected and derived wild hereditary, and where $\Sigma_2$ is derived equivalent to $A_{\ell-1}$, i.e. a branch in the sense of [8, 1]. Following arguments of [13] and [9] it follows moreover that $M_1$ is derived quasi-simple and $M_2$ is derived peripheral. This proves the first part of the statement 3.

Now we show the second part of the statements 1 and 2. So, first let $\Sigma$ be derived equivalent to $k[\mathcal{A}_{\ell}]$ and $M = M' \oplus M''$ a $\Sigma$-module such that $M'$ and $M''$ are indecomposable and form the periphery of a $(p-1, q-1)$-slice $S$ of the component $\mathcal{Z}A_{\ell}$ of $\text{D}^b(\Sigma)$, where $p$ and $q$ are such that $\ell = p + q - 1$. By the first statement of the theorem, we have that $S \oplus \mathcal{M}$ is a tilting complex in $\text{D}^b([M] \Sigma)$ with endomorphism algebra isomorphic to $k[\mathcal{A}_{\ell}]$. Thus $[M] \Sigma$ is derived canonical.

With the same argument we show that $[M] \Sigma$ is derived canonical, when $\Sigma = \Sigma_1 \times \cdots \times \Sigma_s$ and $M = M_1 \times \cdots \times M_s$ where $\Sigma_j$ is derived hereditary and $M_j$ is an indecomposable $\Sigma_j$-module $(j = 1, \ldots, s)$ such that the pair $(\Sigma, M)$ is listed in Table 1.

Let now $\Sigma = \Sigma_1 \times \Sigma_2$, where $\Sigma_1$ is derived canonical and $\Sigma_2$ is derived equivalent to $k[\mathcal{A}_{\ell}]$ for some $\ell \geq 0$. Further let $M = M_1 \times M_2$, where $M_1$ is derived simple and, if $\ell > 0$, then let $M_2$ be derived peripheral. Let $\mathcal{X}(p_1, \ldots, p_\ell)$ be the weighted projective line associated to $\Sigma_1$, where $p = (p_1, \ldots, p_\ell)$ is its weight type. Let $\mathcal{X}$ be the weighted projective line with weight type $(p_1, \ldots, p_{\ell-1}, p_\ell + \ell)$ and with the same parameter sequence $\lambda$ as $\mathcal{X}$. We fix an indecomposable sheaf $E$ of length $\ell + 1$ concentrated at $\lambda_i$ and form the perpendicular category $\mathcal{H} = E^\perp$ of $E$ in $\mathcal{C} = \text{coh} \mathcal{X}$. Then $\mathcal{H} = \mathcal{C} \times \text{mod} \ H$, where $\mathcal{C} = \text{coh} \mathcal{X}$ and $H = k[1 \to \cdots \to \ell]$. Moreover, the middle term of the almost-split sequence $0 \to \tau E \to S \oplus M_2 \to E \to 0$ decomposes into a simple sheaf $S$ in $\mathcal{C}$, concentrated at $\lambda_i$ and in the indecomposable projective-injective $H$-module $M_2$. Next, we realize $\Sigma_1$ as a tilting complex in $\text{D}^b(\mathcal{C})$ so that, by means of the identification $\text{D}^b(\Sigma_1) = \text{D}^b(\mathcal{C})$ the module $M_1$ corresponds to the simple sheaf $S$. Further, we realize the branch $\Sigma_2$ as a tilting complex in $\text{D}^b(H)$ such that, in the identification $\text{D}^b(\Sigma_2) = \text{D}^b(H)$, the $H$-module $M_2$ becomes a (derived peripheral) module over $\Sigma_2$. Following [2], it is easily checked that $E$ together with $\Sigma_1$ and $\Sigma_2$ forms a tilting complex in $\text{D}^b(\text{coh} \mathcal{X})$ with endomorphism algebra $\Sigma = [M_1 \times M_2] \Sigma$. Hence $\Sigma$ is derived canonical of type $\mathcal{X}$.

This completes the proof of the Theorem.\)

\[\square\]
In Theorem 1 we have seen that the request for an algebra $\Sigma$ to admit a derived canonical source or sink extension is very restrictive for $\Sigma$ and for the “extension module” $M$. The information is even more specific if we start the extension procedure with a derived canonical algebra $\Sigma$.

**Corollary 1.** Let $\Sigma$ be derived canonical, and let $M$ be a finite dimensional not necessarily indecomposable $\Sigma$-module.

Then the sink extension $[M]\Sigma$ is derived canonical if and only if $M$ is derived simple, in particular indecomposable.

**Proof.** By Theorem 1, we only need to show that the condition is necessary. So let us assume that $[M]\Sigma$ is derived canonical. As a derived canonical algebra $\Sigma$ is connected, hence Theorem 1 implies that $M$ is derived simple. This uses that a wild hereditary or a representation-finite hereditary algebra is never derived equivalent to a canonical algebra since wild hereditary algebras always have spectral radius greater one and representation-finite hereditary algebras do not admit 1 as a root of their Coxeter polynomial, while canonical algebras have spectral radius one, and 1 is a root of their Coxeter polynomial. Thus cases 1 and 3 of Theorem 1 will not occur.

We now consider the case where a branch $B$ is attached to the extension vertex of $[M]\Sigma$, see [1] for definitions. We denote by $[B,M]\Sigma$ the resulting algebra.

**Corollary 2.** Let $\Sigma$ be a derived canonical algebra, $M$ a $\Sigma$-module and $B$ a branch.

Then $[B,M]\Sigma$ is derived canonical if and only if $M$ is derived simple.

**Proof.** First let $\Sigma = [B,M]\Sigma$ be derived canonical. Clearly, $\Sigma$ is derived equivalent to $\Sigma' = [B',M]\Sigma$, where $B'$ denotes the linearly ordered branch with its sink $\alpha$ as root point. Thus we may identify $\Sigma'$ with the algebra $[M \times P_\alpha](\Sigma \times B')$, where $P_\alpha$ denotes the projective indecomposable associated to the point $\alpha$. Therefore, we may apply Corollary 1. Once again, only the case 2 remains possible, and we infer that $M$ is a derived simple $\Sigma$-module.

The converse is covered by Theorem 1, part 2.

**2.1. Criteria for derived canonical algebras.** Theorem 1 provides us with a necessary condition for an algebra $\Sigma$ to be derived canonical. So we might use this in order to prove that certain algebras are not derived canonical. We shall exhibit this in an example. Let $A_n$ be the algebra given by the linear bound quiver with $n \geq 8$ vertices

```
x   x   x
```

which satisfies the $n-6$ relations $x^7 = 0$. It is not difficult to check that $A_9$, $A_{10}$ and $A_{11}$ are derived canonical of weight type $(2,3,5)$, $(2,3,6)$ and $(2,3,7)$, respectively. Also the Coxeter polynomial of $A_{22}$ has canonical type $(2,7,14)$. However, the algebra $A_{22}$ is not derived canonical.

Using Theorem 1, this can be seen as follows: First, we write $A_{22}$ as a sink extension of the algebra $A_{21}$ by a module $M$ over $A_{21}$, thus $A_{22} = [M]A_{21}$. By Theorem 1, it suffices to show that $A_{21}$ is neither derived equivalent to a representation-finite hereditary algebra, nor derived canonical, nor derived equivalent to a wild hereditary algebra:
Introducing the polynomials

\[ V_n = \frac{T^n - 1}{T - 1}, \]

the Coxeter polynomial \( C(A_{21}) \) of \( A_{21} \) is seen to be \((T - 1)^2V_2V_7V_9V_9/V_4\). In particular, \( C(A_{21}) \) is not of canonical type and hence \( A_{21} \) is not derived canonical. Since all roots of \( C(A_{21}) \) lie on the unit circle in the complex plane and further 1 is a root of \( C(A_{21}) \), the algebra \( A_{21} \) cannot be derived equivalent to a hereditary algebra which is wild or representation-finite.

References


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