ONE-POINT EXTENSIONS AND DERIVED EQUIVALENCE

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Abstract. Work of the first author with de la Peña [1], concerned with the class of algebras derived equivalent to a tubular algebra, raised the question whether a derived equivalence between two algebras can be extended to one-point extensions. The present paper yields a positive answer.

Let \( A \) be a finite-dimensional algebra (associative with 1) over a field \( k \). Modules, for most of this paper, will be finite dimensional right modules, and \( \text{mod} A \) denotes the category of such modules over \( A \). Each \( A \)-module \( M \) we may view as a \((k; A)\)-bimodule \( kM A \), and form the matrix algebra

\[
\begin{bmatrix}
k & M \\
0 & A
\end{bmatrix} = \left\{ \begin{bmatrix}
\alpha & m \\
0 & a
\end{bmatrix} \mid \alpha \in k, m \in M, a \in A \right\}
\]

which is called the one-point extension of \( A \) by \( M \). We denote this algebra by \( \hat{A} \) if \( M \) is clear from the context; moreover \( \hat{M} \) will denote the indecomposable projective \( A \)-module formed by the first row \([k; M] \) of \( \hat{A} \). Note that \( \hat{M} \) has trivial endomorphism ring. Forming the module category (resp. the derived category) over the one-point extension algebra is in a sense inverse to forming the perpendicular category with respect to an exceptional object in a module category [4] (resp. in the derived category of a module category [2]). Both processes are important for induction arguments on the number of isomorphism classes of simple modules. Note that we view modules as stalk complexes concentrated in degree zero. A preprint version of the article has been used by a number of authors [3, 13, 8, 9].

Theorem 1. Let \( A \) and \( B \) be two finite dimensional \( k \)-algebras, \( M \in \text{mod} A \), \( N \in \text{mod} B \) and denote by \( \hat{A} \), \( \hat{B} \) the respective one-point extensions. For any triangulated equivalence \( \Phi : D^b(\text{mod} A) \to D^b(\text{mod} B) \) which maps the module \( M \) to the module \( N \), there exists a triangulated equivalence \( \Phi : D^b(\text{mod} A) \to D^b(\text{mod} B) \) which maps \( \hat{M} \to \hat{N} \) and restricts to a triangulated equivalence from \( D^b(\text{mod} A) \) to \( D^b(\text{mod} B) \).

For an abelian category \( \mathcal{A} \) we denote by \( K^b(\mathcal{A}) \) the homotopy category and by \( D^b(\mathcal{A}) \) the derived category of bounded differential complexes in \( \mathcal{A} \), see [12] for definitions and basic facts. Further, we denote by \( \mathcal{P}_A \) the full subcategory of \text{mod} \( A \) given by the finitely generated projective \( A \)-modules. We identify \( D^b(\text{mod} A) \) with the full subcategory \( M^k = \{ X \mid \text{Hom}(M, X[i]) = 0 \text{ for all } i \} \) of \( D^b(\text{mod} A) \).

Before entering the proof, we recall results from Rickard [10]. Any triangulated equivalence \( \Phi : D^b(\text{mod} A) \to D^b(\text{mod} B) \) induces a triangulated equivalence \( \varphi : K^b(\mathcal{P}_A) \to K^b(\mathcal{P}_B) \), where \( K^b(\mathcal{P}_A) \) refers to the homotopy category of bounded complexes in \( \mathcal{P}_A \). In particular, \( T = \varphi^{-1}(B[0]) \) is a tilting complex, that is, for all \( n \neq 0 \) we have \( \text{Hom}_{K^b(\mathcal{P}_A)}(T, T[n]) = 0 \), and moreover \( \text{add}(T) \), the full subcategory of direct summands of finite direct sums of copies of \( T \), generates \( K^b(\mathcal{P}_A) \).
as a triangulated category. Conversely, a given tilting complex $T$ in $K^b(\mathcal{P}_A)$ with endomorphism algebra $B$, gives rise to a triangulated equivalence from $D^b(\text{mod } A)$ to $D^b(\text{mod } B)$, sending $T$ to $B[0]$.

**Proof.** Note that the canonical projection $\tilde{A} \to A$ induces an embedding $\iota_A : \text{mod } A \to \text{mod } \tilde{A}$ such that $(\ast)$ the two functors $\text{Hom}_{\tilde{A}}(\iota_A \cdot, \tilde{M})$ and $\text{Hom}_A(\cdot, M)$ from $\text{mod } A$ to $\text{mod } k$ are isomorphic and $(\ast\ast)$ $\text{Hom}_{\tilde{A}}(\tilde{M}, \iota_A \cdot)$ is the zero functor.

Let $\varphi : K^b(\mathcal{P}_A) \to K^b(\mathcal{P}_B)$ be the triangulated equivalence induced by $\Phi$ and set $T := \varphi^{-1}(B[0])$ and $\tilde{T} := T \oplus \tilde{M}[0]$. We are going to show that $\tilde{T}$ is a tilting complex in $K^b(\mathcal{P}_A)$. Further, we show that the endomorphism algebra of $\tilde{T}$ is isomorphic to $B$. It then follows from [10], as summarized before, that there is a triangulated equivalence $\tilde{\Phi} : D^b(\text{mod } \tilde{A}) \to D^b(\text{mod } B)$ sending $T$ to $B[0]$ and $\tilde{M}[0]$ to $\tilde{N}[0]$, moreover, in view of $(\ast\ast)$, $\Phi$ extends $\tilde{\Phi}$.

We get a sequence of isomorphisms

$$\text{Hom}_{K^b(\mathcal{P}_A)}(T, \tilde{M}[0]) \cong \text{Hom}_{D^b(\text{mod } A)}(T, M[0])$$

$$\cong \text{Hom}_{D^b(\text{mod } B)}(B[0], N[0])$$

$$\cong \text{Hom}_B(B, N) = N$$

where the first one is due to $(\ast)$ and the second to $\tilde{\Phi}$. By construction we have an isomorphism $\text{End}_{K^b(\mathcal{P}_A)}(T) \cong B$ and, passing to the homotopy categories, we derive from $(\ast\ast)$ that $\text{Hom}_{K^b(\mathcal{P}_A)}(\tilde{M}[0], T) = 0$. Since moreover $\text{End}_{\tilde{A}}(\tilde{M}) = k$ this shows that $\text{End}_{K^b(\mathcal{P}_A)}(\tilde{T})$ is in fact isomorphic to $\tilde{B}$.

Because of $(\ast\ast)$, we have $\text{Hom}_{K^b(\mathcal{P}_A)}(\tilde{M}[0], T[n]) = 0$ for all $n$, and in view of $(\ast)$, we get an isomorphism $\text{Hom}_{K^b(\mathcal{P}_A)}(T, M[n]) \to \text{Hom}_{D^b(\text{mod } A)}(T, M[n])$. The latter term is isomorphic to $\text{Hom}_{D^b(\text{mod } B)}(B[0], N[n])$ and thus is zero for all $n \neq 0$. Similarly, $\text{Hom}_{K^b(\mathcal{P}_A)}(T, T[n]) = 0$ for all $n \neq 0$. Finally, $\text{Hom}_{K^b(\mathcal{P}_A)}(\tilde{M}[0], M[n]) = \text{Ext}_{\tilde{A}}^n(\tilde{M}, M) = 0$ for all $n \neq 0$. Since, obviously, $\tilde{T}$ generates $K^b(\mathcal{P}_A)$ this proves that $\tilde{T}$ is a tilting complex in $K^b(\mathcal{P}_A)$.

Thus we obtain a triangulated equivalence $\varphi : K^b(\mathcal{P}_A) \to K^b(\mathcal{P}_B)$, which maps the tilting complex $\tilde{T}$ to $\tilde{B}[0]$ and its summand $\tilde{M}$ to $\tilde{N}$, and a corresponding triangulated equivalence $\Phi : D^b(\text{mod } A) \to D^b(\text{mod } B)$. Since $\Phi(M) = N$, the functor $\Phi$ further sends $\tilde{M}[\cdot] = D^b(\text{mod } A)$ to $\tilde{N}[\cdot] = D^b(\text{mod } B)$. \hfill $\square$

**Corollary 1.** Let $A$ and $H$ be two finite dimensional $k$-algebras such that there exists a triangulated equivalence $\Phi : D^b(\text{mod } A) \to D^b(\text{mod } H)$. We assume that $H$ is hereditary. Then for every indecomposable $A$-module $M$, there exists an indecomposable $H$-module $N$ such that there is a triangulated equivalence $\tilde{\Phi} : D^b(\text{mod } \tilde{A}) \to D^b(\text{mod } \tilde{H})$, where $\tilde{A}$ and $\tilde{H}$ denote the respective one-point extensions of $A$ and $H$, which restricts to a triangulated equivalence from $D^b(\text{mod } A)$ to $D^b(\text{mod } H)$.

**Proof.** Since $H$ is hereditary, every indecomposable object of $D^b(\text{mod } H)$ is given by a stalk complex $X[i]$ for some indecomposable $H$-module $X$. Modifying $\Phi$ by a suitable shift $[i]$, we may thus assume the existence of an $H$-module $N$ with $\Phi(M) = N$. The assertion now follows from Theorem 1, observing that derived equivalences commute with the shift functors. \hfill $\square$
We mention two further applications. Let $A$ be a derived canonical algebra, that is, $A$ is an algebra which is derived equivalent to a canonical algebra [11]. Note that this includes the case of an algebra derived equivalent to a tame hereditary or a tubular algebra. We call an $A$-module $M$ derived regular if $M$ belongs to a tube $T$ in the derived category $D^b(mod\ A)$. If $M$ has quasi-length $n$ in $T$, we say that $M$ has derived regular length $n$. If moreover $n = 1$ we say that $M$ is derived regular simple.

**Corollary 2.** Let $A$ be a derived canonical algebra, and let $M$ be an $A$-module which is derived regular simple. Then the one-point extension of $A$ by $M$ is again derived canonical.

**Proof.** The assertion holds for a canonical algebra [6], hence by Theorem 1 extends to the derived canonical situation. \hfill $\square$

We recall that any tame hereditary algebra of type $\tilde{D}_n$ is in the same derived class as the canonical algebra of weight type $(2, 2, n - 1)$.

**Corollary 3.** Assume that $A_1$ and $A_2$ are derived canonical of type $(2, 2, n)$ and let $M_i$ be an indecomposable $A_i$-module of derived regular length two taken from a rank $n$ tube of $D^b(mod\ A_i)$, $i = 1, 2$. Then the resulting one-point extensions $\tilde{A}_1$ and $\tilde{A}_2$ are derived equivalent. \hfill $\square$

This implies, in particular, that the (strongly simply connected) polynomial growth critical algebras introduced by Nörenberg and Skowroński [7] with a fixed number of simple modules are in the same derived class, a result formerly requiring a case by case analysis.

**Comments.** (a) Assume that $A$ (and hence also the one-point extension $\tilde{A}$ with respect to the $A$-module $M$) has finite global dimension. Then the category $D^b(mod\ \tilde{A})$ has Auslander-Reiten triangles [5]. We claim that the $A$-module $M$ is isomorphic to the “middle term” $E$ of the Auslander-Reiten triangle in $D^b(mod\ \tilde{A})$

$$\tau\tilde{M} \to E \to \tilde{M} \to \tau\tilde{M}[1].$$

Moreover, if $r : D^b(mod\ \tilde{A}) \to D^b(mod\ A)$ denotes the right adjoint functor to the inclusion $i : D^b(mod\ A) \hookrightarrow D^b(mod\ A)$ (cf. [2] for the existence of $r$), then $M = r\tilde{M}$. Indeed, application of $\text{Hom}(M, -)$ to (1) yields a long exact homology sequence. Invoking Auslander-Reiten duality $\text{Hom}(X, \tau Y[n]) = \text{Hom}_k(\text{Hom}(Y[n - 1], X), k)$ and the exceptionality of $\tilde{M}$, it follows that $\text{Hom}(\tilde{M}, E[n]) = 0$ holds for each $n \in \mathbb{Z}$, thus $E \in D^b(mod\ A)$. Moreover, for each $X \in D^b(mod\ A)$ the segment

$$\text{Hom}(X, \tau\tilde{M}) \to \text{Hom}(X, E) \to \text{Hom}(X, \tilde{M}) \to \text{Hom}(X, \tau\tilde{M}[1])$$

of the long exact homology sequence has vanishing end terms showing that the functors $\text{Hom}(-, E)$ and $\text{Hom}(-, M)$ agree on $D^b(mod\ A)$, hence implying $E \cong r\tilde{M} \cong M$.

(b) The converse of Theorem 1 does not hold. Let $A$ and $B$ be the path algebras of the quivers $Q_A$ and $Q_B$, respectively.
Let $M \in \text{mod}
olimits A$ be given by $M(1) = M(2) = k$, $M(\alpha) = M(\beta) = 1_k$ and $N \in \text{mod}
olimits B$ be given by $N(1) = k$, $N(2) = k^2$, $N(\alpha)$ the diagonal embedding. There does not exist a triangulated equivalence between $\text{D}^b(\text{mod}
olimits A)$ and $\text{D}^b(\text{mod}
olimits B)$, but $\text{D}^b(\text{mod}
olimits A)$ and $\text{D}^b(\text{mod}
olimits B)$ are equivalent as derived categories, since they are both tilted of the hereditary algebra $C$ with quiver $Q_C$: let $T_A = P_1 \oplus P_3 \oplus S$ and $T_B = S \oplus I_1 \oplus I_3$, where $P_x$, respectively $I_x$ denotes the projective cover, resp. injective hull of the simple in $x$ and $S$ is the indecomposable with $S(1) = S(3) = k$ and $S(2) = 0$. Then $\text{End}(T_A) \cong \tilde{A}$ and $\text{End}(T_B) \cong \tilde{B}$.

(c) We finally formulate an infinite variant of Theorem 1. For any ring $A$, denote by $\text{Mod}
olimits A$ the category of all right $A$-modules.

**Theorem 2.** Let $A$ and $B$ be two algebras over a commutative ring $R$ with unit, $M \in \text{Mod}
olimits A$, $N \in \text{Mod}
olimits B$ and denote by $\tilde{A}$, $\tilde{B}$ the respective one-point extensions. For any triangulated equivalence $\Phi : \text{D}^b(\text{Mod}
olimits A) \to \text{D}^b(\text{Mod}
olimits B)$ which maps the module $M$ to the module $N$, there exists a triangulated equivalence $\hat{\Phi} : \text{D}^b(\text{Mod}
olimits A) \to \text{D}^b(\text{Mod}
olimits B)$ which maps $\hat{M}$ to $\hat{N}$ and restricts to a triangulated equivalence from $\text{D}^b(\text{Mod}
olimits A)$ to $\text{D}^b(\text{Mod}
olimits B)$.

By Rickard [10], any triangulated equivalence from $\text{D}^b(\text{Mod}
olimits A)$ to $\text{D}^b(\text{Mod}
olimits B)$ induces a triangulated equivalence from $\text{K}^b(\text{P}
olimits A)$ to $\text{K}^b(\text{P}
olimits B)$. Thus, the proof of Theorem 1 extends to the present setting, replacing each occurrence of $\text{mod}
olimits A$ ($\text{mod}
olimits B$ respectively).

**Acknowledgements.** The authors thank Bernhard Keller for his suggestion to remove the original restriction to the setting of finite global dimension. The first author gratefully acknowledges support of CONACyT, UNAM.

**References**


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