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Generalized transfer and spectral sequences ☆

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Abstract

The purpose of this paper is to show that the generalized fixed-point transfer, as defined by Ulrich and the author, determines a transformation of the Leray–Serre and of the Rothenberg–Steenrod spectral sequences for general homology and cohomology theories, under suitable conditions on the considered fixed point situations.

Keywords: Stable equivariant homotopy; Spectral sequences; Generalized fixed point situations; Fixed point transfer; Metric spaces; Categories

1. Introduction

Recall from [9] the definition and some of the properties of the transfer.

Definition 1.1. Let X be a metric space. A *k-fixed point situation over X*, $k \in \mathbb{Z}$, a k-FPS_X for short, is a commutative diagram

$$\begin{tikzcd} \mathbb{R}^n \times E \supset V \arrow[rr, "f"] \arrow[dr, "p\cdot\mathrm{proj}_E"'] & & \mathbb{R}^{n+k} \times E \arrow[dl, "p\cdot\mathrm{proj}_E"] \\ & X & \end{tikzcd}$$

where $p : E \to X$ is an ENR_X, $n \in \mathbb{Z}$ such that $n+k \geqslant 0$, V is an open subset of $\mathbb{R}^n \times E$ and f is *compactly fixed*, that is, the *coincidence set* $\mathrm{Fix}(f) = \{(y,e) \in V \mid f(y,e) = (0,e)\}$ lies properly over X, i.e., the preimage of every compact set in X is compact in $\mathrm{Fix}(f)$.

To such a k-FPS_X we may assign a transfer, which is to be a certain stable map, according to the definition below. In order to simplify matters, assume that $p : E \to X$

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is trivial, that is, $E = \mathbb{R}^m \times X$. Before stating the definition, let us recall that, since $\mathrm{Fix}(f)$ lies properly over X, there exists a continuous function $\rho : X \to \mathbb{R}^+$ such that $\mathrm{Fix}(f)$ is contained in the tubular neighborhood $D_\rho = \{(y, y', x) \in \mathbb{R}^n \times \mathbb{R}^m \times X \mid \|(y, y')\| < \rho(x)\}$, where $\|\cdot\|$ denotes some norm.

Definition 1.2. Let W be any open neighborhood of $\mathrm{Fix}(f)$. Without loss of generality, we may assume $W \subset V$. Take the diagram of pairs of spaces and maps of pairs.

$$
\begin{tikzcd}
 & (\mathbb{R}^{n+k+m}, \mathbb{R}^{n+k+m} - 0) \times W \arrow[d, equal] \\
(W, W - \mathrm{Fix}(f)) \arrow[r, "(i-f,\mathrm{id})"] \arrow[d, hook, "(1)"'] & (\mathbb{R}^{n+k+m}, \mathbb{R}^{n+k+m} - 0) \times X \times_X \times W \\
(\mathbb{R}^{n+m} \times X, \mathbb{R}^{n+m} \times X - \mathrm{Fix}(f)) & \\
(\mathbb{R}^{n+m} \times X, \mathbb{R}^{n+m} \times X - D_\rho) \arrow[u, hook, "(2)"] \arrow[d, hook, "(3)"'] & \\
(\mathbb{R}^{n+m}, \mathbb{R}^{n+m} - 0) \times X \arrow[ruuu, bend right, "\tau(f)"'] &
\end{tikzcd}
$$

where $i : W \to \mathbb{R}^{n+k+m} \times X$ is such that $i(y, y', x) = (0, y', x) \in \mathbb{R}^{n+k+m} \times X$, (1) and (3) are stably invertible maps, (1) being an excision and (3) being a homotopy equivalence (on the second member of the pair, because one can easily strongly deform $(\mathbb{R}^{n+m} - 0) \times X = \mathbb{R}^{n+m} \times X - 0 \times X$ onto $\mathbb{R}^{n+m} \times X - D_\rho$; hence, inclusion (3) induces isomorphisms for any homology and cohomology theory, being therefore stably invertible). Thus the transfer

$$\tau(f) : X \to W$$

is a stable map of degree k, which induces for any cohomology theory h^* a homomorphism

$$\tau(f)^* : h^*(W) \to h^{*+k}(X).$$

In particular, for $W = E$, one has

$$\tau(f)^* : h^*(E) \to h^{*+k}(X),$$

and, on the other hand, passing to the inverse limit by varying on all neighborhoods W of $\mathrm{Fix}(f)$,

$$\tau(f)^* : \check{h}^*\big(\mathrm{Fix}(f)\big) \to h^{*+k}(X),$$

where $\check{h}^*$ is the *Čechification* of the theory h^*.

Analogously, for any homology theory h_*, the stable transfer determines homomorphisms

$$\tau(f)_* : h_{*+k}(X) \to h_*(W),$$

$$\tau(f)_*: h_{*+k}(X) \to h_*(E),$$

$$\tau(f)_*: h_{*+k}(X) \to \check{h}_*\big(\mathrm{Fix}(f)\big).$$

The transfer for a general k-FPS_X can be defined using the commutativity property [9, 1.15].

In this paper we show that this general transfer is compatible with the spectral sequences of a fibration (Leray–Serre and Atiyah–Hirzebruch–Whitehead) and with the one associated to the Milnor G-resolution of the classifying bundle of a topological group G (Rothenberg–Steenrod). For the case of the transfer of Dold [3] or of Becker–Gottlieb [2], these compatibilities were proved in [5,7].

This transfer, as shown in [9], has several properties, of which, the pullback property will be of special interest in what follows. We formulate all results hereon for the cohomology case. The corresponding results for homology are similar (or better, dual) and their formulation we leave to the reader.

Proposition 1.3. *Let Y be a metric space and $\alpha: X' \to X$ be continuous. If f is a k-FPS_X, then its pullback $f': V' \to \mathbb{R}^{n+k} \times E'$ over α is a k-$\mathrm{FPS}_{X'}$, which has a transfer $\tau(f')^*: h^*(V') \to h^{*+k}(X')$ such that the square*

$$\begin{tikzcd} h^*(V) \arrow[r, "\widetilde{\alpha}^*"] \arrow[d, "\tau(f)^*"'] & h^*(V') \arrow[d, "\tau(f')^*"] \\ h^{*+k}(X) \arrow[r, "\alpha^*"'] & h^{*+k}(X') \end{tikzcd}$$

commutes, where $\widetilde{\alpha}: V' \to V$ is the map induced by α.

Another important property of the transfer, which, as a matter of fact, is a consequence of the pullback property, is the homotopy property.

Proposition 1.4. *If $f: V \to \mathbb{R}^{n+k} \times E$, $V \subset \mathbb{R}^n \times E$, is a k-$\mathrm{FPS}_{X\times I}$, where $I = [0,1]$, and if $f_t: V_t \to \mathbb{R}^{n+k} \times E_t$ is the restriction of f to the slice $X \times \{t\} \approx X$, then the following diagram commutes.*

$$\begin{tikzcd} h^*(V) \arrow[r, "\widetilde{j}_t^*"] \arrow[d, "\tau(f)^*"'] & h^*(V_t) \arrow[d, "\tau(f_t)^*"] \\ h^{*+k}(X \times I) \arrow[r, "j_t^*"'] & h^{*+k}(X) \end{tikzcd}$$

where j_t and $\widetilde{j}_t$ are the corresponding inclusions and the bottom arrow is an isomorphism ($since$ j_t is a homotopy equivalence).

2. The transfer and the spectral sequences of a filtration

In order to study the behavior of the transfer with respect to the Leray–Serre and the Rothenberg–Steenrod spectral sequences, we need to understand how the transfer fits into the spectral sequences determined by filtrations. To that end, we need first to observe that the transfer, as defined in Definition 1.2, induces, in fact, a stable map of pairs. More precisely, let f be a k-FPS_X; if $A \subset X$, and $E_A \to A$, respectively $W_A \to A$ are the restrictions of $E \to X$ to A, that is, $E_A = p^{-1}A \subset E$, $W_A = E_A \cap W$, then $\tau(f)$ induces a stable map of pairs

$$\tau(f) : (X, A) \to (W, W_A)$$

of degree k and therewith homomorphisms

$$\tau(f)^* : h^*(W, W_A) \to h^{*+k}(X, A),$$

or

$$\tau(f)^* : h^*(E, E_A) \to h^{*+k}(X, A),$$

for the particular case $W = E$, and, passing to the inverse limit by varying on all neighborhoods W of $\mathrm{Fix}(f)$,

$$\tau(f)^* : \check{h}^*\big(\mathrm{Fix}(f), \mathrm{Fix}(f_A)\big) \to h^{*+k}(X, A),$$

where $f_A : W_A \to \mathbb{R}^{n+k} \times E_A$ is the restriction of the k-FPS_X f to A.

Let now

$$A = X^{-1} \subset X^0 \subset \cdots \subset X^q \subset X^{q+1} \subset \cdots \subset X^\infty = X$$

be a *filtration of the pair* (X, A) and let

$$E_A = E^{-1} \subset E^0 \subset \cdots \subset E^q \subset E^{q+1} \subset \cdots \subset E^\infty = E$$

be the induced filtration of (E, E_A), $E^q = p^{-1}X^q$, and, more generally,

$$W_A = W^{-1} \subset W^0 \subset \cdots \subset W^q \subset W^{q+1} \subset \cdots \subset W^\infty = W$$

the corresponding filtration of (W, W_A), for any neighborhood W of $\mathrm{Fix}(f)$, $W^q = E^q \cap W$.

Therefore, we have for $q \leqslant r \leqslant s \leqslant t$ the squares

$$\begin{tikzcd}
W^q \arrow[r, hook] \arrow[d] & W^r \arrow[r, hook] \arrow[d] & W^s \arrow[r, hook] \arrow[d] & W^t \arrow[d] \\
X^q \arrow[r, hook] & X^r \arrow[r, hook] & X^s \arrow[r, hook] & X^t
\end{tikzcd}$$

Using Proposition 1.3 we have the commutative square

$$\begin{tikzcd}
h^*(W^t, W^r) \arrow[r] \arrow[d, "\tau"'] & h^*(W^s, W^q) \arrow[d, "\tau"] \\
h^{*+k}(X^t, X^r) \arrow[r] & h^{*+k}(X^s, X^q)
\end{tikzcd} \tag{2.1}$$

where the horizontal arrows are determined by the inclusions and τ denotes the homomorphisms induced by the transfers of the corresponding restrictions of f.

Since the transfer is induced by a stable map, it is compatible also with connecting homomorphisms δ, specially the ones for the triples $W^q \subset W^r \subset W^s$ and $X^q \subset X^r \subset X^s$, thus producing commutative squares

$$\begin{tikzcd} h^*(W^r, W^q) \arrow[r, "\delta"] \arrow[d, "\tau"'] & h^{*+1}(W^s, W^r) \arrow[d, "\tau"] \\ h^{*+k}(X^r, X^q) \arrow[r, "\delta"'] & h^{*+1+k}(X^s, X^r) \end{tikzcd} \tag{2.2}$$

The filtrations given above induce, for h^*, spectral sequences, whose E_1-terms are defined as

$$\begin{aligned} E_1^{r,s}(X, A) &= h^{r+s}\big(X^r, X^{r-1}\big), \\ E_1^{r,s}(E, E_A) &= h^{r+s}\big(E^r, E^{r-1}\big), \\ E_1^{r,s}(W, W_A) &= h^{r+s}\big(W^r, W^{r-1}\big), \end{aligned}$$

respectively. Therefore, using the fact that the two squares (2.1) and (2.2) are commutative, we have the following result.

Theorem 2.1. *The transfer for the k-FPS_X f induces transformations of spectral sequences*

$$\tau : \big\{E_l^{r,s}(E, E_A), d_l\big\} \to \big\{E_l^{r,s+k}(X, X_A), d_l\big\},$$

or, more generally,

$$\tau : \big\{E_l^{r,s}(W, W_A), d_l\big\} \to \big\{E_l^{r,s+k}(X, X_A), d_l\big\}.$$

There is a homology version of all these results, which will be useful in Section 4, but, since the homology results are similar (dual) to these, we leave them out.

3. The transfer and the Leray–Serre spectral sequences

Let $p : E \to X$ be a Hurewicz fibration and assume that $A \subset X$ is such that the pair (X, A) is a 0-connected relative CW-complex and let

$$A = X^{-1} \subset X^0 \subset \cdots \subset X^q \subset X^{q+1} \subset \cdots \subset X^\infty = X$$

be its skeletal filtration and

$$E_A = E^{-1} \subset E^0 \subset \cdots \subset E^q \subset E^{q+1} \subset \cdots \subset E^\infty = E$$

be the induced filtration of the pair (E, E_A). Let either h^* be a strongly additive cohomology theory, or assume that the pair (X, A) has only finitely many cells in each dimension. Then one has a (cohomology) spectral sequence $\{E_l^{r,s}(E, E_A), d_l\}$ associated to the filtration of E and the following result.

Theorem 3.1 (Leray–Serre). *The spectral sequence $\{E_l^{r,s}(E, E_A), d_l\}$ converges to $h^*(E, E_A)$ and for its E_2-term there is an isomorphism*

$$E_2^{r,s}(E, E_A) \cong H^r\big(X, A; h^s(\mathcal{F})\big),$$

where H^* denotes ordinary (cellular) cohomology and its coefficients are taken, as usual, in the local system determined by $h^s(p^{-1}(x))$, $x \in X$.

For a proof, we refer the reader to [11], though we give a description of the isomorphism, which will be of interest to prove our results.

In order to define the local system $h^*(\mathcal{F})$ as a contravariant functor

$$h^*(\mathcal{F}) : \Pi_1(X) \to \mathcal{A}$$

from the fundamental groupoid of X to the category of abelian groups, let first

$$\Gamma : E \times_X X^I \to E^I$$

be a lifting map for the fibration $p : E \to X$. Then

$$h^*(\mathcal{F})(x) = h^*(F_x), \quad F_x = p^{-1}(x)$$

and for a path $\omega : I \to X$ joining x_0 with x_1,

$$h^*(\mathcal{F})(\omega) = \overline{\omega}^*,$$

where $\overline{\omega} : F_{x_0} \to F_{x_1}$ is given by $\overline{\omega}(y) = \Gamma(y, \omega)(1)$.

Recall also that the *cellular cochain complex* with coefficients in the local system $h^*(\mathcal{F})$ associated to the skeletal filtration of (X, A) is defined by

$$C^r\big(X, A; h^s(\mathcal{F})\big) = \prod_{\varphi \in \Phi_r} h^s(F_{\varphi(e_0)}),$$

where Φ_r denotes the set of characteristic maps of the r-cells of (X, A), seen as maps $\varphi : (\mathbb{D}^r, \mathbb{S}^{r-1}) \to (X^r, X^{r-1})$ and $e_0 = (1, 0, 0, \ldots, 0) \in \mathbb{D}^r \subset \mathbb{R}^r$.

If φ is a characteristic map, as above, one can pull back the (restricted) fibration $E^r \to X^r$ over φ to obtain

$$\begin{tikzcd} T_\varphi \arrow[r, "\widetilde{\varphi}"] \arrow[d] & E^r \arrow[d] \\ \mathbb{D}^r \arrow[r, "\varphi"'] & X^r \end{tikzcd}$$

This induced fibration is fiberwise homotopy trivial, i.e., there exists a vertical homotopy trivialization

$$\alpha_\varphi : \mathbb{D}^r \times F_{\varphi(e_0)} \xrightarrow{\simeq} T_\varphi.$$

Then $\kappa : E_1^{r,s}(E, E_A) = h^{r+s}(E^r, E^{r-1}) \to C^r(X, A; h^s(\mathcal{F}))$ is an isomorphism given into each factor by the composite

$$h^{r+s}\left(E^r, E^{r-1}\right) \xrightarrow{\widetilde{\varphi}^*} h^{r+s}\left(T_\varphi, T'_\varphi\right) \xrightarrow[\cong]{\alpha_\varphi^*} h^{r+s}\left(\left(\mathbb{D}^r, \mathbb{S}^{r-1}\right) \times F_{\varphi(e_0)}\right) \xleftarrow[\cong]{\sigma^r} h^s(F_{\varphi(e_0)}),$$

where $T'_\varphi \to \mathbb{S}^{r-1}$ is the restriction to the boundary $\mathbb{S}^{r-1} \subset \mathbb{D}^r$ and σ^r is the suspension isomorphism.

Remark 3.2. In the case $E = X$, $p = \mathrm{id}$, the local system becomes constant with value $h^*(*)$ and the isomorphism κ described above is very simple, namely, one has an isomorphism $\overline{\kappa} : E_1^{r,s}(X, X_A) = h^{r+s}(X^r, X^{r-1}) \to C^r(X, A; h^s(*))$ given into each factor by

$$h^{r+s}\left(X^r, X^{r-1}\right) \xrightarrow{\varphi^*} h^{r+s}\left(\mathbb{D}^r, \mathbb{S}^{r-1}\right) \xleftarrow[\cong]{\sigma^r} h^s(F_{\varphi(e_0)}).$$

This determines the *Atiyah–Hirzebruch–Whitehead* spectral sequence; namely, one has the following special case of Theorem 3.1.

Theorem 3.3 (Atiyah–Hirzebruch–Whitehead). *The spectral sequence* $\{E_l^{r,s}(X, X_A), d_l\}$ *converges to* $h^*(X, X_A)$ *and for its* E_2*-term there is an isomorphism*

$$E_2^{r,s}(X, X_A) \cong H^r\left(X, A; h^s(*)\right),$$

where H^* *denotes ordinary* (*cellular*) *cohomology.*

Take now a Hurewicz fibration $\widetilde{p} : \widetilde{E} \to X \times I$ with lifting map $\widetilde{\Gamma} : (X \times I)^I \times_{X \times I} \widetilde{E} \to \widetilde{E}^I$, call $i_\nu : X \approx X \times \{\nu\} \hookrightarrow X \times I$ the inclusion ($\nu = 0, 1$), and let $E^\nu = \widetilde{E}|_{X \times \{\nu\}} \subset \widetilde{E}$ be the corresponding restriction of $\widetilde{p}$.

There is a homotopy equivalence $\overline{\omega}$ such that one has a commutative triangle

$$\begin{tikzcd} E^0 \arrow[rr, "\overline{\omega}"] \arrow[dr] & & E^1 \arrow[dl] \\ & X & \end{tikzcd}$$

given by

$$\overline{\omega}(e) = \widetilde{\Gamma}(e, \omega_e)(1),$$

where $\omega_e : I \to X \times I$ is such that $\omega_e(t) = (p_X(e), t)$ and $p_X : \widetilde{E} \to X$ is the X-component of $\widetilde{p}$.

We have the following lemma.

Lemma 3.4. *Take a k-$\mathrm{FPS}_{X\times I}$*

$$\begin{tikzcd}
\mathbb{R}^n \times \widetilde{E} \supset \widetilde{V} \arrow[rr, "\widetilde{f}"] \arrow[dr] & & \mathbb{R}^{n+k} \times \widetilde{E} \arrow[dl] \\
& X \times I &
\end{tikzcd}$$

and let $A \subset X$. If $f^\nu : V^\nu \to \mathbb{R}^k \times E^\nu$ is the restriction of f to $X \times \{\nu\}$, then one has a commutative diagram

$$\begin{tikzcd}
h^*(E^0, E^0_A) & & h^*(E^1, E^1_A) \arrow[ll, "\overline{\omega}^*"'] \arrow[dl, "\tau(f^1)^*"] \\
& h^{*+k}(X, A) \arrow[from=ul, "\tau(f^0)^*"'] &
\end{tikzcd}$$

Proof. If $j_\nu : E^\nu \hookrightarrow \widetilde{E}$ is the inclusion, Proposition 1.3 guarantees the commutativity of

$$\begin{tikzcd}
h^*(E^0, E^0_A) \arrow[d, "\tau(f^0)^*"'] & h^*(\widetilde{E}, \widetilde{E}_{A\times I}) \arrow[l, "j_0^*"'] \arrow[r, "j_1^*"] \arrow[d, "\tau(\widetilde{f})^*"] & h^*(E^1, E^1_A) \arrow[d, "\tau(f^1)^*"] \\
h^{*+k}(X, A) & h^{*+k}((X, A) \times I) \arrow[l, "\cong", "i_0^*"'] \arrow[r, "\cong"', "i_1^*"] & h^{*+k}(X, A)
\end{tikzcd}$$

Take $\eta : \widetilde{E} \to E^1$ such that

$$\eta(e) = \widetilde{\Gamma}\big(e, \widetilde{\omega}_e\big)(1),$$

where $\widetilde{\omega}_e(t) = (p(e), (1 - p_I(e))t + p_I(e))$, $p_I : \widetilde{E} \to I$ the I-projection of $\widetilde{p}$. Then $\eta \circ j_1 \simeq \mathrm{id}_{E^1}$ and $\eta \circ j_0 = \overline{\omega}$. Therefore, we can substitute the top arrows of the diagram above by $\overline{\omega}^*$ (pointing to the left). Obviously, the bottom arrows compose to become the identity. Thus, the diagram becomes the desired one. $\square$

Using the previous lemma we obtain the next result.

Proposition 3.5. *Let $f : V \to \mathbb{R}^{n+k} \times E$ be a k-FPS_X, where $p : E \to X$ is a Hurewicz fibration. Then the transfer determines a transformation of local systems*

$$\tau : h^*(\mathcal{F}) \to h^{*+k}(*),$$

such that for each $x \in X$, $\tau_x = \tau(f_x)^* : h^*(F_x) \to h^{*+k}(*)$, where f_x is the restricted k-$\mathrm{FPS}_{\{*\}}$

$$\begin{tikzcd}
\mathbb{R}^n \times F_x \supset V_x \arrow[rr, "f_x"] \arrow[dr] & & \mathbb{R}^{n+k} \times F_x \arrow[dl] \\
& \{*\} &
\end{tikzcd}$$

over $\{*\} \to \{x\} \hookrightarrow X$.

Proof. Take a path $\omega\colon I \to X$ from x_0 to x_1 and let $\widetilde{p}\colon \widetilde{E} \to I$ be the pullback of p over ω. Let $\overline{\omega}\colon F_{x_0} \to F_{x_1}$ be defined as above (for $X = \{*\}$). From Lemma 3.4 we obtain the following commutative diagram

$$\begin{tikzcd}
h^*(F_{x_0}) \arrow[r, "\tau(f_{x_0})^*"] \arrow[d, "\overline{\omega}^*"'] & h^{*+k}(*) \arrow[d, "="] \\
h^*(F_{x_1}) \arrow[r, "\tau(f_{x_1})^*"] & h^{*+k}(*)
\end{tikzcd}$$

which is what we wanted to prove. □

From Lemma 3.4 we deduce the following consequence.

Proposition 3.6. *Let $f: V \to \mathbb{R}^{n+k} \times E$ be a k-FPS_X, where $p: E \to X$ is a Hurewicz fibration and X is contractible and let $A \subset X$. If $\alpha: X \times F \to E$ is a homotopy trivialization of p, then one has a commutative triangle*

$$\begin{tikzcd}
h^*(E, E_A) \arrow[rr, "\alpha^*"] \arrow[dr, "\tau(f)^*"'] & & h^*((X, A) \times F) \arrow[dl, "\tau(\mathrm{id} \times f_0)^*"] \\
& h^{*+k}(X, A) &
\end{tikzcd}$$

where f_0 denotes the restriction of the situation f to the singular space $\{x_0\}$ to which X contracts.

Proof. If $H: X \times I \to X$ is a homotopy such that $H(x,0) = x_0$ and $H(x,1) = x$, take the induced k-$\mathrm{FPS}_{X \times I}$

$$\begin{tikzcd}
\mathbb{R}^n \times \widetilde{E} \supset \widetilde{V} \arrow[rr, "\widetilde{f}"] \arrow[dr] & & \mathbb{R}^{n+k} \times \widetilde{E} \arrow[dl] \\
& X \times I &
\end{tikzcd}$$

over H. Apply Lemma 3.4 to obtain

$$\tau\big(f^0\big)^*\overline{\omega}^* = \tau\big(f^1\big)^*,$$

where f^0 is the restriction of $\widetilde{f}$

$$\begin{tikzcd}
\mathbb{R}^n \times X \times F \supset \widetilde{V}^0 \arrow[rr, "f^0"] \arrow[dr] & & \mathbb{R}^{n+k} \times X \times F \arrow[dl] \\
& X &
\end{tikzcd}$$

and

$$f^0 = \widetilde{f}|_{X \times \{0\}} : (v, x, y) \mapsto \big(0, x, f_0(y)\big)$$

since $H(X \times \{0\}) = \{x_0\}$. On the other hand, the restriction f^1 of $\widetilde{f}$ over $X \times \{1\}$ is, precisely, f. On the other hand, $\overline{\varpi}$ is a trivialization of $E \to B$ which is vertically homotopic to α. Therefore

$$\tau(\mathrm{id} \times f_0)^* \alpha^* = \tau(f)^*. \qquad \square$$

Basically, the main result of this section is contained in the following assertion.

Proposition 3.7. *The diagram*

$$\begin{tikzcd}
h^{r+s}(E^r, E^{r-1}) \arrow[r, "\kappa"] \arrow[d, "\tau(f)^*"'] & C^r(X, A; h^s(\mathcal{F})) \arrow[d, "C^r(X,A;\tau)"] \\
h^{r+s+k}(X^r, X^{r-1}) \arrow[r, "\kappa"'] & C^r(X, A; h^{s+k}(*))
\end{tikzcd}$$

is commutative.

Proof. We consider the following diagram

$$\begin{tikzcd}
h^{r+s}(E^r, E^{r-1}) \arrow[r, "\psi^*"] \arrow[d, "\tau(f)^*"'] & h^{r+s}(T_\varphi, T'_\varphi) \arrow[rr, "\alpha^*_\varphi"] \arrow[dr, dashed, "\tau(f')^*"'] & & h^{r+s}((\mathbb{D}^r, \mathbb{S}^{r-1}) \times F_{\varphi(e_0)}) \arrow[r, "\sigma^{-r}"] \arrow[dl, dashed, "\tau(\mathrm{id}\times f_{\varphi(e_0)})^*"] & h^s(F_{\varphi(e_0)}) \arrow[d, "\tau(f_{\varphi(e_0)})^*"] \\
h^{r+s+k}(X^r, X^{r-1}) \arrow[rr, "\varphi^*"'] & & h^{r+s+k}(\mathbb{D}^r, \mathbb{S}^{r-1}) \arrow[rr, "\sigma^{-r}"'] & & h^s(*)
\end{tikzcd}$$

whose commutativity follows by subdividing it in three diagrams, the first of which commutes by the naturality of the transfer, the second, by Proposition 3.6, and the third, by the stability of the transfer. $\square$

We can now write the main result, which follows immediately from Proposition 3.7.

Theorem 3.8. *Let $f : V \to \mathbb{R}^{n+k} \times E$ be a k-FPS_X such that $p : E \to X$ is a Hurewicz-fibration and (X, A) is a* 0*-connected relative* CW*-complex. Let, moreover, h^* be a strongly additive cohomology theory* (*or assume that the pair (X, A) has finitely many cells in each dimension*). *Then the transfer of f induces a transformation of spectral sequences*

$$\tau(f)^* : \left\{E_l^{r,s}(E, E_A), d_l\right\} \to \left\{E_l^{r,s+k}(X, X_A), d_l\right\}$$

converging to $\tau(f)^* : h^*(E, E_A) \to h^{*+k}(X, A)$, such that at the E_2-term level one has a commutative diagram

$$\begin{tikzcd}
E_2^{r,s}(E, E_A) \arrow[r, phantom, "\cong"] \arrow[d, "\tau(f)^*"'] & H^r(X, A; h^s(\mathcal{F})) \arrow[d, "H^r(X,A;\tau)"] \\
E_2^{r,s+k}(X, X_A) \arrow[r, phantom, "\cong"] & H^r(X, A; h^{s+k}(*))
\end{tikzcd}$$

where the transformation $\tau : h^s(\mathcal{F}) \to h^{s+k}(*)$ between local systems is such that $\tau_x = \tau(f_x)^* : h^s(F_x) \to h^{s+k}(*)$ is the transfer of the restriction of f to each fiber of p.

Proof. Just apply Proposition 3.7. □

There is, of course, a homology version of Theorem 3.8, whose proof can be similarly (dually) given. We leave it to the reader.

4. The transfer and the Rothenberg–Steenrod spectral sequences

Let G be a compact Lie group (it could even be a more general topological group) and let $EG \to BG$ be its classifying (principal) bundle. Moreover, let G act continuously in a space F. In [6] the construction of general Rothenberg–Steenrod spectral sequences which approximate the homology and cohomology of the associated bundle, $h_*(EG \times_G F)$ and $h^*(EG \times_G F)$, for h_* a generalized multiplicative homology theory and an associated generalized multiplicative cohomology theory h^*, (say, both represented by the same spectrum) was given. Since $h_*(*) = h^*(*)$, we shall simply write $h(*)$ for this ring, when there is no danger of confusion. The E_2-terms for these spectral sequences are, respectively, $\mathrm{Tor}^{h_*(G)}(h(*), h_*(F))$ and $\mathrm{Ext}_{h_*(G)}(h(*), h^*(F))$. This second will be the case if h_* and h^* are represented by a spectrum satisfying the finiteness and duality condition 13.3 in [1], which we call the *Adams condition*.[1] (For example, as proved by Adams in *op. cit.*, the classical spectra, such as the sphere spectrum, the Eilenberg–Mac Lane spectrum, MO, MU, MSp, BO, BU, satisfy the Adams condition.)

More precisely, let $E_0 \subset E_1 \subset \cdots \subset E_r \subset E_{r+1} \subset \cdots \subset EG$ be the Milnor G-resolution (cf. [6]). Then one has the corresponding filtration in the base space

$$B_0 \subset B_1 \subset \cdots \subset B_r \subset B_{r+1} \subset \cdots \subset BG$$

as well as the filtration of the total space

$$E_0 \times_G F \subset E_1 \times_G F \subset \cdots \subset E_r \times_G F \subset E_{r+1} \times_G F \subset \cdots \subset EG \times_G F.$$

Let $\{E^l_{r,s}(EG \times_G F), d^l\}$ and $\{E_l^{r,s}(EG \times_G F), d_l\}$ be the corresponding spectral sequences for homology and cohomology. Then one has the following result proven in [6].

Theorem 4.1 (Rothenberg–Steenrod). *The spectral sequences associated to the Milnor G-resolution of BG $\{E^l_{r,s}(EG \times_G F), d^l\}$ and $\{E_l^{r,s}(EG \times_G F), d_l\}$ converge to $h_*(EG \times_G F)$ and $h^*(EG \times_G F)$, respectively. If the homology algebra $h_*(G)$ is $h(*)$-projective, then*

$$E^2_{r,s}(EG \times_G F) \cong \mathrm{Tor}^{h_*(G)}_{r,s}\big(h(*), h_*(F)\big).$$

Moreover, let h_* and h^* be represented by a spectrum which satisfies the Adams condition, then

$$E_2^{r,s}(EG \times_G F) \cong \mathrm{Ext}^{r,s}_{h_*(G)}\big(h(*), h^*(F)\big).$$

[1] The ring spectrum E representing h must be the colimit of finite spectra E_α, for which $E_*(DE_\alpha)$ is projective over $\pi_*(E)$, where DE_α, the S-dual of E_α, satisfies $F^*(DE_\alpha) \cong \mathrm{Hom}^*_{\pi_*(E)}(E_*(DE_\alpha), \pi_*(F))$ for any module-spectrum F over E.

Particularly, if one takes $F = \{*\}$, if the homology algebra $h_*(G)$ is $h(*)$-projective, one has

$$E^2_{r,s}(BG) \cong \mathrm{Tor}^{h_*(G)}_{r,s}\big(h(*), h(*)\big) \Rightarrow h_*(BG),$$

and if, moreover, h_* and h^* are represented by a spectrum which satisfies the Adams condition,

$$E_2^{r,s}(BG) \cong \mathrm{Ext}^{r,s}_{h_*(G)}\big(h(*), h(*)\big) \Rightarrow h^*(BG).$$

As in the case of Theorem 3.1, we will not give the proof of Theorem 4.1, but we shall give a description of the isomorphisms, in order to prove their compatibility with our transfer.

In the case of homology, what one proves is the existence of an isomorphism

$$\beta : h_*(E_r, E_{r-1}) \otimes_{h_*(G)} h_*(F) \to h_*\big((E_r, E_{r-1}) \times_G F\big) \tag{4.1}$$

and for the cohomology, of an isomorphism

$$\overline{\beta} : h^*\big((E_r, E_{r-1}) \times_G F\big) \to \mathrm{Hom}_{h_*(G)}\big(h_*(E_r, E_{r-1}), h^*(F)\big) \tag{4.2}$$

at the E_1-level, which, after taking homology (first derivative) determines the wanted isomorphisms at the E_2-level. We prove, in fact, that the transfer is compatible with the isomorphisms at the E_1-level.

Lemma 4.2. *The exterior* (*homology*) *product*

$$\alpha : h_*(E_r, E_{r-1}) \otimes_{h(*)} h_*(F) \to h_*\big((E_r, E_{r-1}) \times F\big)$$

determines the homomorphism

$$\beta : h_*(E_r, E_{r-1}) \otimes_{h_*(G)} h_*(F) \to h_*\big((E_r, E_{r-1}) \times_G F\big)$$

in such a way that the following diagram is commutative

$$\begin{tikzcd} h_*(E_r, E_{r-1}) \otimes_{h(*)} h_*(F) \arrow[r, "\gamma_1"] \arrow[d, "\alpha"'] & h_*(E_r, E_{r-1}) \otimes_{h_*(G)} h_*(F) \arrow[d, "\beta"] \\ h_*((E_r, E_{r-1}) \times F) \arrow[r, "\gamma_2"'] & h_*((E_r, E_{r-1}) \times_G F) \end{tikzcd}$$

where γ_1 is the canonical epimorphism and γ_2 is induced by the identification $(E_r, E_{r-1}) \times F \to (E_r, E_{r-1}) \times_G F$.

Analogously, one has the next result for cohomology.

Lemma 4.3. *The homomorphism*

$$\overline{\alpha} : h^*\big((E_r, E_{r-1}) \times F\big) \to \mathrm{Hom}_{h(*)}\big(h_*(E_r, E_{r-1}), h^*(F)\big)$$

defined by the slant product determines the homomorphism

$$\overline{\beta} : h^*\big((E_r, E_{r-1}) \times_G F\big) \to \mathrm{Hom}_{h_*(G)}\big(h_*(E_r, E_{r-1}), h^*(F)\big)$$

in such a way that the following diagram is commutative

$$\begin{tikzcd} h^*((E_r, E_{r-1}) \times_G F) \arrow[r, "\overline{\gamma}_2"] \arrow[d, "\overline{\beta}"'] & h^*((E_r, E_{r-1}) \times F) \arrow[d, "\overline{\alpha}"] \\ \mathrm{Hom}_{h_*(G)}(h_*(E_r, E_{r-1}), h^*(F)) \arrow[r, "\overline{\gamma}_1"'] & \mathrm{Hom}_{h(*)}(h_*(E_r, E_{r-1}), h^*(F)) \end{tikzcd}$$

where $\overline{\gamma}_1$ is the canonical monomorphism and $\overline{\gamma}_2$ is induced by the identification $(E_r, E_{r-1}) \times F \to (E_r, E_{r-1}) \times_G F$.

Take now a k-FPS_{BG}

$$\begin{tikzcd} \mathbb{R}^n \times EG \times_G F \supset V \arrow[rr, "f"] \arrow[dr, "p\circ\mathrm{proj}"'] & & \mathbb{R}^{n+k} \times EG \times_G F \arrow[dl, "p\circ\mathrm{proj}"] \\ & BG & \end{tikzcd}$$

and let

$$\tau(f)_*: h_{*+k}(BG) \to h_*(EG \times_G F),$$
$$\tau(f)^*: h^*(EG \times_G F) \to h^{*+k}(BG)$$

be its homology and cohomology transfers, respectively. Then, by Theorem 2.1 applied to h_* and h^* and the filtrations of $EG \times_G F$ and BG, we have the following result.

Proposition 4.4. *The transfer induces transformations of spectral sequences*

$$\tau_*: \{E^l_{r,s+k}(BG), d^l\} \to \{E^l_{r,s}(EG \times_G F), d^l\}$$

and

$$\tau^*: \{E_l^{r,s}(EG \times_G F), d_l\} \to \{E_l^{r,s+k}(BG), d_l\}.$$

In what follows, we shall discuss how the transfers behave in the E_1-terms; more precisely, with respect to the homomorphisms β and $\overline{\beta}$ given in Lemmas 4.2 and 4.3. To that end, recall the *complete* G-resolution of EG (see [6, 1.1])

$$* = D_0 \subset G = E_0 \subset D_1 \subset E_1 \subset D_2 \subset \cdots \subset E_{r-1} \subset D_r \subset \cdots \subset EG,$$

such that the action of G, $EG \times G \to EG$ induces relative homeomorphisms

$$\varphi_r: (D_r, E_{r-1}) \times G \to (E_r, E_{r-1}),$$

(in particular $\varphi_0: D_0 \times G \approx E_0$). The following lemma was proved in [7].

Lemma 4.5. *The diagram*

$$\begin{tikzcd} D_r \times F \arrow[r, "\xi_r"] \arrow[d, "\mathrm{proj}"'] & E_r \times_G F \arrow[d, "p"] \\ D_r \arrow[r, "\overline{\xi}_r"'] & B_r \end{tikzcd}$$

is a pullback square, where ξ_r and $\overline{\xi}_r$ are the restrictions of the quotient maps $E_r \times F \to E_r \times_G F$ and $E_r \to B_r$, respectively.

Let $f_r' : (\mathrm{id}_{\mathbb{R}^n} \times \xi_r)^{-1} V_r \to \mathbb{R}^{n+k} \times D_r \times F$ be the pullback of the restriction the k-FPS_{BG} f to $V_r = V \cap (\mathbb{R}^n \times E_r \times_G F) \to \mathbb{R}^{n+k} \times D_r \times F$. We can decompose this map as follows

$$f_r'\big(y, x, y'\big) = \big(\psi_r\big(y, x, y'\big), x, \varphi_r\big(y, x, y'\big)\big).$$

Let

$$\begin{tikzcd}
\mathbb{R}^n \times F \supset V_0 \arrow[rr, "f_0"] \arrow[dr] & & \mathbb{R}^{n+k} \times F \arrow[dl] \\
& \{*\} &
\end{tikzcd}$$

be the restriction of f to the base point $\{*\} = B_0$ of BG; then, one has the map $f_r'' : \mathbb{R}^n \times D_r \times F \to \mathbb{R}^{n+k} \times D_r \times F$ such that $f_r''(y, x, y') = (f_0^1(y, y'), x, f_0^2(y, y'))$, where $f_0 = (f_0^1, f_0^2)$ and is partially defined for such (y, x, y') that $(y, y') \in V_0$.

Lemma 4.6. $\tau(f_r')_* = \tau(f_r'')_* : h_{*+k}(D_r, E_{r-1}) \to h_*((D_r, D_{r-1}) \times F)$ *and correspondingly for cohomology h^*, (in fact, both transfers are equal as stable maps of degree k).*

Proof. Just recall that $D_r = \mathrm{Cone}(E_{r-1})$ and apply Proposition 3.6. □

Now, from Lemmas 4.5 and 4.6, we obtain the following result.

Proposition 4.7. *There are commutative diagrams*

$$\begin{tikzcd}
h_{*+k}(D_r, E_{r-1}) \arrow[r, "p_{r*}"] \arrow[d, "\tau(\mathrm{id}\times f_0)_*"'] & h_{*+k}(B_r, B_{r-1}) \arrow[d, "\tau(f)_*"] \\
h_*((D_r, E_{r-1}) \times F) \arrow[r, "\xi_{r*}"'] & h_*((E_r, E_{r-1}) \times_G F)
\end{tikzcd}$$

and

$$\begin{tikzcd}
h^*((D_r, E_{r-1}) \times F) \arrow[d, "\tau(\mathrm{id}\times f_0)^*"'] & h^*((E_r, E_{r-1}) \times_G F) \arrow[l, "\xi_r^*"'] \arrow[d, "\tau(f)^*"] \\
h^{*+k}(D_r, E_{r-1}) & h^{*+k}(B_r, B_{r-1}). \arrow[l, "p_r^*"]
\end{tikzcd}$$

The main theorem of this section is the following.

Theorem 4.8. *Take a* k-FPS_{BG}, $f : V \to \mathbb{R}^{n+k} \times EG \times_G F$, $V \subset \mathbb{R}^n \times EG \times_G F$ *and let its restriction to the base point* $B_0 = \{*\}$ *of* BG

$$\begin{tikzcd}
\mathbb{R}^n \times F \supset V_0 \arrow[r, "f_0"] \arrow[dr] & \mathbb{R}^{n+k} \times F \arrow[d] \\
& \{*\}
\end{tikzcd}$$

be such that $\tau(f_0)_* : h_{*+k}(*) \to h_*(F)$, *respectively* $\tau(f_0)^* : h^*(F) \to h^{*+k}(*)$, *is an* $h_*(G)$*-homomorphism, where* $h_*(*)$, $h_*(F)$, $h^*(*)$ *and* $h^*(F)$ *are given the* $h_*(G)$*-module structures induced by the* G*-space structures on* $*$ *and* F. *If the assumptions of Theorem* 4.1 *are fulfilled, then the following diagrams commute*

$$\begin{tikzcd}
E^2_{r,s+k}(BG) \arrow[d, "\tau_*"'] \arrow[r, phantom, "\cong"] & \mathrm{Tor}^{h_*(G)}_{r,s}(h(*), h_{*+k}(*)) \arrow[d, "\mathrm{Tor}(1,\tau(f_0)_*)"] \\
E^2_{r,s}(EG \times_G F) \arrow[r, phantom, "\cong"] & \mathrm{Tor}^{h_*(G)}_{r,s}(h(*), h_*(F))
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_2^{r,s}(EG \times_G F) \arrow[d, "\tau^*"'] \arrow[r, phantom, "\cong"] & \mathrm{Ext}^{r,s}_{h_*(G)}(h(*), h^*(F)) \arrow[d, "\mathrm{Ext}(1,\tau(f_0)^*)"] \\
E_2^{r,s+k}(BG) \arrow[r, phantom, "\cong"] & \mathrm{Ext}^{r,s}_{h_*(G)}(h(*), h^{*+k}(*))
\end{tikzcd}$$

Proof. We have to prove that the following diagrams are commutative. They are the corresponding diagrams at the E_1-level.

$$\begin{tikzcd}
h_*(E_r, E_{r-1}) \otimes_{h_*(G)} h_{*+k}(*) \arrow[r, "\beta_{\{*\}}"] \arrow[d, "1\otimes\tau(f_0)_*"'] & h_{*+k}(B_r, B_{r-1}) \arrow[d, "\tau(f)_*"] \\
h_*(E_r, E_{r-1}) \otimes_{h_*(G)} h_*(F) \arrow[r, "\beta_F"'] & h_*((E_r, E_{r-1}) \times_G F)
\end{tikzcd} \tag{4.3}$$

and

$$\begin{tikzcd}
h^*((E_r, E_{r-1}) \times_G F) \arrow[r, "\overline{\beta}_F"] \arrow[d, "\tau(f)^*"'] & \mathrm{Hom}_{h_*(G)}(h_*(E_r, E_{r-1}), h^*(F)) \arrow[d, "\mathrm{Hom}(1,\tau(f_0)^*)"] \\
h^{*+k}(B_r, B_{r-1}) \arrow[r, "\overline{\beta}_{\{*\}}"'] & \mathrm{Hom}_{h_*(G)}(h_*(E_r, E_{r-1}), h^{*+k}(*))
\end{tikzcd} \tag{4.4}$$

One can check these commutativities by analyzing the corresponding diagrams with the α-s instead of with the β-s. Namely, in the case of homology one has a composed diagram

$$\begin{tikzcd}
h_{*+k}(D_r, E_{r-1}) \arrow[d, "\tau(\mathrm{id}\times f_0)"'] & h_*(D_r, E_{r-1}) \otimes h_{*+k}(*) \arrow[l, "\alpha"', "\cong"] \arrow[r, "i_*\otimes 1"] \arrow[d, "1\otimes\tau(f_0)_*"] & h_*(E_r, E_{r-1}) \otimes h_{*+k}(*) \arrow[r, "\gamma_1"] \arrow[d, "1\otimes\tau(f_0)_*"] & h_*(E_r, E_{r-1}) \otimes_{h_*(G)} h_{*+k}(*) \arrow[d, "1\otimes\tau(f_0)_*"] \\
h_*((D_r, D_{r-1}) \times F) & h_*(D_r, D_{r-1}) \otimes h_*(F) \arrow[l, "\cong", "\alpha"'] \arrow[r, "i_*\otimes 1"'] & h_*(E_r, E_{r-1}) \otimes h_*(F) \arrow[r, "\gamma_1"'] & h_*(E_r, E_{r-1}) \otimes_{h_*(G)} h_*(F)
\end{tikzcd} \tag{4.5}$$

The left square commutes, since α constitutes a natural transformation of homology theories; the other diagrams commute obviously.

If one now glues diagram (4.3) on the right side of diagram (4.5), the top row composes to produce the isomorphism p_{r*} and the bottom row produces ξ_{r*}. Therefore, the glued diagram, by Proposition 4.7, commutes, and, since the rows of (4.5) are isomorphisms, (4.3) is commutative. Dually, one can show that (4.4) is also commutative, thus proving the theorem. □

Remark 4.9. The assumption of Theorem 4.8 that $\tau(f_0)_*: h_{*+k}(*) \to h_*(F)$, respectively $\tau(f_0)^*: h^*(F) \to h^{*+k}(*)$, has to be an $h_*(G)$-homomorphism holds, for example, if, say, $V = \mathbb{R}^n \times EG \times_G F$ and $f(y,x,y') = (f_0^1(y,y'), x, f_0^2(y,y')) \in \mathbb{R}^{n+k} \times EG \times_G F$ and $f_0 = (f_0^1, f_0^2): \mathbb{R}^n \times F \to \mathbb{R}^{n+k} \times F$ is a G-map, or if $f(y,x,y') = (f_0^1(y,y'), g(x), f_0^2(y,y')) \in \mathbb{R}^{n+k} \times EG \times_G F$ and $f_0 = (f_0^1, f_0^2)$ and $g: EG \to EG$ are G-maps and G is abelian.

Remark 4.10. The Dold–Lashof's approach to Milnor's G-resolution (see [4]) holds as well for (strictly) associative H-spaces G. Therefore, our results hold in fact in a more general set up.

5. Applications and comments

Let G be a discrete group. Then, the filtration induced in BG by the Milnor G-resolution coincides with the skeletal filtration of the CW-complex BG. Therefore, the Rothenberg–Steenrod spectral sequences (Theorem 4.1) which approximate the homology and cohomology of $EG \times_G F$ and BG coincide with the corresponding Leray–Serre–Atiyah–Hirzebruch–Whitehead spectral sequences (Theorem 3.1) for the fibration $EG \times_G F \to BG$. We have the following result.

Theorem 5.1. *Let G be a discrete group and assume the hypotheses of Theorem* 4.1 *are satisfied. Then for the cohomology case*

$$H^*\big(BG; h^*(\mathcal{F})\big) \cong \operatorname{Ext}^{**}_{h_*(G)}\big(h_*(*), h^*(F)\big)$$

and, in particular

$$H^*\big(BG; h^*(*)\big) \cong \operatorname{Ext}^{**}_{h_*(G)}\big(h_*(*), h^*(*)\big).$$

And for the homology case

$$H_*\big(BG; h_*(\mathcal{F})\big) \cong \operatorname{Tor}^{h_*(G)}_{**}\big(h_*(*), h_*(F)\big)$$

and, in particular

$$H_*\big(BG; h_*(*)\big) \cong \operatorname{Tor}^{h_*(G)}_{**}\big(h_*(*), h_*(*)\big).$$

All four isomorphisms are of bigraded groups.

Observe that, being G discrete, the $h(*)$-algebra $h_*(G)$ is the group algebra $h(*)(G)$ of G.

In this case, a k-FPS$_{BG}$ as in Theorem 4.8 induces transformations both for the Leray-Serre spectral sequence as for the Rothenberg–Steenrod spectral sequence producing commutative squares. That is, we have the following result.

Theorem 5.2. *Let G be a discrete group. Given a k-FPS$_{BG}$ f, the following squares commute.*

$$\begin{array}{ccc}
H^*(BG; h^*(\mathcal{F})) & \cong & \operatorname{Ext}^{**}_{h_*(G)}(h_*(*), h^*(F)) \\
\Big\downarrow{\scriptstyle H^*(BG,\tau)} & & \Big\downarrow{\scriptstyle \operatorname{Ext}^{**}(h_*(*),\tau(f_0))} \\
H^*(BG; h^*(*)) & \cong & \operatorname{Ext}^{**}_{h_*(G)}(h_*(*), h^*(*)),
\end{array}$$

$$\begin{array}{ccc}
H_*(BG; h_*(*)) & \cong & \operatorname{Tor}^{h_*(G)}_{**}(h_*(*), h_*(*)) \\
\Big\downarrow{\scriptstyle H_*(BG,\tau)} & & \Big\downarrow{\scriptstyle \operatorname{Tor}_{**}(h_*(*),\tau(f_0))} \\
H_*(BG; h_*(\mathcal{F})) & \cong & \operatorname{Tor}^{h_*(G)}_{**}(h_*(*), h_*(F)),
\end{array}$$

where τ represents the corresponding transfers for local coefficient systems and $\tau(f_0)$ denotes the transfer for the restriction of f to the base point.

As shown in [9], given a k-FPS$_X$ f, the formula $\tau(f)^* \circ p^* = \smile I(f): h^*(X) \to h^{*+k}(X)$ holds, where $I(f) \in h^k(B)$ is the *fixed point index* of f. For the restriction to the fiber over $x \in X$ f_x, a similar formula holds, namely, $\tau(f_x)^* \circ p_x^* = \smile I(f_x): h^*(*) \to h^{*+k}(*)$. If, for example, the element $I(f_x) \in h^k(*)$ is an invertible element, then Theorem 3.8 implies that $\tau(f)^* \circ p^*$ is an isomorphism, that is, p^* is a split monomorphism. More generally, if X is connected and for some (hence any) $x \in X$, $I(f_x) \in h^k(*)$ is such that $\smile I(f_x): h^*(*) \to h^{*+k}(*)$ is isomorphic, then the same holds.

Example 5.3. Let G be a compact Lie group and let N be the normalizer of its maximal torus. It is well known that the Euler-characteristic $\xi(G/N) = 1$. For the 0-FPS$_{BG}$ $\mathrm{id}_{EG\times_G N}$ (in the fibration $EG \times_G N \to BG$) we have by the Rothenberg–Steenrod spectral sequence, that its transfer is determined in the E_2-term by $\tau_0 = \tau(\mathrm{id}_{G/N})$. But $\tau_0 \circ p_0^* = I(\mathrm{id}_{G/N}) = \xi(G/N) = 1$, therefore, $\tau(\mathrm{id}_{EG\times_G N}) \circ p^*: h^*(BG) \to h^*(BG)$ is an isomorphism, and p^* is a split monomorphism, where $p: BN = EG \times N \to BG$ is the canonical fibration.

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