Fixed point theory and framed cobordism*

Carlos Prieto
Instituto de Matemáticas, UNAM, 04510 México, D.F., MEXICO

Abstract

The Thom-Pontryagin construction is studied from the point of view of fixed point situations, and a very natural correspondence between framed cobordism classes and fixed point situations is given. Since fixed point classes integrate a cohomology theory, called FIX, which generalizes in a natural way to an equivariant theory, this sheds light into possible approaches to equivariant cobordism.

1 FIX-cohomology

In this paragraph we recall the definitions of [5] of the FIX functors as a cohomology theory. All spaces we shall consider will be metric spaces. If X is a metric space, then we say that a space over X, $p : E \to X$, is a *euclidean neighborhood retract over* X, or an ENR_X, if there is an embedding $i : E \hookrightarrow \mathbb{R}^q \times X$, such that $\mathrm{proj}_X \circ i = p$, an open neighborhood U of $i(E)$ in $\mathbb{R}^q \times X$ and a retraction $r : U \to E$, i.e., $r \circ i = \mathrm{id}_E$, such that $p \circ r = \mathrm{proj}_X |_U$.

Definition 1.1. Let $p : E \to X$ be an ENR_X and let m, n be nonnegative integers. An (m, n)*-fixed point situation over* X, or an (m, n)-FPS_X, is a commutative triangle

$$\begin{tikzcd}
\mathbb{R}^n \times E \supset V \arrow[rr, "f"] \arrow[dr, "p\circ\mathrm{proj}_E"'] & & \mathbb{R}^m \times E \arrow[dl, "p\circ\mathrm{proj}_E"] \\
& X\,, &
\end{tikzcd}$$

*2000 Math. Subj. Class.: Primary 54H25; Secondary 55N22, 55M20
Keywords and phrases: Cobordism, fixed point index, generalized cohomology
The author was partially supported by CONACYT grant Nr. 400333-25406E

where f is *properly fixed*, that is, such that the set $F = \mathrm{Fix}(f) = \{(y,e) \in V \mid f(y,e) = (0,e) \in \mathbb{R}^n \times E\}$ lies properly over X, namely, $(p \circ \mathrm{proj}_E)|_F : F \to X$ is a proper map.

Given two (m,n)-FPS$_X$,

$$\mathbb{R}^n \times E_0 \supset V_0 \xrightarrow{f_0} \mathbb{R}^m \times E_0 \quad , \qquad \mathbb{R}^n \times E_1 \supset V_1 \xrightarrow{f_1} \mathbb{R}^m \times E_1$$

we declare them as *equivalent* if there is an (m,n)-FPS$_{X \times \mathrm{I}}$, $\mathrm{I} = [0,1]$,

$$\begin{tikzcd}
\mathbb{R}^n \times E \supset V \arrow[rr, "f"] \arrow[dr, "p \circ \mathrm{proj}_E"'] & & \mathbb{R}^m \times E \arrow[dl, "p \circ \mathrm{proj}_E"] \\
& X \times \mathrm{I}\,, &
\end{tikzcd}$$

such that *restricted* to $X \times \{\iota\}$, that is, taking its pullback to $X \times \{\iota\} \approx X$, gives us f_ι, $\iota = 0,1$. We denote the equivalence class of f_0, simply as $[f_0]$.

Again, given two (m,n)-FPS$_X$,

$$\mathbb{R}^n \times E \supset V \xrightarrow{f} \mathbb{R}^m \times E\,, \qquad \mathbb{R}^n \times E' \supset V' \xrightarrow{f'} \mathbb{R}^m \times E'\,,$$

one can take their *sum* as the (m,n)-FPS$_X$

$$\begin{tikzcd}
\mathbb{R}^n \times (E \sqcup E') \supset V \sqcup V' \arrow[rr, "f \sqcup f'"] \arrow[dr, "(p,p') \circ (\mathrm{proj}_{E \sqcup E'})"'] & & \mathbb{R}^m \times (E \sqcup E') \arrow[dl, "(p,p') \circ (\mathrm{proj}_{E \sqcup E'})"] \\
& X\,; &
\end{tikzcd}$$

we denote its equivalence class by $[f] + [f']$.

Given an (m,n)-FPS$_X$, and an (m',n')-FPS$_{X'}$

$$\mathbb{R}^n \times E \supset V \xrightarrow{f} \mathbb{R}^m \times E\,, \qquad \mathbb{R}^{n'} \times E' \supset V' \xrightarrow{f'} \mathbb{R}^{m'} \times E'\,,$$

one can take their *exterior product* as the $(m+m', n+n')$-FPS$_{X \times X'}$

$$\begin{tikzcd}
\mathbb{R}^{n+n'} \times (E \times E') \supset V \times V' \arrow[rr, "f \times f'"] \arrow[dr, "(p \times p') \circ (\mathrm{proj}_{E \times E'})"'] & & \mathbb{R}^{m+m'} \times (E \times E') \arrow[dl, "(p \times p') \circ (\mathrm{proj}_{E \times E'})"] \\
& X \times X'\,, &
\end{tikzcd}$$

up to an obvious shuffling of coordinates; we denote its equivalence class by $[f] \times [f']$. In particular, one may take the *interior product* of the situations above, when $X = X'$ by taking the pullback of the $(m+m', n+n')$-FPS$_{X\times X}$ to a $(m+m', n+n')$-FPS$_X$ over the diagonal map $\Delta : X \to X \times X$; in other words we obtain $[f] \smile [f']$ as the class of

$$\begin{tikzcd}[column sep=small]
\mathbb{R}^{n+n'} \times (E \times_X E') \supset V \times_X V' \arrow[rr, "f \times_X f'"] \arrow[dr, "(p,p')\circ(\mathrm{proj}_{E\times_X E'})"'] & & \mathbb{R}^{m+m'} \times (E \times_X E') \arrow[dl, "(p,p')\circ(\mathrm{proj}_{E\times_X E'})"] \\
& X\,, &
\end{tikzcd}$$

where $\times_X$ denotes the fibered product over X.

Definition 1.2. The sum provides the set of equivalence classes of (m, n)-FPS$_X$, $[f]$, with a group structure. We denote the (bigraded) group by $\mathrm{FIX}^{m,n}(X)$; moreover, the exterior and interior product define a multiplicative structure in this bigraded group

$$\times : \mathrm{FIX}^{m,n}(X) \otimes \mathrm{FIX}^{m',n'}(X') \to \mathrm{FIX}^{m+m',n+n'}(X \times X')\,,$$

$$\smile : \mathrm{FIX}^{m,n}(X) \otimes \mathrm{FIX}^{m',n'}(X) \to \mathrm{FIX}^{m+m',n+n'}(X)\,.$$

REMARK 1.3. If $\mu : \mathbb{S}^1 \to \mathbb{S}^1$ has degree -1, then, seeing this map as a $(0,0)$-FPS$_*$ ($* =$ a point), $-[f] = [\mu] \times [f] = [\mu \times f]$, for any (m, n)-FPS$_X$, f.

One easily shows as in [6, 2.4], the following.

Lemma 1.4. *For any natural number k, there is a natural isomorphism*

$$\mathrm{FIX}^{m,n}(X) \to \mathrm{FIX}^{m+k,n+k}(X)\,.$$

□

Therefore, one has the following consequence.

Corollary 1.5. $\mathrm{FIX}^{m,n}(X)$ *depends only on the difference $m-n$. Therefore, for any integer k, one can define the groups* $\mathrm{FIX}^{k}(X)$ *as* $\mathrm{FIX}^{k+n,n}(X)$ *for some (any) natural number n, such that $k+n \geq 0$.* □

EXAMPLE 1.6. The map $\nu : \mathbb{R} \to \mathbb{R}$, such that $x \mapsto -x$ is a $(0,0)$-FPS$_*$ representing the same element as the map μ in Remark 1.3 (one might also take the map ν such that $\nu(x) = 2x$). Also the map $\nu' : \mathbb{C} \to \mathbb{C}$ ($\mathbb{C} \approx \mathbb{R}^2$), such that $\nu'(z) = \overline{z}$ (complex conjugate) is a $(0,0)$-FPS$_*$ such that $[\nu'] = [\nu] = [\mu] \in \mathrm{FIX}^{0,0}(*)$. Moreover, the map ν can also be seen as a $(1,1)$-FPX$_*$ and ν' as a $(2,2)$-FPX$_*$.

The equivalence classes which form $\mathrm{FIX}^k(X)$ are very ample. One can always find good representatives, according to our needs. One possibility is the following.

Let $\mathbb{R}^n \times E \supset V \xrightarrow{f} \mathbb{R}^m \times E$ be a representative of an arbitrary element in $\mathrm{FIX}^{m-n}(X)$. Let the embedding $i : E \hookrightarrow \mathbb{R}^q \times X$, such that $\mathrm{proj}_X \circ i = p$, the open neighborhood U of $i(E)$ in $\mathbb{R}^q \times X$ and the retraction $r : U \to E$, i.e., $r \circ i = \mathrm{id}_E$, represent E as an ENR$_X$; there is another (m,n)-FPS over X

$$\mathbb{R}^n \times E' \supset V' \xrightarrow{f'} \mathbb{R}^m \times E'$$

where $E' = \mathbb{R}^q \times X$, $V' = (\mathrm{id}_{\mathbb{R}^n} \times r)^{-1}(V)$ and $f' = (\mathrm{id}_{\mathbb{R}^m} \times i) \circ f \circ (\mathrm{id}_{\mathbb{R}^n} \times r)$, such that, as one easily checks, $\mathrm{Fix}(f') \approx \mathrm{Fix}(f)$. Thus, in other words, one has an (m,n)-FPS

$$\mathbb{R}^n \times \mathbb{R}^q \times X \supset V' \xrightarrow{f'} \mathbb{R}^m \times \mathbb{R}^q \times X$$

and one may prove (see [5, 1.15]) the following.

Lemma 1.7. $[f'] = [f] \in \mathrm{FIX}^{m-n}(X)$. □

The elements $[f] \in \mathrm{FIX}^{m,n}(X)$ depend only on the behavior of f around $\mathrm{Fix}(f)$, one has the following.

Proposition 1.8. *Given an (m,n)-FPS over X, $f : V \to \mathbb{R}^n \times E$, and $\mathrm{Fix}(f) \subset W \subset V$, $W \subset \mathbb{R}^m \times E$ open, then $[f|_W] = [f] \in \mathrm{FIX}^{m,n}(X)$.* □

Let $A \subset X$ be closed. By taking an (m,n)-FPS$_X$, f, such that $\mathrm{Fix}(f) \cap (\mathbb{R}^n \times p^{-1}A) = \varnothing$, that is, if $\mathrm{Fix}(f)$ lies over the difference set $X - A$, we obtain what we call an (m,n)-FPS *over the pair* (X,A), or an (m,n)-FPS$_{(X,A)}$; defining a homotopy between two (m,n)-FPS$_{(X,A)}$ as an (m,n)-FPS$_{(X,A)\times \mathrm{I}}$, we obtain homotopy classes $[f]$ which belong to a group $\mathrm{FIX}^{m,n}(X,A)$ which, in turn, as it was the case in 1.5, depends only on the difference $m-n$.

There are also convenient smooth representatives in $\mathrm{FIX}^{m,n}(X,A)$, when these make sense. We have the following two results, when X is an open set in a euclidean space $\mathbb{R}^p$, which are also equally valid if X is a smooth q-manifold instead.

Proposition 1.9. *Let $X \subset \mathbb{R}^p$ be open and let $A \subset X$ be closed (in X). Given an (m,n)-$\mathrm{FPS}_{(X,A)}$,*

$$\mathbb{R}^n \times \mathbb{R}^q \times X \supset V \xrightarrow{f} \mathbb{R}^m \times \mathbb{R}^q \times X\,,$$

there exists a smooth (m,n)-$\mathrm{FPS}_{(X,A)}$,

$$\mathbb{R}^n \times \mathbb{R}^q \times X \supset V \xrightarrow{f'} \mathbb{R}^m \times \mathbb{R}^q \times X\,,$$

equivalent to f, such that it is transverse to $j = j_V : V \to \mathbb{R}^m \times \mathbb{R}^q \times X$, $j(y,y',x) = (0,y',x)$.

Proposition 1.10. *Let $X \subset \mathbb{R}^p$ be open and let $A \subset X$ be closed (in X). Given an (m,n)-$\mathrm{FPS}_{(X,A)\times \mathrm{I}}$,*

$$\mathbb{R}^n \times \mathbb{R}^q \times X \times \mathrm{I} \supset V \xrightarrow{f} \mathbb{R}^m \times \mathbb{R}^q \times X \times \mathrm{I}\,,$$

such that its restrictions f_ι to $V_\iota = V \cap (\mathbb{R}^n \times \mathbb{R}^q \times X \times \{\iota\})$ are smooth and transverse to $j_\iota : V_\iota \to \mathbb{R}^n \times \mathbb{R}^q \times X \times \{\iota\}$; then there exists a smooth (m,n)-$\mathrm{FPS}_{(X,A)\times I}$,

$$\mathbb{R}^n \times \mathbb{R}^q \times X \times \mathrm{I} \supset V \xrightarrow{f'} \mathbb{R}^m \times \mathbb{R}^q \times X \times \mathrm{I}$$

equivalent to f, such that it is transverse to $j : V \to \mathbb{R}^n \times \mathbb{R}^q \times X \times \mathrm{I}$ and $f'|_{V_\iota} = f_\iota$.

Proof of 1.9. Since $\mathrm{Fix}(f) \subset V$ lies properly over X and maps into the open set $X - A$, by taking a smaller neighborhood of $\mathrm{Fix}(f)$, we may assume without loss of generality that the closure $\overline{V}$ itself lies properly over X and is contained in $\mathbb{R}^n \times \mathbb{R}^q \times (X - A)$. Let $\varepsilon : X \to \mathbb{R}^+$ be such that the fiberwise distance $\mathrm{d}(\mathrm{Fix}_x(f), \mathbb{R}^m \times \mathbb{R}^q \times \{x\} - V_x) > \varepsilon(x) > 0$.

Using, for instance, [3, 1.4.4], we know there is a smooth fiberwise $f'' : V \to \mathbb{R}^m \times \mathbb{R}^q \times X$ (approximate only the first two euclidean components) such that $|f''(y,y',x) - f(y,y',x)| < \varepsilon(x)/2$. By perturbing f'' to a map f', if necessary, in less than another $\varepsilon(x)/2$, we may assume that $0 \in \mathbb{R}^n \times \mathbb{R}^q$ is a regular value of $\mathrm{proj}_{\mathbb{R}^n \times \mathbb{R}^q} \circ (j - f')$, so that we have that $|f'(y, y', x) - f(y, y', x)| < \varepsilon(x)/2$, and f' is transverse to j.

Let now $g : V \times \mathrm{I} \to \mathbb{R}^n \times \mathbb{R}^q \times X \times \mathrm{I}$ be such that $g(y, y', x, t) = (tf'(y, y', x) + (1-t)f(y, y', x), t)$. $\mathrm{Fix}(g) \subset V \times \mathrm{I}$, therefore, it lies properly and maps into $(X - A) \times \mathrm{I}$; moreover, $g|_{V\times\{0\}} = f$ and $g|_{V\times\{1\}} = f'$. □

The *proof* of 1.10 is very similar using [3, 1.4.8] instead. □

There is a suspension isomorphism. Namely, let

$$\mathbb{R}^n \times E \supset V \xrightarrow{f} \mathbb{R}^m \times E$$

be an (m, n)-$\mathrm{FPS}_{(X,A)}$; then let

$$\mathbb{R}^n \times (\mathbb{R} \times E) \supset W \xrightarrow{f^\Sigma} \mathbb{R} \times \mathbb{R}^m \times (\mathbb{R} \times E)$$

be such that $W = \{(y, s, e) \in \mathbb{R}^n \times (\mathbb{R} \times E) \mid (y, e) \in V\}$ and $f^\Sigma(y, s, e) = (-s, \mathrm{proj}_{\mathbb{R}^m} f(y, e), s, \mathrm{proj}_E f(y, e))$. This is an $(m+1, n)$-FPS over the pair $(\mathbb{R}, \mathbb{R} - (-1, 1)) \times (X, A) = (\mathbb{R} \times X, (\mathbb{R} - (-1, 1)) \times X \cup \mathbb{R} \times A)$ and we have the following.

Proposition 1.11. *The function* $\Sigma : \mathrm{FIX}^k(X, A) \to \mathrm{FIX}^{k+1}((\mathbb{R}, \mathbb{R} - (-1, 1)) \times (X, A))$, *such that* $\Sigma([f]) = [f^\Sigma]$, *is a well defined isomorphism of groups. It is called the* suspension isomorphism. □

Given an (m, n)-$\mathrm{FPS}_{(X,A)}$, f, and a map $\varphi : (Y, B) \to (X, A)$, where $B \subset Y$ is closed, we can pull back f to an (m, n)-$\mathrm{FPS}_{(Y,B)}$, $\varphi * (f)$. This induces a homomorphism

$$\varphi* : \mathrm{FIX}^k(X, A) \to \mathrm{FIX}^k(Y, B)\,,$$

thus turning FIX^k into a (contravariant) functor. We can summarize all said as follows.

Theorem 1.12. *The groups* $\mathrm{FIX}^*(X, A)$ *constitute a multiplicative cohomology theory in the category of pairs* (X, A) *of metric spaces modulo closed subspaces.* □

2 Generic elements in FIX*

In this paragraph we shall use Dold's results in [1]. To start with, we recall the definition of a σ-structure on a manifold.

Definition 2.1. Let $\mathfrak{Vect}$ be the category of smooth vector bundles $\pi : E \to X$ over smooth manifolds X, with bundle maps $\widetilde{f} : E' \to E$ covering smooth maps $f : X' \to X$, i.e. such that the diagram

$$\begin{tikzcd} E' \arrow[r, "\widetilde{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

is a pullback diagram where, fiberwise, $\widetilde{f}$ induces linear isomorphisms. A contravariant functor

$$\sigma : \mathfrak{Vect} \to \mathfrak{Set}$$

together with a natural transformation

$$s : \sigma(E) \to \sigma(\mathbb{R} \times E)$$

(where $\mathbb{R} \times E \to X$ is the obvious map) is called a *structure functor* if it satisfies the following conditions:

(a) **Homotopy invariance**. The projection $\widetilde{q} : \mathbb{R} \times E \to E$ (viewed as a bundle map covering the projection $q : \mathbb{R} \times X \to X$) induces bijections $\widetilde{q}^* : \sigma(E) \cong \sigma(\mathbb{R} \times E)$, for any bundle $E \to X$.

(b) **Mayer-Vietoris property**. Let $E \to X$ be a smooth vector bundle and $X = X_1 \cup X_2$ for $X_1, X_2 \subset X$ open subsets. Denote by $E_\iota \to X_\iota$, $\iota = 1, 2$, the restrictions of the bundle. If $u_\iota \in \sigma(E_\iota)$ are such that $u_1|E_1 \cap E_2 = u_2|E_1 \cap E_2$, then there exists $u \in \sigma(E)$ such that $u|E_\iota = u_\iota$, where the restriction means the element induced by the inclusion.

(c) **Additivity**. If $E \to X$ is a bundle such that $X = \coprod_{n=1}^{\infty} X_n$, then the canonical function $\sigma(E) \to \prod_{n=1}^{\infty} \sigma(E_n)$ is bijective.

(d) **Stability**. The transformation s is an equivalence of sets

$$s : \sigma(E) \cong \sigma(\mathbb{R} \times E)\,.$$

For the scope of this paper, we shall be interested in the case where $\sigma(E)$ is the set of homotopy classes of stable trivializations of the form $\mathbb{R}^m \times E \cong \mathbb{R}^N \times X$.

Definition 2.2. A smooth manifold X is a *σ-manifold* if its tangent bundle $\tau(X) \to X$ is σ-structured.

EXAMPLE 2.3. Seen as a bundle over a point, $\mathbb{R}^l$ can always be σ-structured (unless $\sigma \equiv \varnothing$), namely, one just defines one structure on $\mathbb{R}$ (or on a point $* = \mathbb{R}^0$); hence, any product bundle $\mathbb{R}^l \times X \to X$ is also σ-structured and, with it, any trivial bundle has a σ-structure induced by the trivialization. Therefore, also *any open subset $X \subset \mathbb{R}^l$ is canonically a σ-manifold.*

REMARK 2.4. Given a smooth l-manifold X, then, by definition, if $\nu(X) \to X$ denotes its normal bundle, one has the direct sum $\nu(X) \oplus \tau(X) = \mathbb{R}^n \times X$. Therefore, this sum has a canonical σ-structure. Now, assuming that we have a trivialization of the normal bundle, namely, $\mathbb{R}^{n-l} \times X \cong \nu(X)$, this trivialization determines a stable trivialization of the tangent bundle as follows

$$\mathbb{R}^{n-l} \times \tau(X) = (\mathbb{R}^{n-l} \times X) \oplus \tau(X) \cong \nu(X) \oplus \tau(X) = \mathbb{R}^n \times X\,;$$

thus, this trivialization induces on $\mathbb{R}^{n-l} \times \tau(X)$ a σ-structure and, by the stability property of the σ-structure, also one on $\tau(X)$.

Now, for simplicity, we may assume X to be an open set in some euclidean space $\mathbb{R}^l$, but we could also assume it to be an l-dimensional manifold without boundary as will be the case later. Let $A \subset X$ be a (locally) closed set. Recall that a *proper σ-manifold over* the pair (X, A) is a continuous proper map $p : M \to X$ such that M is a smooth manifold, $p(M) \subset X - A$, and the bundle $\nu(M) \oplus p^*(\tau(X))$ is σ-structured. Let $E = \mathbb{R}^q \times X$ and let $\mathbb{R}^n \times E \supset V \xrightarrow{f} \mathbb{R}^m \times E$ be an (m, n)-$\mathrm{FPS}_{(X,A)}$ representing an arbitrary element in $\mathrm{FIX}^k(X, A)$, $k = m - n$. Let $j : V \to \mathbb{R}^m \times E$ be such that $j(y, e) = (0, e)$. We may assume f is smooth and *transverse* to j, that is, the difference map $j - f : V \to \mathbb{R}^m \times E$ such that

$$(j - f)(y, y', x) = (-y, y' - \mathrm{proj}_{\mathbb{R}^q} f(y, e), x)$$

is transverse to the submanifold $0 \times X \subset \mathbb{R}^q \times E = \mathbb{R}^{q+m} \times X$. Then $p|_M : M = \mathrm{Fix}(f) = (j - f)^{-1}(0 \times X) \to X$ is a proper manifold of dimension $n - m + l$ over X, whose image lies in $X - A$.

Since the normal bundle of $0 \times X$ in $\mathbb{R}^m \times E$ is trivial, namely $\nu(0 \times X) = \mathbb{R}^m \times \mathbb{R}^q \times X = \mathbb{R}^{m+q} \times X$, it has a canonical σ-structure, which can be pulled back to obtain a natural σ-structure on $\nu(M) \oplus g^*(\tau(X))$. For example, the trivialization of the normal bundle is a universal σ-structure, determining framed cobordism groups. In particular, pulling back the trivialization one obtains a framing for the proper manifold $g : M \to X$. Thus, it defines an element $[M \to X] \in \Omega_\sigma^{m-n}(X, A)$, where Ω_σ denotes the σ-cobordism functor as defined by Dold in [1] (see also [8]) and we will have a function

$$I : \mathrm{FIX}^{m-n}(X, A) \to \Omega_\sigma^{m-n}(X, A)$$

which we shall call the (*fixed point*) *index* and shall analyze in what follows. We will prove, namely, the following.

Theorem 2.5. *The assignment* $[f] \mapsto [\mathrm{Fix}(f') \to X]$, *where* $f' \in [f]$ *is a smooth representative, transverse to* j, *determines a well defined group homomorphism*

$$I : \mathrm{FIX}^{m-n}(X, A) \to \Omega_\sigma^{m-n}(X, A)\,.$$

Moreover, this homomorphism is a natural transformation of multiplicative cohomology theories.

In order to prove this theorem we shall proceed in steps. That the function is well defined follows from the next result.

Lemma 2.6. $[\mathrm{Fix}(f') \to X]$ *is independent of the choice of the smooth map* $f' \in [f]$.

Proof. Let f'_0 and f'_1 be both representatives of the class. We may assume that $f'_0 : V_0 \to \mathbb{R}^m \times \mathbb{R}^q \times X$ and $f'_1 : V_1 \to \mathbb{R}^m \times \mathbb{R}^q \times X$ are smooth and transverse to j_{V_ι}, $\iota = 0, 1$, and that $V_0, V_1 \subset \mathbb{R}^n \times \mathbb{R}^q \times X$ are open. By 1.10, we know that there is a smooth (m,n)-$\mathrm{FPS}_{(X,A)\times \mathrm{I}}$, $g : V \to \mathbb{R}^m \times \mathbb{R}^q \times X \times \mathrm{I}$, transverse to j_V.

Therefore, $W = \mathrm{Fix}(g) \subset \mathbb{R}^n \times \mathbb{R}^q \times X \times \mathrm{I} \to X \times \mathrm{I}$ is a proper σ-structured manifold with boundary $\partial W = \mathrm{Fix}(g) \cap (\mathbb{R}^n \times \mathbb{R}^q \times X \times \{0, 1\}) = \mathrm{Fix}(f'_0) \sqcup \mathrm{Fix}(f'_1) = M_0 \sqcup M_1$. Moreover, the σ-structure in W restricts to the corresponding σ-structure in M_0 and, since the inner normal vector of $M_1 \subset W$ points downward, to the opposite structure on M_1. Hence, $[M_0 \to X] = [M_1 \to X] \in \Omega_\sigma^{m-n}(X, A)$. ☐

The other results we need are the following three lemmas.

Lemma 2.7. *The function* $I : \mathrm{FIX}^{m-n}(X,A) \to \Omega_\sigma^{m-n}(X,A)$ *is additive and multiplicative.*

Proof. By definition of the additive structure, the sum of two elements of $\mathrm{FIX}^{m-n}(X,A)$ represented by $f_\iota : V_\iota \to \mathbb{R}^m \times E_\iota$, $\iota = 1,2$, is given by taking their disjoint union (topological sum). Therefore, if both are smooth and transverse to (the corresponding) j, then also $f_1 \sqcup f_2$ is smooth and transverse to j, and $\mathrm{Fix}(f_1 \sqcup f_2) = \mathrm{Fix}(f_1) \sqcup \mathrm{Fix}(f_2)$ has the obvious (by the σ-structure on each summand induced) σ-structure. This represents, by definition of the additive structure in $\Omega^{m-n}(X,A)$, the sum of $I([f_1])$ and $I([f_2])$. Therefore,

$$I([f_1] + [f_2]) = I([f_1 \sqcup f_2]) = I([f_1]) + I([f_2]) \,.$$

By definition of the multiplicative structure, the product of two elements in $\mathrm{FIX}^*(X,A)$ is given by taking the fibered product over X of their representatives f_1 and f_2. Anaglogously to the case of addition, one has that $\mathrm{Fix}(f_1 \times_X f_2) = \mathrm{Fix}(f_1) \times_X \mathrm{Fix}(f_2)$, so that, if both maps are smooth and transverse to (the corresponding) j, then also $f_1 \times_X f_2$ is smooth and transverse to j and its fixed point set (manifold) inherits the obvious σ-structure. Therefore,

$$I([f_1] \smile [f_2]) = I([f_1 \times_X f_2]) = I([f_1]) \smile I([f_2]) \,.$$

$\square$

Lemma 2.8. *The function* $I : \mathrm{FIX}^{m-n}(X,A) \to \Omega_\sigma^{m-n}(X,A)$ *is natural.*

Proof. Let $\varphi : (X,A) \to (Y,B)$ be continuous and let $f : V \to \mathbb{R}^m \times E$ represent an element in $\mathrm{FIX}^{m-n}(X,A)$. Assume f is smooth and transverse to j and deform φ as to make it transverse to f; then $\varphi^* : \mathrm{FIX}^{m-n}(X,A) \to \mathrm{FIX}^{m-n}(Y,B)$ sends the class $[f]$ to the class of the pullback $[\varphi^* f]$, which, in turn, is sent by I to the class $[\mathrm{Fix}(\varphi^* f) \to Y] = [\varphi^* \mathrm{Fix}(f) \to Y] = \varphi^*([\mathrm{Fix}(f) \to X]) \in \Omega^{m-n}(Y,B)$. Therefore

$$\varphi^* I([f]) = I \varphi^*([f]) \,.$$

$\square$

Finally, we need the following step.

Lemma 2.9. *The function* $I : \mathrm{FIX}^*(X,A) \to \Omega_\sigma^*(X,A)$ *commutes with the suspension isomorphism.*

Proof. This assertion is immediate if one recalls that, given an element of $\Omega^k_\sigma(X,A)$ represented by a σ-structured manifold over (X,A), $p: M^{l-k} \to X$ (if X is open in $\mathbb{R}^l$ or is an l-manifold), its *suspension* $\Sigma[M \to X] \in \Omega^{k+1}_\sigma((\mathbb{R}, \mathbb{R}-(-1,1)) \times (X,A))$ is represented by the map $(0,p): M \to \mathbb{R} \times X$, such that $(0,p)(m) = (0,p(m))$, with the obvious induced σ-structure (given by the stability property of the structure σ).

Since, on the other hand, given an $(n, n+k)$-$\mathrm{FPS}_{(X,A)}$,

$$\mathbb{R}^n \times E \supset V \xrightarrow{f} \mathbb{R}^{n+k} \times E \ ,$$

representing an element of $\mathrm{FIX}^k_\sigma(X,A)$, the fixed point set of its suspension $\mathrm{Fix}(f^\Sigma) \approx \mathrm{Fix}(f)$ (as it was defined just before 1.11) maps into $\mathbb{R} \times X$ precisely in 0 in the $\mathbb{R}$-component and as the original projection of $\mathrm{Fix}(f)$ into the X-component, then we have that, indeed, the index commutes with the suspension isomorphisms. □

We then have that the index is a natural transformation of cohomology theories from FIX-cohomology to σ-cobordism.

The main result of this paper states that the index yields a natural isomorphism between FIX^* and framed cobordism, Ω^*_{fr}, as multiplicative cohomology theories. There are two ways of proving this. One is through a shortcut established by two facts; on the one hand that FIX^* and stable cohomotopy π^*_{st} are isomorphic ([5]), and, on the other hand, the fact that, via the Pontryagin-Thom construction, Ω^*_{fr} and π^*_{st} are isomorphic.

The other way of proving the main result is by directly defining an inverse of the index $I: \mathrm{FIX}^{m-n}(X,A) \to \Omega^{m-n}_{\mathrm{fr}}(X,A)$.

In the next paragraph, we follow the first way.

3 The Pontryagin-Thom construction

Let X be an l-dimensional manifold without boundary and let $A \subset X$ be a closed set. Let $M^{l-k} \to X$ be a proper manifold over (X,A). A *trivialization of its normal bundle* is an isomorphism

$$\varphi : \nu(M) \oplus p^*(\tau(X)) \cong M \times \mathbb{R}^{l+l'} \ ,$$

where $\nu(M)$ is the normal bundle of M, that is, the normal bundle of an embedding $e: M \hookrightarrow \mathbb{R}^{l+l'-k}$, and $p^*(\tau(X))$ is the pullback to M of the tangent bundle of X (which is trivial for open sets $X \subset \mathbb{R}^l$). By taking the vertical embedding $e' = (e, p) : M \hookrightarrow \mathbb{R}^{l+l'-k} \times X$ (with its image inside $\mathbb{R}^{l+l'-k} \times (X - A)$), we have that its normal bundle $\nu(e')$ is the bundle $\nu(M) \oplus p^*(\tau(X))$, which is homeomorphic to a tubular neighborhood of M in $\mathbb{R}^{l+l'-k} \times X$, so that the Pontryagin-Thom construction for this embedding, that is the result of adding to $\mathbb{R}^{l+l'-k} \times (X - A)$ a point at infinity and then collapsing the complement in it of the tubular neighborhood of M (which is homeomorphic to $M \times \mathbb{R}^{l+l'}$ via φ), together with the projection $M^+ \to \mathbb{S}^0$, yields a map

$$\begin{tikzcd}
\Sigma^{l+l'-k}(X/A) \arrow[rrr] \arrow[d, equal] & & & \mathbb{S}^{l+l'} \arrow[d, equal] \\
(\mathbb{R}^{l+l'-k} \times (X - A))^* \arrow[r] & (\nu(M) \oplus p^*(\tau(X)))^* \approx \Sigma^{l+l'}(M^+) \arrow[rr] & & \Sigma^{l+l'}\mathbb{S}^0 ;
\end{tikzcd}$$

this is, namely, a representative of an element in $\pi^k_{\mathrm{st}}(X, A)$.

If we call $\Omega^k_{\mathrm{fr}}(X, A)$ the cobordism group of proper manifolds over (X, A), structured with a trivialization of their normal bundle, we obtain a function

$$\Phi : \Omega^k_{\mathrm{fr}}(X, A) \to \pi^k_{\mathrm{st}}(X, A) ,$$

which we call the *Pontryagin-Thom homomorphism.*

Conversely, given a pointed map

$$f : \Sigma^{l+l'-k}(X/A) \to \mathbb{S}^{l+l'} = \mathbb{R}^{l+l'} \cup \infty ,$$

representing an element in $\pi^k_{\mathrm{st}}(X, A)$, we may deform it to a smooth pointed map having 0 as a regular value. Hence $M = f^{-1}(0) \subset \Sigma^{l+l'-k}(X/A)$ is a manifold of dimension $l-k$, $l = \dim(X)$, lying, in fact, in $\mathbb{R}^{l+l'-k} \times (X - A) = \Sigma^{l+l'-k}(X/A) - \{*\}$. The restriction of the projection $p : M \to X - A \subset X$ is proper and the canonical trivialization of the normal bundle of $0 \subset \mathbb{R}^{l+l'-k}$ pulls back to a trivialization of the normal bundle of the embedding $M \hookrightarrow \mathbb{R}^{l+l'-k} \times X$, namely of $\nu(M) \oplus p^*\tau(X)$. We have therefore an element in $\Omega^k_{\mathrm{fr}}(X, A)$ and thus a function

$$\Psi : \pi^k_{\mathrm{st}}(X, A) \to \Omega^k_{\mathrm{fr}}(X, A) ,$$

which is the inverse of the Pontryagin-Thom construction.

Proposition 3.1. *For any (X, A) and every k, the following triangle commutes.*

$$\begin{tikzcd} \mathrm{FIX}^k(X,A) \arrow[rr, "I_\pi"] \arrow[dr, "I_\Omega"'] & & \pi^k_{\mathrm{st}}(X,A) \\ & \Omega^k_{\mathrm{fr}}(X,A) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Therefore, if two of the arrows are isomorphisms so is the third.

Proof. Let $[f] \in \mathrm{FIX}^{m-n}(X, A)$ be represented by a smooth map

$$\mathbb{R}^n \times \mathbb{R}^q \times X \xrightarrow{f} \mathbb{R}^m \times \mathbb{R}^q \times X$$

transverse to $j(y, y', x) = (0, y', x)$. Therefore, its image $I_\Omega[f] \in \Omega^{m-n}_{\mathrm{fr}}(X, A)$ is represented by the projection $p : M = \mathrm{Fix}(f) = (j - f)^{-1}(0 \times 0 \times X) \to X$, together with the trivialization of its normal bundle $\nu(M) \oplus p^*(\tau(X)) \cong \mathbb{R}^{m+q} \times M$ given by the restriction of the derivative $D(j - f) : \mathbb{R}^{n+q} \times \tau(X) \to \mathbb{R}^{m+q} \times \tau(X)$. Since the Pontryagin-Thom construction applied to $[M \to X]$ collapses the complement of the tubular neighborhood $T \approx \nu(M) \oplus p^*(\tau(X)) \cong \mathbb{R}^{m+q} \times M$ of M in $\mathbb{R}^{n+q} \times X$, the derivative determines a map $\widehat{f} : \Sigma^{n+q}(X/A) \to \Sigma^{m+q}(M^+) \to \Sigma^{m+q}(\mathbb{S}^0) = \mathbb{S}^{m+q}$. This map represents the element $[\widehat{f}] = \Phi \circ I_\Omega[f] \in \pi^{m-n}_{\mathrm{st}}(X, A)$.

On the other hand, the index of f, $I_\pi[f] \in \pi^{m-n}_{\mathrm{st}}(X, A)$ is represented by the following composite of maps of pairs.

$$\begin{tikzcd} (\mathbb{R}^{n+q} \times X, \mathbb{R}^{n+q} \times X - M) \arrow[r, "(j-f)'"] & (\mathbb{R}^{m+q}, \mathbb{R}^{n+q} - 0) \\ (\mathbb{R}^{n+q} \times X, (\mathbb{R}^{n+q} \times X - B_\rho) \cup \mathbb{R}^{n+q} \times A) \arrow[u, hook] \arrow[d, hook] & \\ (\mathbb{R}^{n+q}, \mathbb{R}^{n+q} - \mathbb{D}) \times (X, A) \arrow[d] & (\mathbb{R}^{m+q}, \mathbb{R}^{m+q} - \mathbb{D}) \arrow[uu, hook] \arrow[d] \\ (\Sigma^{n+q}(X/A), *) \arrow[r, dashed, "\widetilde{f}"] & (\mathbb{S}^{m+q}, *)\,, \end{tikzcd}$$

where all vertical arrows, except the one on the top left, are homotopy equivalences, some induced by inclusions and some by identifications; here B_ρ represents a neighborhood of the zero-section in $\mathbb{R}^{n+q} \times X$ built by balls of variable radius $\rho : X \to [1,\infty]$ containing the proper manifold M, $\mathbb{D}$ represents the unit ball in the corresponding euclidean space and $(j-f)'$ represents the euclidean part of $j-f$ (cf. [7]). That the classes $[\widehat{f}]$ constructed before and the class $[\widetilde{f}]$ just constructed coincide inside $\pi_{\mathrm{st}}^{m-n}(X,A)$ follows from the fact that the derivative $D(j-f)$ approximates $j-f$ around M. Therefore, $\Phi \circ I_\Omega[f] = [\widehat{f}] = [\widetilde{f}] = I_\pi[f]$ □

4 The inverse of the index

In this section we prove directly that the index

$$I : \mathrm{FIX}^k(X,A) \to \Omega^k_{\mathrm{fr}}(X,A)$$

is an isomorphism by defining an inverse

$$J : \Omega^k_{\mathrm{fr}}(X,A) \to \mathrm{FIX}^k(X,A)\,.$$

Let a proper $(l-k)$-manifold over an l-manifold X, $g : M^{l-k} \to X^l$, and a trivialization $\varphi : \nu(M) \oplus g^*(\tau(X)) \cong \mathbb{R}^{q+l} \times M$ represent an element in $\Omega^k_{\mathrm{fr}}(X,A)$. The normal bundle $\nu(M)$ can be realized as a tubular neighborhood of an embedding $e : M \hookrightarrow \mathbb{R}^N$. Making this embedding fiberwise, that is, taking $e' : M \hookrightarrow \mathbb{R}^N \times X$, such that $e'(y) = (e(y), g(y))$, the image $e'(M) \subset \mathbb{R}^N \times X$ has a tubular neighborhood $V \subset \mathbb{R}^N \times X$, which is homeomorphic to $\nu(M) \oplus g^*(\tau(X))$; therefore, the bundle isomorphism φ determines a homeomorphism $\psi : V \approx \mathbb{R}^{N+k} \times M$. Consider the map

$$\mathbb{R}^N \times X \supset V \xrightarrow{f} \mathbb{R}^{N+k} \times X$$

such that $f = (\mathrm{id}_{\mathbb{R}^{N+k}} \times g) \circ \psi$. One easily verifies that f is a map over X such that $f(w,x) = (0,x)$ if and only if $(w,x) \in e'(M) \subset \mathbb{R}^N \times X$; that is, taking the ENR_X $E = X$, the map f is an $(N+k,N)$-$\mathrm{FPS}_{(X,A)}$ such that $\mathrm{Fix}(f) = e'(M)$. In other words, f represents an element $[f] \in \mathrm{FIX}^k(X,A)$. Moreover, pulling back the canonical trivialization of the normal bundle of $0 \times X \subset \mathbb{R}^N \times X$, we recover the given trivialization φ of $\nu(M) \oplus g^*(\tau(X))$.

Let now $g : W \to X$ denote a proper framed cobordism between two proper framed $(l-k)$-manifolds over (X,A), $g_0 : M_0 \to X$ and $g_1 : M_1 \to X$. Now, there is an embedding $e : W \hookrightarrow \mathbb{R}^{N+1}$ such that its image $e(W)$ is such that $e(W) \subset \mathbb{R}^N \times \mathrm{I}$, and $(\mathbb{R}^N \times \{\iota\}) \cap e(W) = e(M_\iota)$, $\iota = 0,1$. If we now extend this embedding to a fiberwise one $e' : W \hookrightarrow \mathbb{R}^N \times \mathrm{I} \times X = \mathbb{R}^N \times X \times \mathrm{I}$, as above, it is easy to see that this embedding has a normal bundle homeomorphic to a tubular neighborhood $\widetilde{V}$ of it. Therefore, if $\widetilde{\psi} : \widetilde{V} \approx \mathbb{R}^{N+k} \times W$ is the corresponding trivialization of the tubular neighborhood, we can consider the map

$$\mathbb{R}^N \times X \times \mathrm{I} \supset \widetilde{V} \xrightarrow{\widetilde{f}} \mathbb{R}^{N+k} \times X \times \mathrm{I}\,,$$

such that $\widetilde{f} = (\mathrm{id}_{\mathbb{R}^{N+k}} \times \widetilde{g}) \circ \widetilde{\psi}$, where $\widetilde{g} : W \to X \times \mathrm{I}$ is such that $\widetilde{g}(w) = (g(w), t(w))$ and $t(w) \in \mathrm{I}$ is the component of $e'(w)$ corresponding to the factor I. As before, $\mathrm{Fix}(\widetilde{f}) = e'(W)$, so that $\widetilde{f}$ is an $(N+k, N)$-$\mathrm{FPS}_{(X,A)\times \mathrm{I}}$ and the restrictions $\widetilde{f}|_{X\times\{\iota\}} = f|_\iota$, $\iota = 0, 1$. Therefore, cobordant proper framed manifolds over a pair (X, A) determine equivalent fixed point situations over (X, A).

From the discussion above, we have a well defined homomorphism

$$J : \Omega^k_{\mathrm{fr}}(X, A) \to \mathrm{FIX}^k(X, A)\,,$$

which is a (left) inverse to the index I defined in section 2. We have the following.

Theorem 4.1. *The fixed point index* $I : \mathrm{FIX}^k(X, A) \to \Omega^k_{\mathrm{fr}}(X, A)$ *is an isomorphism with inverse* $J : \Omega^k_{\mathrm{fr}}(X, A) \to \mathrm{FIX}^k(X, A)$.

Proof. We already saw that $I \circ J = \mathbf{1}_{\Omega^k_{\mathrm{fr}}(X,A)}$. In order to verify that $J \circ I = \mathbf{1}_{\mathrm{FIX}^k(X,A)}$, we have to start with any $(N+k, N)$-$\mathrm{FPS}_{(X,A)}$, $f : V \to \mathbb{R}^{N+k} \times E$, where, by proposition 1.9 without loss of generality we may assume that $E = \mathbb{R}^n \times X$ and that f is smooth and transverse to $j : V \to \mathbb{R}^{N+k} \times E$. Hence, the derivative $D(j - f) : \tau(V) = \mathbb{R}^N \times \mathbb{R}^n \times \tau(X) \to \tau(\mathbb{R}^{N+k} \times E) = \mathbb{R}^{N+k} \times \mathbb{R}^n \times \tau(X)$ induces a trivialization φ of $\nu(\mathrm{Fix}(f)) \oplus g^*(\tau(X))$, (where in this case $g : E \to X$ is the projection). Now, since locally $D(j - f)$ is a linear approximation of $j - f$, one easily verifies that $j - f$ and $(\mathrm{id}_{\mathbb{R}^{N+k}} \times g) \circ \varphi$ represent the same class in $\mathrm{FIX}^k(X, A)$. $\square$

5 Generalizations and remarks

There is a symmetric version of the FIX-cohomology for the action of any compact Lie group ([6, 7]); namely, an equivariant cohomology theory FIX^*_G can be defined as follows.

If X is a metric G-space, that is, a metric space with a continuous G-action, then we say that a space over X, $p : E \to X$, is a *G-euclidean neighborhood retract over X*, or a G-ENR_X, if p is an equivariant map and there is an equivariant embedding $i : E \hookrightarrow Q \times X$, where Q is a real G-module (say, $\mathbb{R}^q$ with a linear orthogonal G-action), such that $\mathrm{proj}_X \circ i = p$, an invariant open neighborhood U of $i(E)$ in $Q \times X$ and an equivariant retraction $r : U \to E$.

Definition 5.1. Let $p : E \to X$ be a G-ENR_X and let M, N be G-modules. An (M, N)*-fixed point situation over* X, or an (M, N)-FPS_X, is a commutative triangle

$$\begin{tikzcd} N \times E \supset V \arrow[rr, "f"] \arrow[dr, "p \circ \mathrm{proj}_E"'] & & M \times E \arrow[dl, "p \circ \mathrm{proj}_E"] \\ & X, & \end{tikzcd}$$

where the equivariant map f is *properly fixed*, that is, it is such that the set $F = \mathrm{Fix}(f) = \{(y, e) \in V \mid f(y, e) = (0, e) \in \mathbb{R}^n \times E\}$ lies properly over X.

Using this equivariant version of a fixed point situation, one can define in a completely analogous way groups $\mathrm{FIX}^*_G(X, A)$ which constitute an $\mathrm{RO}(G)$-graded equivariant cohomology theory (which is isomorphic to equivariant stable cohomotopy).

REMARK 5.2. We saw that the fixed point theory, i.e. FIX-theory, provides an alternative approach to cobordism. In a sense, manifolds, as inverse images of zeroes which are regular values of smooth maps, are exchanged by fixed point sets of continuous maps. Transversality theorems are generally false when one deals with symmetries, since, in general, one may deform a symmetric (equivariant) map only loosing symmetry. Since FIX-theory admits a symmetric version for the action of any compact Lie group G, our approach provides an equivariant alternative to (framed) cobordism.

REMARK 5.3. There is a dual approach to FIX^*, which produces the corresponding homology theory FIX_*, which happens to be isomorphic to framed bordism; directly, cap and slant products can be defined in a natural way, thus enriching FIX-theory. In particular, duality theorems such as Poincaré's or Alexander's can be proved. This FIX-homology has its own equivariant version too. Details will appear elsewhere.

REMARK 5.4. We have worked out all theory for X either an open set of a euclidean space or, more generally, for a smooth manifold. Departing from the case of the open set, there is also a generalization for X a locally closed subset of a euclidean space, in particular, for ENRs. This is done by extending the situations considered from X to an open neighborhood of it in which X is closed, in the same spirit as Dold's construction of the geometric cobordism groups in [1].

REMARK 5.5. Dold [2] constructed the isomorphism $\mathrm{FIX}^0(X) \cong \pi^0_{\mathrm{st}}(X)$, which is a special case of the one given in [5]. On the other hand Koźniewski [4] proved the special case $\mathrm{FIX}^0(X) \cong \Omega^0_{\mathrm{fr}}(X)$ of our main result 4.1.

References

[1] A. DOLD, *Geometric cobordism and the fixed point transfer*, Algebraic Topology – Proceedings, Vancouver 1977, Lect. Notes Math. 673, 32-87, Springer-Verlag, Berlin Heidelberg New York 1978

[2] A. DOLD, *The fixed point index of fibre-preserving maps*, Inventiones math. 25, (1974) 281–297

[3] A. DOLD, Teoría de Punto Fijo – Volumen I, Monografías del Instituto de Matemáticas 18, UNAM, México 1984

[4] T. KOŹNIEWWSKI, *The category of submersions*, Bull. Acad. Pol. Sci., Sér. Sci. Math. 27 No. 3–4, (1979) 321–326

[5] C. PRIETO, *Coincidence index for fiber-preserving maps. An approach to stable cohomotopy*, manuscripta math. 47 (1984) 233–249

[6] C. PRIETO, *$KO(B)$-graded stable cohomotopy over B and $RO(G)$-graded G-equivariant cohomotopy: A fixed point theoretical approach to the Segal conjecture*, Contemporary Math. 58 (1987) 89–107

[7] C. PRIETO, H. ULRICH, *Equivariant fixed point index and fixed point transfer in nonzero dimensions*, Trans. Amer. Math. Soc. 328 (1991) 731–745

[8] D. QUILLEN, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Adv. in Math. 7 (1971) 29–56

cprieto@math.unam.mx