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THE UNSTABLE EQUIVARIANT FIXED POINT INDEX AND THE EQUIVARIANT DEGREE

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Abstract

A correspondence between the equivariant degree introduced by Ize, Massabó, and Vignoli and an unstable version of the equivariant fixed point index defined by Prieto and Ulrich is shown. With the help of conormal maps and properties of the unstable index, a sum decomposition formula is proved for the index and consequently also for the degree. As an application, equivariant homotopy groups are decomposed as direct sums of smaller groups of fixed orbit types, and a geometric interpretation of each summand is given in terms of conormal maps.

Introduction

In this paper we study the equivariant degree defined by Ize, Massabó, and Vignoli by comparing it with the equivariant fixed point index defined by Prieto and Ulrich. In what follows, we define an unstable equivariant fixed point index with nice properties, which is helpful in proving some results about the degree.

In the first two sections, in order to establish our notation, we recall the definitions of the equivariant degree [**14**] and of the equivariant fixed point index [**21**]. After comparing both concepts in the next section, we use the degree to define in Section 4 an unstable version of the fixed point index as an element of some unstable equivariant homotopy group. Its properties allow us to extend the unstable index to G-euclidean neighbourhood retracts. Using this, in Section 5 we prove a sum formula, similar to the already proved formula for the stable index [**21**, 2.13], which reflects the stratification of a G-euclidean neighbourhood retract in different orbit types. This formula in turn provides a corresponding one for the degree, which was obtained by Balanov and Krawcewicz using different techniques [**2**]. To do this, we introduce the notion of conormal map, which in a sense is dual to the notion of normal map used by others. We show that any equivariant map with compact fixed point set is equivariantly homotopic to a conormal map that is unique up to conormal homotopy. As an application, in Section 6 we give a direct sum decomposition of equivariant homotopy groups, and illustrate how our sum formula can be easily used to prove Segal's theorem stating that the G-equivariant 0th stable homotopy group of a point is isomorphic to the Burnside ring of G, namely, $\pi_G^{\mathrm{st}\,0}(*) \cong A(G)$.

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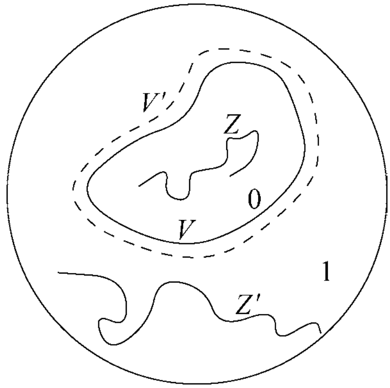

FIGURE 1.

1. *The equivariant degree*

In this section, we provide the definition of the equivariant degree, as given in [**14**]. Let G be a compact Lie group, and let M and N denote G-modules with orthogonal linear actions of G, of dimensions m and n, respectively. Denote by $\mathbb{S}^N$, $\mathbb{S}^M$ the n- and m-dimensional spheres obtained as one-point compactifications of $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$, with the corresponding G-actions. Later in the paper we use the unit spheres of G-modules M, which we denote by $S(M)$. Note that there is a canonical equivariant homeomorphism (stereographic projection) between $\mathbb{S}^M$ and $S(M \oplus \mathbb{R})$ that sends the point at $\infty \in \mathbb{S}^M$ to $(0,1) \in M \times \mathbb{R}$, where $\mathbb{R}$ has trivial G-action.

DEFINITION 1.1. For an equivariant map $f : V \longrightarrow M$, where $V \subset N$ is an open G-invariant set such that $Z = f^{-1}(0)$ is compact, the following is done.

(0) Shrinking V if necessary, we may assume that V is bounded, f is defined in $\overline{V}$, and that $f^{-1}(0) \subset V$.

(1) Take R large enough such that $\overline{V} \subset B_R$, where B_R denotes the open ball centred at the origin with radius R in N.

(2) Using the Tietze–Gleason extension theorem, extend f to an equivariant map $\widehat{f} : B_R \longrightarrow M$. Denote by $\widehat{Z}$ the zero-set of $\widehat{f}$. $\widehat{f}^{-1}(0) = \widehat{Z} = Z \cup Z'$, where $Z' \subset B_R - \overline{V}$.

(3) After taking an open set V' such that $\overline{V} \subset V' \subset B_R$, $\overline{V'} \cap \widehat{Z} = Z$, using an equivariant version of Urysohn's lemma, construct a G-invariant map $\varphi : B_R \longrightarrow [0,1] = \mathrm{I}$, such that $\varphi|_{B_R - V'} = 1$ and $\varphi|_{\overline{V}} = 0$.

(4) Define $F : \mathrm{I} \times B_R \longrightarrow \mathbb{R} \times M$ by

$$F(t,x) = (2t + 2\varphi(x) - 1, \widehat{f}(x)).$$

(5) Since $F(t,x) = 0$ if and only if $t = 1/2$ and $x \in Z$, F has no zeros on the boundary $\partial(\mathrm{I} \times B_R) \approx \mathbb{S}^N$ and therefore F determines, by restriction, a map

$$F' : \mathbb{S}^N \longrightarrow \mathbb{R} \times M - 0 \longrightarrow \mathbb{S}^M,$$

where the second map is the usual retraction onto the unit sphere in $\mathbb{R} \times M$, which is canonically G-homeomorphic to $\mathbb{S}^M$, as we mentioned above.

By definition, the *unstable* class $\deg_G(f) = [F'] \in [\mathbb{S}^N, \mathbb{S}^M]_G$ is the *equivariant degree* of f. Figure 1 illustrates the construction.

REMARK 1.2. The excision property of the degree [**14**, (c), p. 443] guarantees that the definition above is independent of the equivariant shrinking mentioned in Section 0.

2. *The equivariant fixed point index*

In this section, we recall the definition of the *stable* equivariant fixed point index, as given in [**21**], but in a special case.

Let G be a compact Lie group. Given an equivariant map $\varphi : V \longrightarrow K \oplus M'$, where K, M' and N' are G-modules and $V \subset K \oplus N'$ is an open and G-invariant set such that the fixed point set $F = \mathrm{Fix}(\varphi) = \{(y,z) \in V \subset K \oplus N' \mid \varphi(y,z) = (y,0) \in K \oplus M'\} \subset V$ is compact, one has an *equivariant fixed point index*, $I^G(\varphi)$, which is an element of the $(M-N)$-homology group $h_{M-N}(*)$, where h^G is some $\mathrm{RO}(G)$-graded equivariant homology theory and $M - N \in \mathrm{RO}(G)$ is the element in the real representation ring of G represented by the (virtual) difference of $M = K \oplus M'$ and $N = K \oplus N'$ (cf. [**18**]).

DEFINITION 2.1. The *fixed point index* of φ is defined as follows. Consider the diagram in Figure 2 where $j : V \longrightarrow M$ is such that $j(y,z) = (y,0) \in K \oplus M'$, $(y,z) \in V \subset K \oplus N'$. Since F is closed and V is open in N, then (1) is an excision, and (2) is a homotopy equivalence in the second term of the pair; thus both induce isomorphisms in homology. Therefore, the dotted arrow i_φ induces a well-defined homomorphism

$$(i_\varphi)_* : h^G_{\rho+N}(N, N-0) \longrightarrow h^G_{\rho+N}(M, M-0),$$

where $\rho \in \mathrm{RO}(G)$, which, after desuspending, determines a homomorphism

$$I^G_\varphi : h^G_\rho(*) \longrightarrow h^G_{\rho+N-M}(*)$$

and, taking the image of the element $1 \in h^G_0(*)$, also an element $I^G(\varphi) = I^G_\varphi(1) \in h^G_{N-M}(*)$.

Particularly interesting is the case where h^G is the equivariant stable homotopy π^G_{st}. Then the index $I^G(\varphi)$ is a *stable* element in $\pi^G_{\mathrm{st}\,N-M}(*) = \{\mathbb{S}^N, \mathbb{S}^M\}_G = \mathrm{colim}_K[\mathbb{S}^{N\oplus K}, \mathbb{S}^{M\oplus K}]_G$, where K varies over a cofinal set of G-modules. Note that this homotopy group can also be considered as the *cohomotopy* group $\pi_G^{\mathrm{st}\,M-N}(*)$.

$$\begin{tikzcd}
(V, V-F) \arrow[r, "j-\varphi"] \arrow[d, hook, "(1)"'] & (M, M-0) \\
(N, N-F) & \\
(N, N-B_r) \arrow[u, hook] \arrow[d, hook, "(2)"'] & \\
(N, N-0) \arrow[uuur, dashed, "i_\varphi"'] &
\end{tikzcd}$$

FIGURE 2.

$$\begin{tikzcd}
(\mathbb{R},\mathbb{R}-0)\times(V,V-Z) \arrow[r, "\mathrm{id}\times f"] \arrow[d, hook, "\simeq"'] & (\mathbb{R},\mathbb{R}-0)\times(M,M-0) \\
(\mathbb{R},\mathbb{R}-0)\times(N,N-Z) \arrow[ur, "\widetilde{F}"] &
\end{tikzcd}$$

FIGURE 3.

$$\begin{tikzcd}
(\mathbb{R},\mathbb{R}-0)\times(N,N-Z) \arrow[r, "\widetilde{F}"] & (\mathbb{R},\mathbb{R}-0)\times(M,M-0) \arrow[d, equal] \\
(\mathbb{R}\times N,\mathbb{R}\times N-B_R) \arrow[u, hook] \arrow[r, dashed, "d_f"'] & (\mathbb{R}\times M,\mathbb{R}\times M-0)
\end{tikzcd}$$

FIGURE 4.

3. *Comparison of the degree with the fixed point index*

Recall Section 1, where given a map $f\colon V\longrightarrow M$, $V\subset N$ open G-invariant such that $Z=f^{-1}(0)$ is compact, we defined the degree $\deg_G(f)$ as the equivariant homotopy class of a map $F'\colon\mathbb{S}^N\longrightarrow\mathbb{S}^M$.

In order to compare the construction of the equivariant degree with the one for the equivariant fixed point index, first, using the linear homeomorphism $\mathbb{D}^1=[-1,1]\longrightarrow \mathrm{I}$, $t\longmapsto (t+1)/2$, we change the map F in Definition 1.1(4) to a map $G\colon\mathbb{D}^1\times B_R\longrightarrow\mathbb{R}\times M$. Thus

$$G(t,x)=(t+2\varphi(x),\widehat{f}(x)).$$

Then we can extend the map G further to a map $\widetilde{F}\colon\mathbb{R}\times N\longrightarrow\mathbb{R}\times M$, say by taking first

$$\widetilde{F}(t,x)=\begin{cases} G(t,x) & |t|\leqslant 1 \text{ and } |x|\leqslant R\\ G\Big(\dfrac{t}{|t|},x\Big) & |t|\geqslant 1 \text{ and } |x|\leqslant R\\ G\Big(t,R\dfrac{x}{|x|}\Big) & |t|\leqslant 1 \text{ and } |x|\geqslant R\\ G\Big(\dfrac{t}{|t|},R\dfrac{x}{|x|}\Big) & |t|\geqslant 1 \text{ and } |x|\geqslant R. \end{cases}$$

Then, the zero set $\widetilde{Z}=\widetilde{F}^{-1}(0)=\{0\}\times Z$ and we have indeed a map of pairs $\widetilde{F}\colon(\mathbb{R},\mathbb{R}-0)\times(N,N-Z)\longrightarrow(\mathbb{R}\times M,\mathbb{R}\times M-0)$. The triangle in Figure 3 commutes up to equivariant homotopy of pairs, since if $(t,x)\in\mathbb{R}\times V$, then $\widetilde{F}(t,x)=(t,f(x))$ if $|t|\leqslant 1$ and $=(t/|t|,f(x))$, if $|t|\geqslant 1$.

One has the map of pairs $d_f\colon(\mathbb{R}\times N,\mathbb{R}\times N-B_R)\longrightarrow(\mathbb{R}\times M,\mathbb{R}\times M-0)$ defined by the diagram in Figure 4.

PROPOSITION 3.1. *The map $d_f\colon(\mathbb{R}\times N,\mathbb{R}\times N-B_R)\longrightarrow(\mathbb{R}\times M,\mathbb{R}\times M-0)$ induces in homotopy classes the element*

$$\deg_G(f)\in[(\mathbb{R}\times N,\mathbb{R}\times N-0);(\mathbb{R}\times M,\mathbb{R}\times M-0)]_G\cong[\mathbb{S}^N,\mathbb{S}^M]_G.$$

$$\begin{tikzcd}
 & k_j(\mathbb{R}\times N-0) & k_j(\mathbb{S}^N) \arrow[l, "\cong"'] \arrow[ddd, "d'_f"] \\
k_{j+1}((\mathbb{R},\mathbb{R}-0)\times(N,N-B_R)) \arrow[r, "\cong", "(1)"'] \arrow[d] \arrow[dd, bend right=60, "d_f"'] & k_j(\mathbb{R}\times(N-B_R)\cup(\mathbb{R}-0)\times N) \arrow[u, "\cong"', "(3)"] \arrow[d] & \\
k_{j+1}((\mathbb{R},\mathbb{R}-0)\times(N,N-Z)) \arrow[r] \arrow[d, "\widetilde{F}_*"'] & k_j(\mathbb{R}\times N-(0\times Z)) \arrow[d, "(\widetilde{F}|)_*"] & \\
k_{j+1}((\mathbb{R},\mathbb{R}-0)\times(M,M-0)) \arrow[r, "\cong", "(2)"'] & k_j(\mathbb{R}\times M-0) & k_j(\mathbb{S}^M) \arrow[l, "\cong"']
\end{tikzcd}$$

FIGURE 5.

Proof. Let k_* be any graded reduced homotopy functor with a natural exact sequence for pairs of spaces, such as either equivariant homotopy groups π_*^G (see [**1**]), or any equivariant reduced homology theory $\widetilde{h}_*^G$. Take the diagram in Figure 5 where the horizontal arrows on the left ladder are given by the corresponding connecting homomorphisms, and the two on the right by inclusions. The horizontal arrows (1) and (2) are natural isomorphisms, since $k_{j+1}(\mathbb{R}\times N) = k_{j+1}(\mathbb{R}\times M) = k_j(\mathbb{R}\times N) = k_j(\mathbb{R}\times M) = 0$ and the vertical arrow (3) is an isomorphism given by a canonical homotopy equivalence. The curved arrow on the left is the homomorphism d_f defined above. The two isomorphisms on the right-hand side ladder follow because the inclusion of the unit spheres in $\mathbb{R}\times N-0$, respectively $\mathbb{R}\times M-0$, are equivariant homotopy equivalences, and these spheres are equivariantly homeomorphic to $\mathbb{S}^N$ and $\mathbb{S}^M$, respectively.

In the special case $k_j = \pi_N^G = [\mathbb{S}^N, -]_G$, the homomorphism d'_f corresponds to a homomorphism

$$[\mathbb{S}^N, \mathbb{S}^N]_G \longrightarrow [\mathbb{S}^N, \mathbb{S}^M]_G,$$

which sends $[\mathrm{id}_{\mathbb{S}^N}]$ to $\deg_G(f)$. □

Given any element $[\alpha] \in [\mathbb{S}^N, \mathbb{S}^M]_G$, it induces a homomorphism $\alpha_* : \widetilde{h}_*^G(\mathbb{S}^N) \longrightarrow \widetilde{h}_*^G(\mathbb{S}^M)$.

COROLLARY 3.2. *If* $1 \in h_0^G(*) \cong \widetilde{h}_0^G(\mathbb{S}^0) = \widetilde{h}_N^G(\mathbb{S}^N)$, *then* $\deg_G(f)_*(1) = I^G(j-f) \in \widetilde{h}_N^G(\mathbb{S}^M) \cong \widetilde{h}_{N-M}^G(\mathbb{S}^0) \cong h_{N-M}^G(*)$. *In particular, if* $\widetilde{h}_*^G$ *is equivariant stable homotopy, then* $\deg_G(f)_*(1) \in \{\mathbb{S}^N, \mathbb{S}^M\}_G$ *is the* stabilization *of* $\deg_G(f) \in [\mathbb{S}^N, \mathbb{S}^M]_G$, *which we call the* stable degree.

Proof. Figures 3 and 4, together, give us Figure 2 suspended by taking the product with $(\mathbb{R},\mathbb{R}-0)$ on the left, and taking $K=0$; therefore, $j=0$, and $\varphi = j-f$. Then $F=Z$, that is, $\mathrm{Fix}(\varphi) = f^{-1}(0)$. Hence, taking $k_j = \widetilde{h}_N^G$, the homomorphism d'_f in Figure 5 sends 1 to $I^G(j-f) \in \widetilde{h}_N^G(\mathbb{S}^M) \cong \widetilde{h}_{N-M}^G(\mathbb{S}^0) \cong h_{N-M}^G(*)$. □

4. *The unstable fixed point index*

In this section we redefine the equivariant fixed point index to obtain an *unstable* version of it. We shall use the equivariant degree instead of Figure 2 in Section 2, which was used to define the stable index.

DEFINITION 4.1. Let M, N and K be G-modules and let $V \subset N \times K$ be open and invariant. If $\varphi : V \longrightarrow M \times K$ is such that $F = \mathrm{Fix}(\varphi) = \{(y,e) \in V \mid \varphi(y,e) = (0,e)\}$ is compact, then, if $j : V \longrightarrow M \times K$ is such that $j(y,z) = (0,z)$ and $f(y,z) = (j - \varphi)(y,z)$, define the *unstable equivariant fixed point index* of φ by

$$I_G^u(\varphi) = \deg_G(f) \in [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G.$$

This group is abelian if $\dim(N^G \oplus K^G) > 0$ (see [**14**] or [**12**]).

This unstable index has the following properties which are either direct consequences of the corresponding properties of the equivariant degree, or can be obtained by a slight modification of the corresponding proofs in [**21**] for the stable index (cf. also [**14**, (c), (b), (e) p. 443]).

(a) *Localization* (corresponding to the excision property of the degree): If $W \subset V$ is open and G-invariant and $F \subset W$, then

$$I_G^u(\varphi) = I_G^u(\varphi|_W) \in [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G.$$

(b) *G-homotopy*: Let $\varphi_\tau : V_\tau \longrightarrow M \times E$ be such that $F_\tau = \mathrm{Fix}(\varphi_\tau) = \{(y,e) \in V_\tau \mid \varphi_\tau(y,e) = (0,e)\}$ is compact for every $\tau \in \mathrm{I}$, then

$$I_G^u(\varphi_\tau) = I_G^u(\varphi_0) \in [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G, \qquad \tau \in \mathrm{I}.$$

Such a homotopy φ_τ will be called *admissible*.

(c) *Additivity*: Let $\varphi_\nu : V_\nu \longrightarrow M \times K$, $\nu = 1, 2$, $V_\nu \subset N \times K$ open and G-invariant, be such that the fixed point sets $F_\nu = \mathrm{Fix}(f_\nu)$ are compact and disjoint. By the localization property, one can thus assume that the domains V_ν are also disjoint. If $V = V_1 \cup V_2$ and $\varphi : V \longrightarrow M \times K$ is such that $\varphi|_{V_\nu} = \varphi_\nu$, then φ has a compact fixed point set $F = \mathrm{Fix}(\varphi) = F_1 \cup F_2$ and

$$\Sigma\big(I_G^u(\varphi)\big) = \Sigma\big(I_G^u(\varphi_1)\big) + \Sigma\big(I_G^u(\varphi_2)\big) \in [\mathbb{S}^{N \oplus K+1}, \mathbb{S}^{M \oplus K+1}]_G,$$

where $\Sigma : [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G \longrightarrow [\mathbb{S}^{N \oplus K+1}, \mathbb{S}^{M \oplus K+1}]_G$ is the suspension homomorphism and, for any L, $\mathbb{S}^{L+1}$ denotes the one-point compactification of the G-module $L \oplus \mathbb{R}$. (The additivity holds, thus, already after one suspension.)

Moreover, the unstable index has a property that the degree does not have.

(d) *Commutativity* (corresponding to [**21**, 1.15]): Let M, N, K, and K' be G-modules and let $U \subset N \times K$, $W \subset K'$ be open invariant sets. If $\alpha : U \longrightarrow M \times K'$ and $\beta : W \longrightarrow K$ are continuous equivariant maps such that the map

$$N \times K \supset \alpha^{-1}(M \times W) \xrightarrow{(1_M \times \beta)\alpha} M \times K$$

has a compact fixed point set $F = \mathrm{Fix}((1_M \times \beta)\alpha)$, then also the map

$$N \times K' \supset (i_N \times \beta)^{-1}(U) \xrightarrow{\alpha(1_N \times \beta)} M \times K'$$

has a compact fixed point set $F' = \mathrm{Fix}(\alpha(1_N \times \beta))$. Moreover, both F and F' are homeomorphic and

$$\Sigma^{\overline{K}} I_G^u((1_M \times \beta)\alpha) = \Sigma^{\overline{K}'} I_G^u(\alpha(1_N \times \beta)) \in [\mathbb{S}^{N \oplus L}, \mathbb{S}^{M \oplus L}]_G,$$

where L is the smallest G-module, such that $K \oplus \overline{K} = L$ and $K' \oplus \overline{K}' = L$ and Σ denotes the corresponding suspension homomorphism. In particular, if $K = K'$, one can take $L = K = K'$ and then one does not need to suspend in order to have the commutativity property.

Using this last property, as in [**21**], one can extend the definition of the unstable index to more general situations.

To that purpose, let E be a *G-euclidean neighbourhood retract*, namely $E \subset U \subset K$, where U is open and G-invariant, and there is an equivariant retraction $r : U \longrightarrow E$ (see [**15**, **23**] for general properties of G-euclidean neighbourhood retracts). Let $i : E \hookrightarrow K$ be the inclusion.

Definition 4.2. Let $V \subset N \times E$ be open, invariant and let $\varphi : V \longrightarrow M \times E$ be such that $F = \mathrm{Fix}(\varphi) = \{(y, e) \in V \mid \varphi(y, e) = (0, e)\}$ is compact. Then we define the *unstable equivariant fixed point index* of φ taking $\widetilde{\varphi} : \widetilde{V} \longrightarrow M \times K$, such that $\widetilde{V} = (1_N \times r)^{-1}(V) \subset N \times K$ and $\widetilde{\varphi} = (1_M \times i) \circ \varphi \circ (1_N \times r)$ and putting

$$I^u_G(\varphi) = I^u_G(\widetilde{\varphi}) = \deg_G(j - \widetilde{\varphi}) \in [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G,$$

where, as before, $j : \widetilde{V} \longrightarrow M \times K$ is such that $j(y, z) = (0, z)$.

This general unstable equivariant index for maps (partially) defined on G-euclidean neighbourhood retracts is well defined and has all properties (a)–(d), which the previous case has.

Remark 4.3. Consider a map φ, partially defined on $N \times K$ and with image in $M \times K$ with a compact fixed point set. For the sake of notational simplicity, one might simply write $I^u_G(\widetilde{\varphi}) \in [\mathbb{S}^{N \oplus K \oplus L}, \mathbb{S}^{M \oplus K \oplus L}]_G$ for the unstable index $I^u_G(\widetilde{\varphi}) \in [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G$, instead of the suspension $\Sigma^L I^u_G(\widetilde{\varphi})$, since from the term $\oplus L$ in the homotopy set one can infer that one is dealing with the L-suspension.

Similarly, for a map f, partially defined on $N \times K$ and with image in $M \times K$ with a compact zero-set, one might simply write $\deg_G(f) \in [\mathbb{S}^{N \oplus K \oplus L}, \mathbb{S}^{M \oplus K \oplus L}]_G$ for the degree $\deg_G(f) \in [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G$, instead of the suspension $\Sigma^L \deg_G(f)$.

Even though the unstable equivariant fixed point index is defined via the equivariant degree, it allows us to extend the definition of the degree to a more general situation, which will be useful later on in Theorem 5.6.

Definition 4.4. Given a G-retract E of an open invariant set U in a G-module K with retraction $r : U \longrightarrow E$, such that $0 \in E$, and a map $f : N \times E \longrightarrow M \times E$, such that $Z = f^{-1}(0)$ is compact, one may define $\deg_G(f) = I^u_G(\varphi) \in [\mathbb{S}^{N \oplus K}, \mathbb{S}^{M \oplus K}]_G$, if $\varphi = j - f(1 \times r) : N \times U \longrightarrow M \times K$, where $j : N \times U \longrightarrow M \times K$ is such that $j(y, z) = (0, z)$.

5. *Sum decomposition formula*

In this section we show that the unstable equivariant index decomposes as a sum of elements, each corresponding to one orbit type. This leads to a decomposition of the group $[\mathbb{S}^N, \mathbb{S}^M]_G$ into a direct sum, as was shown by Balanov and Krawcewicz [**2**] using the equivariant degree. Our approach is based on a method used in [**21**], where the formula was proved for the stable index (see also [**20**]), and it originated in [**23**]. This approach is simpler since it does not need any G-transversality as it was the case in [**2**], and thus it works in more general situations (G-euclidean neighbourhood retracts).

We begin by recalling a few notions of compact transformation group theory. Let X be any G-space and $H \subset G$ be a closed subgroup. We use the following notation of [**21**]:

$$X^{(H)} = \{x \in X \mid (H) \subset (G_x)\},$$

$$X^{(\underline{H})} = \{x \in X \mid (H) \subsetneq (G_x)\},$$

$$X_{(H)} = \{x \in X \mid (H) = (G_x)\},$$

where $(H) \subset (H')$ means that some conjugate of H is contained in H'. Therefore, $X_{(H)} = X^{(H)} - X^{(\underline{H})}$ and consists of points of isotropy groups in (H), that is, of orbit type (G/H). For simplicity we may call the orbit type of these points (H) instead. The set of all orbit types of X, that is, of conjugacy classes (H) such that $X_{(H)} \neq \varnothing$, will be denoted by $\mathrm{Or}(X)$.

Note that for every G-euclidean neighbourhood retract X, the set $\mathrm{Or}(X)$ is finite, since by definition, X is an equivariant retract of an open invariant set $V \subset M$; thus $\mathrm{Or}(X) \subset \mathrm{Or}(V) \subset \mathrm{Or}(M) = \mathrm{Or}(S(M))$. However $\mathrm{Or}(S(M))$ is finite, because the unit sphere $S(M)$ in M is a smooth, compact G-manifold (cf. [**4**, IV.1.2]).

Next, observe that for a G-space X with a finite set of orbit types there is an ordered indexing (H_j) of $\mathrm{Or}(X)$ such that

$$(H_j) \subset (H_i) \implies j \leqslant i. \tag{5.1}$$

Indeed, we may enumerate the minimal elements of $\mathrm{Or}(X)$ in an arbitrary way and subtract them from $\mathrm{Or}(X)$, then enumerate the minimal elements of the remaining set, and continue this procedure.

For such an indexing, we define a filtration of X by

$$X_i = \bigcup_{i \leqslant j} X^{(H_j)}. \tag{5.2}$$

Note that for the difference sets of the filtration (5.2) we have $X_i - X_{i-1} = X_{(H_i)}$.

If we now take $X = E$ to be a G-euclidean neighbourhood retract, then every E_i is a closed G-euclidean neighbourhood retract subspace of E, because for every H the set $E^{(H)}$ is a closed G-euclidean neighbourhood retract subspace of E (cf. [**15**, **23**]).

Now we state the main technical step (cf. [**23**, II.5.2], see also [**21**, 2.11]) that we use below, which when adapted to our situation reads as follows.

Proposition 5.1. *Let E be a G-euclidean neighbourhood retract. Consider $\mathbb{R}^m \times E$ and $\mathbb{R}^n \times E$, where $\mathbb{R}^m$ and $\mathbb{R}^n$ have trivial actions, and let $\varphi : V \longrightarrow \mathbb{R}^m \times E$ be a G-map with a compact fixed point set $F = \mathrm{Fix}(\varphi) \subset V$, $V \subset \mathbb{R}^n \times E$. Moreover, let $D \subset E$ be a closed G-euclidean neighbourhood retract subspace such that $\varphi(V \cap (\mathbb{R}^n \times D)) \subset \mathbb{R}^m \times D$.*

Then there exists a G-map $\varphi_D : V \longrightarrow \mathbb{R}^m \times E$, homotopic to φ relative to V^D by an admissible homotopy, that is, a homotopy with a compact fixed point set, of the form

$$\varphi_D = \varphi \circ r,$$

where $r|_{\overline{U}} : \overline{U} \longrightarrow D$ is an equivariant deformation retraction for some open invariant set $U \supset D$.

Proof. By the localization property of the unstable index, we may restrict φ to a numerically open G-invariant set V with compact closure. Thus V and V^D are G-euclidean neighbourhood retracts and so the inclusion $V^D \hookrightarrow V$ is a G-cofibration (see [**1**, 4.2.13]); hence there exists a G-deformation $d_\tau : V \longrightarrow V$ relative to V^D such that $d_1^{-1}(V^D)$ is a G-neighbourhood of V^D (see [**1**, 4.1.16(b)]).

We can make d_τ stationary outside of a G-neighbourhood U of V^D as follows. Take U such that $\overline{U} \subset d_1^{-1}(V^D)$ (that is, U is a shrinking of $d_1^{-1}(V^D)$), and take W to be an open G-neighbourhood of $\overline{U}$ in V. Then take $\sigma : V \longrightarrow \mathrm{I}$ to be an Urysohn G-function such that

$$\sigma|_{\overline{U}} = 1 \quad \text{and} \quad \sigma|_{V-W} = 0$$

and modulate d by taking $(v,\tau) \longmapsto d_{\sigma(v)\tau}(v)$ instead. Call this deformation again d_τ. Now $d_0 = \mathrm{id}_V$ and $d_\tau|_{V-W} = \mathrm{id}_{V-W}$, thus d is now stationary outside of W. We may assume $\overline{W}$ to be compact and contained in V.

The map $\varphi \circ d_\tau : V \longrightarrow \mathbb{R}^m \times E$ is a G-homotopy of φ relative to $(V^D) \cup (V - W)$, and its fixed point set is a closed subset of $\overline{W} \times I \cup \mathrm{Fix}(\varphi) \times I$ and it is thus compact. Take $r = d_1$. Then the map $\varphi_D = \varphi \circ r$ satisfies all the requirements of the statement. □

Proposition 5.1 leads us to the notion of a conormal map, which is dual to the notion of a normal map that was used to study the equivariant degree and was first introduced in [**8**] for $G = \mathbb{S}^1$ (see [**9**, **2**] and the references therein for the general case).

Definition 5.2. Let E be a G-euclidean neighbourhood retract and $\psi : V \longrightarrow \mathbb{R}^m \times E$ be a G-map with a compact fixed point set $F = \mathrm{Fix}(\psi) \subset V$, $V \subset \mathbb{R}^n \times E$, where $\mathbb{R}^m$ and $\mathbb{R}^n$ have trivial actions. We say that ψ is *conormal* if for every $(H) \in \mathrm{Or}(E)$ there exist an open invariant neighbourhood U of $V^{(\underline{H})}$ in $V^{(H)}$ and an equivariant retraction $r : \overline{U} \longrightarrow V^{(\underline{H})}$ such that for the restricted map $\psi^{(H)} = \psi|_{V^{(H)}}$ we have

$$\psi^{(H)}|_{\overline{U}} = \psi \circ r : \overline{U} \longrightarrow \mathbb{R}^m \times E. \tag{5.3}$$

As a direct consequence of the definition we get the following.

Proposition 5.3. *Let $\psi : V \longrightarrow \mathbb{R}^m \times E$ be a conormal map and $F = \mathrm{Fix}(\psi)$. Then for every orbit type (H) we have*

$$\overline{F \cap V_{(H)}} \cap V^{(\underline{H})} = \varnothing.$$

Moreover, we have

$$I_G^u\big(\psi^{(H)}\big) = I_G^u\big(\psi^{(\underline{H})}\big) + I_G^u\big(\psi_{(H)}\big) \in [\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G,$$

where $\psi_{(H)} = \psi|_{V_{(H)}}$.

Proof. Indeed, for every $x \in \overline{U} \subset V^{(H)}$, if $\psi(x) = x$ then $x \in V^{(\underline{H})}$. This shows the first part of the statement.

Now take U and $U' = V^{(H)} - \overline{U}$. By the additivity and localization properties of the unstable index, we have

$$I_G^u\big(\psi^{(H)}\big) = I_G^u(\psi|_U) + I_G^u(\psi|_{U'}) = I_G^u(\psi|_U) + I_G^u\big(\psi_{(H)}\big),$$

because all the fixed points of $\psi|_{V_{(H)}}$ lie in U'. On the other hand, by the commutativity property of the index and since ψ is conormal, namely of the form (5.3), $I^u_G(\psi|_U) = I^u_G(\psi|V^{(\underline{H})}) = I^u_G(\psi^{(\underline{H})})$. □

For any given map, the following theorem states the existence and uniqueness of homotopic conormal maps.

THEOREM 5.4. *Let E be a G-euclidean neighbourhood retract and let $\varphi : V \longrightarrow \mathbb{R}^m \times E$ be a G-map with a compact fixed point set $F = \mathrm{Fix}(\varphi) \subset V$, $V \subset \mathbb{R}^n \times E$, where $\mathbb{R}^m$ and $\mathbb{R}^n$ have trivial actions. Then we have the following.*

(a) *φ is equivariantly homotopic by an admissible homotopy φ_τ to a conormal map*
$\psi = \varphi_1 : V \longrightarrow \mathbb{R}^m \times E$. Moreover, if $A \subset V$ is a closed G-euclidean neighbourhood retract subspace, then this homotopy can be taken relative to A.

(b) *Furthermore, if φ_0 and φ_1 are equivariantly homotopic by an admissible homotopy, and each of them is equivariantly homotopic by an admissible homotopy to two conormal maps $\psi_0, \psi_1 : V \longrightarrow \mathbb{R}^m \times E$, respectively, then these two maps are equivariantly homotopic by an admissible conormal homotopy.*

Note that in the second part of (a), ψ is conormal, provided it is conormal on A. Otherwise it is conormal relative to A only. On the other hand, what (b) really states is that any two homotopic conormal maps can be deformed into each other by a conormal homotopy.

Proof of Theorem 5.4. By induction over the length of the filtration E_i of E defined in (5.2). For $E = E_1$ the statement is trivial and the required conormal map is $\psi_1 = \varphi$. Now let $E = E_2$ and take $D_1 = E_1 \cup A$. We apply Proposition 5.1. Let $U_1 = U$, $W_1 = W$, and $d^1_\tau = d_\tau : \overline{U_1} : \longrightarrow D_1$ be as in the proof of Proposition 5.1. Then $\psi_2 = \psi_1 \circ d^1_1 = \varphi \circ r_1$ is a conormal map.

Assume now that the result has been proved up to length $n-1$ and take $E = E_n$. Assume that $\psi_{n-1} : V \longrightarrow \mathbb{R}^m \times E$ is the already constructed conormal map for E_{n-1} such that $\psi_{n-1} = \psi_{n-2} \circ d^{n-1}_1 = \varphi \circ r_1 \circ r_2 \circ \ldots \circ r_{n-1}$, where $r_1, r_2, \ldots, r_{n-1}$ are the corresponding local retractions. We now take $D_n = E_{n-1} \cup A \subset E_n$ and apply Proposition 5.1 again. Thus we have $U_n = U$, $W_n = W$, and $d^n_\tau = d_\tau : \overline{U_n} \longrightarrow D_n$ as in the proof of Proposition 5.1. Take $\psi_n = \psi_{n-1} \circ d^n_1 = \varphi \circ r_1 \circ r_2 \circ \ldots \circ r_n$.

In order to see that $\psi = \psi_n$ is a conormal map, note that by its construction ψ is equivariantly homotopic to φ, relative to A and $V - \bigcap_{i=1}^n W_i$; thus it is homotopic via an admissible homotopy. Suppose that for a given orbit type (H) we have $(H) = (H_{i+1})$ in the ordering (5.1). As U we can take $U_i \cap V^{(H_i)}$ and as the retraction $r_i|_{V^{(H_i)}}$. r_i is equivariant and $r_i(\overline{U_i}) \subset V^{(H_i)} \cap E_i = V^{(\underline{H}_i)}$, so that we have completed the proof of (a).

To prove (b), it is enough to apply (a) to the following situation. Take $E \times \mathbb{R}$ instead of E as the given G-euclidean neighbourhood retract; instead of the map φ take the homotopy φ_τ between φ_0 and φ_1, defined on the open set $\tilde{V} = V \times (-\varepsilon, 1+\varepsilon)$. Moreover, take the homotopies from φ_0 to ψ_0 and from φ_1 to ψ_1. Thus there is a homotopy that we call φ_τ between the two conormal maps ψ_0 and ψ_1 which can be extended constantly over $(-\varepsilon, 0]$ and $[1, 1+\varepsilon)$. As the closed subset A we take $V \times \{0\} \cup V \times \{1\}$. Thus (a) provides the desired conormal homotopy. □

We should point out that an analogous statement has been shown by Komiya [**16**, Lemma 1] for $m = n = 0$ and E a compact, smooth G-manifold.

We are in position to prove our main theorem on the decomposition of the unstable fixed point index which corresponds to [**21**, 2.13] for the stable fixed point index.

THEOREM 5.5. *Let $\varphi\colon V \longrightarrow \mathbb{R}^m \times E$, $V \subset \mathbb{R}^n \times E$ open G-invariant, E a G-euclidean neighbourhood retract, be a G-map with compact fixed point set, and let $\psi\colon V \longrightarrow \mathbb{R}^m \times E$ be a homotopic conormal map by an admissible homotopy. Then*

$$I^u_G(\varphi) = \sum_{(H)} I^u_G(\psi_{(H)}) = \sum_{(H)} \big(I^u_G(\varphi^{(H)}) - I^u_G(\varphi^{(\underline{H})})\big) \in [\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G,$$

where the sum runs over $(H) \in \operatorname{Or}(V)$. Additionally, for every fixed $(H_0) \in \operatorname{Or}(V)$ we have

$$I^u_G(\varphi^{(H_0)}) = \sum_{(H)} I^u_G(\psi_{(H)}) = \sum_{(H)} \big(I^u_G(\varphi^{(H)}) - I^u_G(\varphi^{(\underline{H})})\big) \in [\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G,$$

where the sum now runs over $(H) \in \operatorname{Or}(V)$ such that $(H) \subset (H_0)$. This decomposition agrees with the additive structure of $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G$, in the sense that every (H)-coordinate of the sum of two elements φ, φ' is given by the sum of their corresponding coordinates.

Proof. We start proving a sum formula for a conormal map. We do it by induction over the filtration (5.2) and the explicit form of a conormal map given in the proof of Theorem 5.4. Suppose that this formula holds for all (H_j), $j \leqslant i$. Note that the map $\psi = \varphi \circ r_1 \ldots \circ r_l$ preserves this filtration and $\psi|_{E_{i+1}} = \varphi \circ r_1 \ldots \circ r_{i-1} \circ r_i$, where r_i is the end of a G-homotopy defined on V_{i+1} relative to V_i such that the restriction $r_i\colon \overline{U_i} \longrightarrow V_i$ is a retraction, for some invariant neighbourhood U_i of V_i. Repeating the argument of the proof of Proposition 5.3, we get

$$I^u_G(\psi|_{V_{i+1}}) = I^u_G(\psi|_{V_i}) + I^u_G(\psi|_{V_{i+1}-V_i}).$$

However $V_{i+1} - V_i = V_{(H_{i+1})}$, and consequently $I^u_G(\psi|_{V_{i+1}-V_i}) = I^u_G(\psi^{(H_{i+1})}) - I^u_G(\psi^{(\underline{H_{i+1}})})$, by Proposition 5.3. The sum formula is thus proved for a conormal map.

By Theorem 5.4, any equivariant map $\varphi\colon V \longrightarrow \mathbb{R}^m \times E$ is G-homotopic to a conormal map ψ. Thus $I^u_G(\varphi) = I^u_G(\psi)$, and $I^u_G(\varphi^{(H)}) = I^u_G(\psi^{(H)})$, $I^u_G(\varphi^{(\underline{H})}) = I^u_G(\psi^{(\underline{H})})$. This proves the first sum formula of the statement. The second sum formula follows from the first, when applied to the G-equivariant map $\varphi^{(H_0)}$.

As for the last assertion of the statement, it follows from the fact that any sum of two fixed point indices can be realized as the fixed point index of one map, by taking a disjoint union. This is always possible in our case, since we are dealing with suspensions by taking the product with $\mathbb{R}$ (which has no action), using the additivity property. □

We shall call the first equation $I^u_G(\varphi) = \sum_{(H)} I^u_G(\psi_{(H)})$ of Theorem 5.5 the *decomposition formula*, because it decomposes $I^u_G(f)$ into a sum of indices (of another map, in general) each of which corresponds to the index on the nonsingular open part of the natural invariant stratification $\{E^{(H)}\}$ of E.

We shall call the second equation $I^u_G(\varphi) = \sum_{(H)} I^u_G(\psi_{(H)})$ of Theorem 5.5, or the equation of Theorem 5.6 below, the *sum formula*, because it shows the numerical value of each term of the above-mentioned decomposition.

We now apply our decomposition and sum formulae of Theorem 5.5 to get similar formulae for the equivariant degree. Since our spaces are not open subsets of a G-module, but only retracts of them, we use here the concept of equivariant degree given in Definition 4.4. If $f : V \longrightarrow \mathbb{R}^m \times K$ is a G-map such that $V \subset \mathbb{R}^n \times K$ is open and invariant and the zero-set $Z = f^{-1}(0)$ is compact, then $\deg_G(f) = I^u_G(\varphi)$, where $\varphi = j - f$, $j : V \longrightarrow \mathbb{R}^m \times K$ such that $j(y, z) = (0, z)$. Thus, since $I^u_G(\varphi) = \sum(I^u_G(\varphi_i) - I^u_G(\varphi_{i-1}))$, we have

$$\begin{aligned}\deg_G(f) &= \sum I^u_G\big(\varphi_{(H_i)}\big) = \sum \big(I^u_G\big(\varphi^{(H_i)}\big) - I^u_G\big(\varphi^{(\underline{H}_i)}\big)\big)\\ &= \sum \big(\deg_G\big(f^{(H_i)}\big) - \deg_G\big(f^{(\underline{H}_i)}\big)\big)\end{aligned}$$

and we obtain the desired decomposition formula for the equivariant degree. Thus we have the following.

THEOREM 5.6. *Let $f : V \longrightarrow \mathbb{R}^m \times K$ be a G-map such that $V \subset \mathbb{R}^n \times K$ is an open invariant set and the zero-set $Z = f^{-1}(0)$ is compact. Then*

$$\deg_G(f) = \sum \big(\deg_G\big(f^{(H)}\big) - \deg_G\big(f^{(\underline{H})}\big)\big) \in [\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G,$$

where $f_{(H)} = j - \varphi_{(H)}$, and the sum is taken over all orbit types $(H) \in \mathrm{Or}(V)$.

Moreover, under the same hypotheses as above, for any (fixed) subgroup $H_0 \subset G$,

$$\deg_G\big(f^{(H_0)}\big) = \sum \big(\deg_G\big(f^{(H)}\big) - \deg_G\big(f^{(\underline{H})}\big)\big) \in [\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G,$$

where the sum is taken over all orbit types $(H) \in \mathrm{Or}(V)$ such that $(H) \subset (H_0)$.

REMARK 5.7. Using techniques of differential topology, namely the notion of a regular normal map, Balanov and Krawcewicz [**2**] obtained the decomposition formula for the equivariant degree, which corresponds to the equation $I^u_G(\varphi) = \sum_{(H)} I^u_G(\psi_{(H)})$ in Theorem 5.5 stated as

$$\deg_G(f, V) \;=\; \sum_{(H)} \deg_G\big(f_{(H)}, V\big) \tag{5.4}$$

provided that f is *normal*. However, they do not have the sum formula of Theorem 5.6, because they and previous authors did not have defined degrees in the more general context that we have in Definition 4.4. On the other hand, we must add that if f is *regular normal*, by a transversality argument, it follows that in formula (5.4) [**2**, (2.1)] there are no terms that correspond to (H) such that $\dim W(H) > n - m$, where $W(H) = N(H)/H$ is the Weyl group of H. We could not show that using conormal map techniques.

To finish this section we include an algebraic scheme that allows to compute the coordinates of the decomposition theorems, Theorems 5.5 and 5.6. Recall that for any poset $(X, \leqslant)$, one can define a function ζ by

$$\zeta(x, y) = \begin{cases} 1 & x \leqslant y\\ 0 & \text{otherwise.}\end{cases}$$

This produces an 'upper triangular matrix' Z with 'entries' $Z^x_y = \zeta(x,y)$ and 1s along the diagonal. Thus there is (see, for instance [**3**, 7.5.2]) another 'upper triangular matrix' M, known as the *Moebius matrix* of the poset, such that it is an inverse matrix, in the sense that $MZ = I$ and $ZM = I$, or entrywise, such that

$$\sum_z M^x_z Z^z_y = \delta^x_z \quad \text{and} \quad \sum_z Z^x_z M^z_y = \delta^x_z,$$

where δ^x_z is the Kronecker δ-function. Call $\mu(x,y)$ the entries M^x_y of this matrix. μ is the so-called *Moebius function* of the poset.

Thus, given any two abelian group-valued functions $\alpha, \beta : X \longrightarrow \Gamma$ such that

$$\alpha(y) = \sum_{x \leqslant y} \beta(x)\,, \quad \text{then } \beta(y) = \sum_{x \leqslant y} \mu(x,y)\alpha(x). \tag{5.5}$$

This last is called the *Moebius inversion formula*. Applying (5.5) to the second sum formula of Theorem 5.6, we obtain the following.

THEOREM 5.8. *Under the same hypotheses as the previous results*

$$I^u_G\big(\varphi^{(H_0)}\big) - I^u_G\big(\varphi^{(\underline{H})}\big) = \sum \mu((H),(H_0))\, I^u_G\big(\varphi^{(H)}\big),$$

$$\deg_G\big(f^{(H_0)}\big) - \deg_G\big(f^{(\underline{H_0})}\big) = \sum \mu((H),(H_0))\, \deg_G\big(f^{(H)}\big),$$

where the sum is taken over the orbit types $(H) \in \mathrm{Or}(V)$ *such that* $H \subset H_0$*, and* μ *is the Moebius function of the poset* $\{(H) \mid H \text{ is a subgroup of } G\}$.

REMARK 5.9. A similar formula using the generalized Moebius function obviously holds also for the fixed point index using the sum formula for the index as in [**21**, 2.13] instead. Komiya [**16**] deals with a similar formula for the classical equivariant fixed point index, which in our terms corresponds to the case $m = n = 0$, and applies it to an equivariant fixed point problem.

REMARK 5.10. Making use of a GAP programming package, one may derive the Moebius function μ for the poset of conjugacy classes of subgroups of G, provided that the group G is included in the library of the package.

6. *Direct sum decomposition of equivariant homotopy groups*

To begin this section we show that our decomposition theorem leads to already known decompositions of unstable as well as stable equivariant homotopy groups graded by integers.

DEFINITION 6.1. Given n, $m \in \mathbb{N} \cup \{0\}$, a G-module K, and an orbit type $(H) \in \mathrm{Or}(\mathbb{R}^n \oplus K) = \mathrm{Or}(K)$, we define the subset $[\mathbb{S}^{n+K}, \mathbb{S}^{m+K}]_{G,(H)} \subset [\mathbb{S}^{n+K}, \mathbb{S}^{m+K}]_G$, as the set of elements of the form

$$I^u_G(\psi, V) = I^u_G(\psi) \in [\mathbb{S}^{n+K}, \mathbb{S}^{m+K}]_G,$$

where $\psi : V \longrightarrow \mathbb{R}^m \oplus K$ is a conormal map with compact fixed point set $\mathrm{Fix}(\psi) \subset V_{(H)}$ and V is an open invariant subset of $\mathbb{R}^n \oplus K$.

We have the following theorem (cf. [**2**]).

THEOREM 6.2. *Suppose that $m > 0$ or $\dim K^G > 0$. Then for every $(H) \in \mathrm{Or}(K)$, the set $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_{G,(H)}$ is a subgroup of $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G$, and*

$$[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G \cong \bigoplus_{(H)} [\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_{G,(H)},$$

where the sum is taken over all $(H) \in \mathrm{Or}(K)$.

Moreover $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_{G,(H)} = 0$, if $\dim W(H) > n - m$, where $W(H)$ is the Weyl group of H.

Proof. The fact that $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_{G,(H)}$ is a subgroup follows from the decomposition theorem (Theorem 5.5), since the (H)-coordinate of the sum of two elements is the sum of their corresponding (H)-coordinates. In order to see that it is a decomposition as a direct sum, suppose that $[\varphi] \neq 0$ lies in $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_{G,(H_1)}$ as well as in $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_{G,(H_2)}$. Then it is of the form $I^u_G(\psi_1, V_1)$, as well as $I^u_G(\psi_2, V_2)$, where ψ_ν, $\nu = 1, 2$, are conormal maps and $\mathrm{Fix}(\psi_1) \subset V_{1\,(H_1)}$, $\mathrm{Fix}(\psi_2) \subset V_{2\,(H_2)}$. Using the localization property of the index, we may assume that $V_1 = V_2 = V$ by taking $V = V_1 \cup V_2$. By Theorem 5.4, ψ_1 and ψ_2 are homotopic by a conormal homotopy. On the other hand, it is easy to check that a conormal homotopy does not change the orbit type, that is $\mathrm{Fix}(\psi_1) \cap V_{(H)} \neq \varnothing$ if and only if $\mathrm{Fix}(\psi_2) \cap V_{(H)} \neq \varnothing$. This shows that $(H_1) = (H_2)$, which completes the proof of the decomposition.

Theorem 5.5 shows that every element of the form $I^u_G(\varphi, V)$ belongs to the above direct sum. We are left with the task of showing that every element in $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G$ is of the form $I^u_G(\varphi, V)$. Since $(\mathbb{R}^{m+1} \oplus K)^G \neq \{0\}$, we can construct an equivariant isotopy on $\mathbb{S}^{m+K+1}$ that takes any given point $x_0 \in \mathbb{S}^{m+K+1}$ to ∞. Consequently, every class $[f] \in [\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G$ has a representative f such that $f(\infty) = \infty$. Take $V = \mathbb{R}^{m+1} \oplus K = \mathbb{S}^{m+K+1} - \{\infty\}$ and $\varphi = j - f$. Since $f(\infty) = \infty$, ∞ is not an accumulation point of zeros of f, thus neither of fixed points of φ. Consequently $I^u_G(\varphi, V) = \deg_G(f, V) = [f]$.

To show that $[\mathbb{S}^{n+K}, \mathbb{S}^{m+K}]_{G,(H)} = 0$ if $\dim W(h) > n - m$, one needs a transversality argument (cf. [**2**]). □

REMARK 6.3. We re-proved a theorem about the decomposition of the groups $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_G$ into a direct sum of subgroups $[\mathbb{S}^{n+K+1}, \mathbb{S}^{m+K+1}]_{G,(H)}$. Our interpretation of each element of the latter as an index seems to make the construction of some special elements easier. Note that we need only construct a conormal map on an open invariant set.

Moreover, besides the decomposition, we have the sum formula of Theorems 5.6 and 5.8, which give the 'numerical' values of the coordinates of this decomposition.

Of course all the decompositions and sums for the unstable index and degree imply, after stabilizing, the corresponding results in the stable range.

To gain confidence on the above results, we established a connection between our decomposition and sum formulae and the Segal theorem which states that the stable cohomotopy group $\pi_G^{\mathrm{st}\,0}(*)$ is isomorphic to the Burnside ring $A(G)$ of G. This theorem was proved independently by Hauschild [**10**] (see also [**11**]), Kosniowski [**17**], and Rubinsztein [**22**], with a correction of a gap in the latter by Dancer [**5**] (see also [**12**]). Recall that the Burnside ring $A(G)$ of a finite group G is an additively free group with generators given by the orbits (G/H), that is, every element $\alpha \in A(G)$ can be uniquely written as $\alpha = \sum_{(H)} k_{(H)}(G/H)$, $k_{(H)} \in \mathbb{Z}$. Recall that the unit sphere $S(K \oplus \mathbb{R})$ coincides with the one-point compactification $\mathbb{S}^K$ of K and the point $(0,1)$ in the former corresponds to the point at infinity ∞ in the latter. Either of these points is taken as the natural base point. We denote by $[S(K \oplus \mathbb{R}), S(K \oplus \mathbb{R})]_G^*$ (or $[\mathbb{S}^K, \mathbb{S}^K]_G^*$) the set of pointed equivariant homotopy classes. We also set $V_\infty = \mathbb{S}^K - \{\infty\} = K$.

Suppose that G is finite, K is a complex representation of G, and $f : \mathbb{S}^K \longrightarrow \mathbb{S}^K$ is an equivariant (pointed) map. We assign to f an element $\omega(f)$ of $A(G) \otimes \mathbb{Q}$ by

$$\omega(f) = \sum_{(H)} \frac{I^u\big((j-f)^{(H)}, V_\infty^{(H)}\big) - I^u\big((j-f)^{(\underline{H})}, V_\infty^{(\underline{H})}\big)}{|G/H|}(G/H)\,, \tag{6.1}$$

where in the numerator of the fraction, we write nonequivariant (unstable) indices, whose difference is an integer, and the sum runs over all $(H) \in \mathrm{Or}(V)$. Note that $\omega(f)$ is a well-defined equivariant homotopy invariant, that is, it depends only on $[f]$. Furthermore, $\omega(f_1 + f_2) = \omega(f_1) + \omega(f_2)$.

Proposition 6.4. *The element $\omega(f)$ lies in the Burnside ring $A(G)$; that is, all coefficients $k_{(H)}$ in* (6.1) *are integers, and $\omega(f)$ determines the homotopy class $[f]$ of f.*

In other words, the mapping $[f] \longmapsto \omega(f)$ defines a monomorphism from $[\mathbb{S}^K, \mathbb{S}^K]_G$ to $A(G)$.

Proof. The first statement follows from the fact that $I^u((j-f)^{(H)}, V_\infty^{(H)}) - I^u((j-f)^{(\underline{H})}, V_\infty^{(\underline{H})})$ is divisible by $|G/H|$ (cf. [**23**]), consequently $\omega(f) \in A(G)$. Next we recall that an element $\alpha \in A(G)$ is uniquely determined by the collection $\{\chi^H(\alpha)\}$ of values of some homomorphisms $\chi^H : A(G) \longrightarrow \mathbb{Z}$, $(H) \in \mathrm{Or(G)}$ (cf. [**6**, **7**] for the definitions and properties of χ^H). One can show that for the element $\omega(f)$ we have $\chi^H(\omega(f)) = \deg(f^H)$ for every subgroup $H \subset G$ (cf. [**23**]).

On the other hand, by a theorem of tom Dieck it follows that the collection $\{\deg f^H\}$, $H \in \mathrm{Or(G)}$, determines the homotopy class of f, provided that $\dim K^L - \dim K^{L'} \geqslant 2$ for every two subgroups $L \subsetneq L' \subset G$ (see [**6**]). This latter condition is satisfied if K is complex. $\square$

Lemma 6.5. *Let $\psi : V \longrightarrow K$, $V \subset K$, be an equivariant conormal map such that* $\mathrm{Fix}(\psi) \subset V_{(H)}$. *Then*

$$I_G^u(\psi) = I_G^u\big(\psi^{(H)}\big) - I_G^u\big(\psi^{(\underline{H})}\big) = \frac{I^u\big((\psi)^{(H)}\big) - I^u\big((\psi)^{(\underline{H})}\big)}{|G/H|}.$$

Consequently, formula (6.1) is the sum formula of Theorems 5.5 and 5.6 if we understand elements of $[\mathbb{S}^K, \mathbb{S}^K]_G$ as elements of $A(G)$ by Proposition 6.4.

Proof of Lemma 6.5. The statement follows once more by comparing all values of χ^L, $L \in \mathrm{Or}(V)$, with $\deg(j-\psi)^L$, both as elements of $A(G)$. $\square$

Now we show that our sum formula allows us to see any element of $A(G)$ as an index of an equivariant map, which consequently leads to the subsequent result.

PROPOSITION 6.6. *Let G be a finite group. Let K be the complex regular representation of G or any other complex unitary representation of G that contains all irreducible representations of G as summands. Then the mapping given in Proposition* 6.4

$$\omega : [\mathbb{S}^K, \mathbb{S}^K]^*_G = [S(K \oplus \mathbb{R}), S(K \oplus \mathbb{R})]^*_G \longrightarrow A(G)$$

yields an epimorphism. Consequently $[\mathbb{S}^{K+1}, \mathbb{S}^{K+1}]^*_G \cong A(G)$, *and thus also* $\widetilde{\pi}^{\mathrm{st}\,0}_G(*) \cong A(G)$.

Proof. We apply Lemma 6.5. Since the sum formulae of Theorems 5.5 and 5.6 are additive with respect to the addition in $[\mathbb{S}^K, \mathbb{S}^K]_G$, at least after one suspension, it is enough to construct, for a fixed (H), a conormal map $\psi : V \longrightarrow K$, $V \subset K$ open G-invariant, with only one fixed orbit $(G/H) \approx Gx \subset V_{(H)}$ and such that $I^u(\psi, V) = \pm|G/H|$. Note that by the localization property, we may assume that $V = G \times_H D_\varepsilon(x)$, where $D_\varepsilon(x)$ is a small disk around x considered as an $H = G_x$-space. On the other hand, the G-maps from $G \times_H D_\varepsilon(x)$ to V are in one–one correspondence with H-maps from $D_\varepsilon(x)$ to V (cf. [**4**, **6**]). Consequently our task is to find an H-map $\widetilde{\psi} : D_\varepsilon(x) \longrightarrow V$ with $\mathrm{Fix}(\widetilde{\psi}) = \{x\}$ and $I^u(\widetilde{\psi}) = \pm 1$. To that end, take the H-equivariant projection $p_H : D_\varepsilon(x) \longrightarrow D_\varepsilon(x)^H$ and compose it with any map $\overline{\psi} : D_\varepsilon(x)^H \longrightarrow D_\varepsilon(x)^H$ such that $\mathrm{Fix}(\overline{\psi}) = \{x\}$ and $I^u(\overline{\psi}, D_\varepsilon(x)^H) = \pm 1$. This map $\widetilde{\psi} = \overline{\psi} \circ p_H$ is the required map and consequently also provides ψ. □

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