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Computation of the equivariant 1-stem

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Abstract

Let G be a compact Lie group. In this paper, combining a short exact sequence obtained by Balanov and Krawcewicz with some additional topological techniques, we complete the computation of the first equivariant stem $\pi_1^{G\,\mathrm{st}}$. Using the exact sequence and a property of nonabelian connected compact Lie groups, whose proof was suggested to us by R. Oliver, we show that this group is finite if and only if G is finite.

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1. Introduction

A description of the homotopy classes, or of the stable homotopy classes of maps between two topological spaces has been a classical question in topology. Particularly, the stable homotopy classes of (pointed) maps between spheres, namely the so-called stable stems, π_*^{st}, have been important objects to study. Historically, via the Brouwer degree theory, the

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0-stem was computed, namely $\pi_0^{\mathrm{st}} \cong \mathbb{Z}$. The Hopf map and the Pontryagin theorem provided $\pi_1^{\mathrm{st}} \cong \mathbb{Z}_2$. Let us mention that the Brouwer degree is the main tool in the Krasnoselsky (local) and P. Rabinowitz (global) bifurcation theorems. Moreover, the nontriviality of the H. Hopf map is the topological ingredient of the E. Hopf theorem on the bifurcation of periodic solutions.

A variant of the question arises when we assume that a compact Lie group G acts on all spaces involved and that all the maps considered commute with the group action, namely, that the maps are G-equivariant, or G-maps for short. Then the corresponding question is to provide a description of the stable G-homotopy classes between G-spaces. Especially, the stable homotopy classes of maps between unit spheres of orthogonal representations pose an important question. It is quite easy to show that the negative G-stems are zero, that is $\pi_k^{G\,\mathrm{st}} = 0$ if $k < 0$. In 1970, Segal [26], stated that for any finite group G, $\pi_0^{G\,\mathrm{st}} \cong A(G)$, where $A(G)$ is the Burnside ring of G. This result was proved by Kosniowski [18], and independently by Rubinsztein [24] with a gap that was filled later by Dancer [6]. Tom Dieck [8] proved the same result for a general compact Lie group G, giving a convenient definition of the Burnside ring $A(G)$ for this case.

The groups $\pi_k^{G\,\mathrm{st}}$, $k > 0$, have been studied intensively by people working on nonlinear analysis. First, tackling the question about the multiplicity of periodic solutions of a nonlinear problem, one had to study $\mathbb{S}^1$-equivariant maps. Second, they provide very interesting applications to problems on bifurcations with symmetries (see [13]). Ize et al. have made many computations of $\pi_*^{G\,\mathrm{st}}$ when G is abelian [13,17]. We should emphasize that for the applications in nonlinear analysis, not only the form of $\pi_k^{G\,\mathrm{st}}$ is of importance, but also a knowledge of which element of this group corresponds to a given (unstable) map $S(V \oplus \mathbb{R}^k) \longrightarrow S(V)$.

Balanov and Krawcewicz [1] showed for a general compact Lie group G that there is a direct sum decomposition

$$\pi_k^{G\,\mathrm{st}} \cong \bigoplus_{(H)} \Pi_k(H), \tag{1.1}$$

where $\Pi_k(H)$ denotes the subgroup of $\pi_k^{G\,\mathrm{st}}$ corresponding to the isotropy type (H), for a subgroup $H \subset G$; the sum ranks over all (H) such that $\dim W(H) \leqslant k$. Here $W(H) = NH/H$ is the Weyl group of H. Moreover, this splitting is in the unstable range (see [21] for an alternative proof of this fact), unlike that given in [19, V.9.1]. Following computations made in [10], where a construction of the equivariant degree is given, one obtains that if $\dim W(H) = k$, then $\Pi_k(H) \cong \mathbb{Z}$ or $\mathbb{Z}_2$, depending on whether $W(H)$ is biorientable or not. On the other hand, in the treatment of $\pi_1^{G\,\mathrm{st}}$ made in [1] it was shown that for $\Pi_1(H) \subset \pi_1^{G\,\mathrm{st}}$, with $\dim W(H) = 0$, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Pi_1(H) \longrightarrow W(H)_{\mathrm{ab}} \longrightarrow 0, \tag{1.2}$$

where $W(H)_{\mathrm{ab}}$ denotes the abelianization of the Weyl group. In [2], Balanov, Krawcewicz, and Steinlein, using results of Ize and purely algebraic arguments, proved that this sequence splits when G is abelian.

Cruickshank [5] has also considered stable equivariant homotopy groups of spheres. One should beware, however, that his concept of equivariant 1-stem differs from that of our first equivariant stem.

In the first part of the paper we give another geometrical interpretation of the kernel in the short exact sequence (1.2) and then show that the sequence always splits (Theorem 2.19). This, together with well-known facts, leads to a complete description of $\pi_1^{G\,\mathrm{st}}$ for any compact Lie group G in Theorem 2.7. It is worth to point out that this theorem works in the unstable range, provided that the representation fulfills some conditions (see Proposition 2.18).

After analyzing some examples in Section 3, we prove that the first G-stem is finite if and only if G is finite; otherwise, the first G-stem is not even finitely generated. This will be a consequence of the decomposition (1.1), following a suggestion of Bob Oliver. Note that one can deduce this result also from the short exact sequence (1.2).

Finally, we would like to remark that, despite this sort of splitting theorems have been studied for years, for instance, by Kosniowski [18], tom Dieck [7], and Hauschild [11], who have proved results in this direction in the seventies as well as by Lewis, Jr., May, and McClure, who proved a general result in [19, V.10.1] in 1986, we did not find in the literature any description of $\pi_1^{G\,\mathrm{st}}$ as the one given in Theorem 2.7 of the next section. We believe that this explicit description might be applied to bifurcation problems with symmetry as were studied by Chossat and coll. in [4].

We wish to thank W. Krawcewicz, who, after reading a preliminary version of this paper, pointed out a mistake in the proof of Proposition 2.18. We also thank Bob Oliver for giving us the proof of Proposition 2.18.

2. Computation of the first G-stem

In this section, by showing that the short exact sequence (1.2) always splits, we compute the first equivariant stem for any compact Lie group G.

Given any orthogonal representation V of G, $\mathbb{S}^V$ will denote the one-point compactification of V with the induced G-action.

Definition 2.1. We define the kth *equivariant stem for a compact Lie group G* or briefly the kth *G-stem*, $k = 0, 1, 2, \ldots,$ by

$$\pi_k^{G\,\mathrm{st}} = \operatorname*{colim}_V\, [\mathbb{S}^{V+k}, \mathbb{S}^V]_G,$$

where V varies along a cofinal set of orthogonal G-representations and $[-,-]_G$ denotes the set of pointed G-homotopy classes of pointed G-maps. Of course, $V+k$ denotes the orthogonal representation $V \oplus \mathbb{R}^k$ with G acting trivially in the second summand.

Remark 2.2. For the concept of a colimit in general, we may refer the reader to Mac Lane's book on categories [20]. Observe, anyway, that the elements of $\pi_k^{G\,\mathrm{st}}$ can be represented by maps of pairs

$$\alpha : (V \times \mathbb{R}^k, V \times \mathbb{R}^k - 0) \longrightarrow (V, V - 0)$$

for some orthogonal representation V of G (see also [23]).

Remark 2.3. In the language of modern algebraic topology, $\pi_k^{G\,\mathrm{st}}$ can be considered as the kth homotopy group of the infinite loop space $Q_G = \Omega_G^\infty \mathbb{S}^\infty = \mathrm{colim}_V\, \Omega^V \mathbb{S}^V$, where $\Omega^V \mathbb{S}^V = \mathrm{Maps}_G(\mathbb{S}^V, \mathbb{S}^V)$. This follows from the adjunction $[\mathbb{S}^V X, \mathbb{S}^W Y] \cong [X, \Omega^V \mathbb{S}^V Y]$. For the case of nonlinear analysis, it is more convenient to use the definition that we gave above. For instance, in the Schauder approximation of a map of the form $L + \varphi$, L linear Fredholm and φ completely continuous, we are lead in a natural way to the form of our previous definition.

Let $W(H)$ denote the *Weyl group* of $H \subset G$, defined by $W(H) = NH/H$, where $NH \subset G$ is the normalizer of H in G.

For the kth G-stem, one has the following decomposition formula derived using an equivariant transversality argument in [1, 2.8]; namely

$$\pi_k^{G\,\mathrm{st}} \cong \bigoplus_{\substack{(H)\in \mathrm{Or}_G \\ \dim W(H) \leqslant k}} \Pi_k(H). \tag{2.4}$$

Recall that a compact Lie group Γ is said to be *biorientable* if it has an orientation invariant under left and right translations (see [1,10], or [22]). From considerations in [10] (see also [22]) the following can be proved:

Proposition 2.5. *Let* $\dim W(H) = k$. *Then*

$$\Pi_k(H) \cong \begin{cases} \mathbb{Z} & \textit{if } W(H) \textit{ is biorientable}, \\ \mathbb{Z}_2 & \textit{otherwise.} \end{cases}$$

Note 2.6. For instance, a compact Lie group Γ is biorientable if it is either finite, abelian, or connected (cf. [10]). The simplest nonbiorientable group (of dimension 1) is $O(2)$.

In what follows, by showing that the short exact sequence (1.2) always splits, we shall compute the subgroups $\Pi_1(H)$ of the first G-stem to obtain a full description of it. Combining this with Proposition 2.5 and the decomposition (2.4), we shall obtain the main result of this section as follows.

Theorem 2.7. *There is a sum decomposition of the first G-stem*

$$\pi_1^{G\,\mathrm{st}} = \bigoplus_{\substack{(H)\in \mathrm{Or}_G \\ \dim W(H) \leqslant 1}} \Pi_1(H).$$

Here, if $\dim W(H) = 0$,

$$\Pi_1(H) \cong \mathbb{Z}_2 \oplus W(H)_{\mathrm{ab}}, \tag{2.8}$$

where $W(H)_{\mathrm{ab}}$ *is the abelianization of* $W(H)$, *and, if* $\dim W(H) = 1$,

$$\Pi_1(H) \cong \begin{cases} \mathbb{Z} & W(H) \textit{ is biorientable}, \\ \mathbb{Z}_2 & \textit{if } W(H) \textit{ is not biorientable}. \end{cases} \tag{2.9}$$

In view of Proposition 2.5, we only need to prove Eq. (2.8). For doing this, we shall make some general considerations.

Assumption 2.10. V denotes a G-module and the elements in $\pi_k^{G\ \mathrm{st}}$ are represented by maps $(V\times\mathbb{R}^{l+k}, V\times\mathbb{R}^{l+k}-0)\longrightarrow(V\times\mathbb{R}^l, V\times\mathbb{R}^l-0)$, with $l\geqslant k+3$, where G acts trivially on the second factor.

In the rest of the paper, we denote the Weyl group $W(H)$ of $H\subset G$ by Γ_H, or simply by Γ when there is no danger of confusion. Note that Γ acts effectively on V^H. We denote by U the representation $V^H\times\mathbb{R}^{l+k}$ of Γ, with the obvious action, and by U_0 the representation $V^H\times\mathbb{R}^l$. Let (P) be the principal orbit type of the action of Γ on U, and let $U_P=U-S$, where S consists of all points in U with isotropy group type different from (P) (see [8]).

Note 2.11.

1. The set U_P is in general disconnected; however, it is connected, provided that $\dim(U-U_P)\leqslant\dim U-2$. This holds if $\dim U^{\Gamma'}\leqslant\dim U-2$ for any $(\Gamma')>(P)$, and this can always be attained in the stable range. For this, it is enough to replace V by $V\oplus V$.
2. Even being U_P connected, it need not be simply connected. By Lefschetz duality, U_P will be simply connected if $\dim(U-U_P)\leqslant\dim U-3$. This holds if $\dim U^{\Gamma'}\leqslant\dim U-3$ for any $(\Gamma')>(P)$. For this, it is enough to replace V by $V\oplus V\oplus V$.
3. Increasing the size of V further (summing again with itself) we may also assume that U_P has an orientation-preserving Γ-action.

Denote by $\Omega_k^{\Gamma\ \mathrm{fr}}(U_P)$ the group of bordism classes of Γ-framed k-submanifolds of U_P. For the definition and more details about the equivariant bordism, refer to [1]. One has the following result of Balanov and Krawcewicz.

Proposition 2.12 (*Balanov and Krawcewicz [1, 3.2]*). *Let* $\dim\Gamma\leqslant k$. *Then* $\Pi_k(H)\cong\Omega_k^{\Gamma\ \mathrm{fr}}(U_P)$.

To focus on the proof of Eq. (2.8), assume in what follows that $\dim\Gamma=0$; that is, Γ is a finite group. Note that Γ acts effectively on U, but since Γ is finite, the principal orbit type corresponds to trivial isotropy, i.e., the action of Γ on U_P is in fact free.

Let $\overline{U}_P$ denote the quotient space U_P/Γ. There is a homomorphism $\Phi_k:\Omega_k^{\Gamma\ \mathrm{fr}}(U_P)\longrightarrow\Omega_k(\overline{U}_P)$, where Ω_k denotes the usual oriented bordism of k-submanifolds. The image of the canonical homomorphism $\Omega_k^{\Gamma\ \mathrm{fr}}(U_P)\longrightarrow\Omega_k^{\mathrm{fr}}(U_P)$ lies inside $\Omega_k^{\mathrm{fr}}(U_P)^{\Gamma}$, where $\Omega_k^{\mathrm{fr}}(U_P)$ has the action of Γ induced by that on U_P. Consequently, if $[M,\eta]\in\Omega_k^{\mathrm{fr}\ \Gamma}(U_P)$, then M is a framed Γ-submanifold of U_P (and η is a Γ-trivialization of the normal bundle), and thus Γ acts freely on M and $\overline{M}=M/\Gamma$ is an oriented submanifold of $\overline{U}_P$. By the Steenrod–Thom theorem, we know that there is a homomorphism $\Omega_k(\overline{U}_P)\longrightarrow H_k(\overline{U}_P;\mathbb{Z})$ that is an isomorphism for $k\leqslant 3$ and an epimorphism for $k=4$ (see [25]). In the case $k=1$, that we are concerned with, we thus have an isomorphism.

An essential step in deriving $\Pi_1(H)$ when $\dim\Gamma=0$ was done in [1, 4.3], where details on the previous comments can be seen; namely we have the following.

Theorem 2.13. $\ker \Phi \cong \mathbb{Z}_2$ *and thus one has a short exact sequence*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \xrightarrow{\Phi} H_1(\overline{U}_P; \mathbb{Z}) \longrightarrow 0. \tag{2.14}$$

Moreover, $\ker \Phi$ *consists of those bordism classes of G-framed invariant manifolds* $[M, \eta] \in \Omega_1^{\Gamma\ \mathrm{fr}}(U_P)$, *where* $M \approx_{\mathrm{diff}} \mathbb{S}^1$ *and* η *is an equivariant trivialization of the normal bundle such that the quotient manifold* $\overline{M} = M/\Gamma \subset \overline{U}_P$, $\overline{M} \approx \mathbb{S}^1$, *is nullbordant.*

In [2, 2.5], it is shown that if G is abelian, then the sequence (2.14) splits. Their argument is purely algebraic and makes use of the computation in [14] of $\Pi_1(H)$ as a product of p-factors (see also [17]), p prime. We show in what follows that (2.14) *always* splits.

Note 2.15. There is an isomorphism

$$\Omega_*^{\Gamma\ \mathrm{fr}}(\Gamma x) = \Omega_*^{\Gamma\ \mathrm{fr}}(\Gamma) \cong \Omega_*^{\mathrm{fr}}(*), \tag{2.16}$$

that is a consequence of the following well-known fact (see [8]). Namely, there is a bijection $[\mathbb{S}^V \wedge X, \mathbb{S}^V \wedge Y \wedge \Gamma^+]_\Gamma \cong [\mathbb{S}^V \wedge X, \mathbb{S}^V \wedge Y]$, that provides the isomorphism (2.16), since the homology theory $\Omega_*^{\Gamma\ \mathrm{fr}}$ is equivalent to the theory $\pi_*^{\Gamma\ \mathrm{st}}$. In particular,

$$\Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma x) \cong \Omega_1^{\mathrm{fr}}(*) \cong \mathbb{Z}_2.$$

Here X and Y represent topological spaces with some distinguished base point, and the $+$-sign means adding an isolated point as a base point.

Lemma 2.17. *Take* $x \in U_P$. *If* $i_x^\Gamma : \Gamma \cong \Gamma x \hookrightarrow U_p$ *is the inclusion and* $i_* = i_{x\,*}^\Gamma : \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma) \longrightarrow \Omega_1^{\Gamma\ \mathrm{fr}}(U_P)$ *is the induced homomorphism, then*

$$\ker \Phi = \mathrm{im}(i_*).$$

Proof. Recall first that $d = \dim U_P = \dim \overline{U}_P \geqslant 3$ (see Assumption 2.10), and assume that we have a metric on U_P that is Γ-invariant and take $\varepsilon > 0$ sufficiently small, that $\pi^{-1}(D_\varepsilon(\overline{x})) = \bigsqcup_{\gamma \in \Gamma} \gamma D_\varepsilon(x) \approx \Gamma \times D_\varepsilon(x)$, where D_ε denotes the corresponding d-balls of radius ε, and let $\overline{M}$ be the boundary $\partial D^2_{\varepsilon/2}(\overline{x}) \subset \overline{U}_P$ of a 2-disk of radius $\varepsilon/2$ contained in $D_\varepsilon(\overline{x})$. Hence $\overline{M}$ is diffeomorphic to $\mathbb{S}^1$.

Let $\overline{\eta}_0, \overline{\eta}_1 : \nu(\overline{M}) \longrightarrow \overline{M} \times \mathbb{R}^{d-1}$ be trivializations of the normal bundle of $\overline{M}$ such that $[\overline{M}, \overline{\eta}_0] = 0 \in \Omega_1^{\mathrm{fr}}(D_\varepsilon(\overline{x}))$ and $[\overline{M}, \overline{\eta}_1] \neq 0 \in \Omega_1^{\mathrm{fr}}(D_\varepsilon(\overline{x}))$. Let $j_x : D_\varepsilon(\overline{x}) \longrightarrow D_\varepsilon(x)$ be the inverse diffeomorphism to that induced by π, and call $M_x = j_x(\overline{M})$. Define $M = \bigsqcup_{\gamma \in \Gamma} \gamma M_x \subset U_P$. M is homeomorphic to $\Gamma \times \overline{M}$.

Note that $\nu(M_x) \subset D_\varepsilon(x)$ is diffeomorphic to $\nu(\overline{M})$ via the mapping $(m, v) \mapsto (\pi(m), D\pi(m)v)$. On the other hand,

$$\gamma M_x \subset \gamma D_\varepsilon(x) = D_\varepsilon(\gamma x) \quad \text{and} \quad \nu(\gamma M_x) = \gamma(\nu(M_x)),$$

since γ induces a diffeomorphism, because it is a linear orthogonal map. Consequently, the tubular neighborhood

$$\nu(M) = \bigsqcup_{\gamma\in\Gamma} \gamma(\nu(M_x))$$

and thus we can define an equivariant trivialization $\eta_i : \nu(M) \longrightarrow M \times U_0$, $i = 0, 1$, by

$$\eta_i(\gamma m, \gamma v) = (\gamma m, \gamma\overline{\eta}_i(\pi(m), D\pi(m)v))$$

for $m \in M_x$ and $v \in \nu_m(M_x)$. Observe that η_i is equivariant, since for $\mu \in \Gamma$ we have

$$\begin{aligned}\eta_i(\mu(\gamma m, \gamma v)) &= \eta_i((\mu\gamma)m, (\mu\gamma)v)\\ &= ((\mu\gamma)m, (\mu\gamma)(\overline{\eta}_i(\pi(m), D\pi(m)v)))\\ &= (\mu(\gamma m), \mu(\gamma\overline{\eta}_i(\pi(m), D\pi(m)v)))\\ &= \mu\eta_i(\gamma m, \gamma v).\end{aligned}$$

Hence we get that $[M, \eta_0], [M, \eta_1] \in \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma \times D_\varepsilon(x)) \subset \Omega_1^{\Gamma\ \mathrm{fr}}(U_P)$. Consequently, $\overline{M} = M/\Gamma$ is nullbordant, thus implying that $[M, \eta_0], [M, \eta_1] \in \ker \Phi$. By construction, they lie in $\mathrm{im}(i_*) = \mathrm{im}(i^\Gamma_{D_\varepsilon *})$, where $i^\Gamma_{D_\varepsilon} : \Gamma \times D_\varepsilon \hookrightarrow U_P$, and obviously, $[M, \eta_1] \neq 0$ in $\Omega_1^{\Gamma\ \mathrm{fr}}(U_P)$. $\square$

Proposition 2.18. *If Γ is finite and U_P is connected, then*

$$\Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \cong \mathbb{Z}_2 \oplus H_1(\overline{U}_P; \mathbb{Z}).$$

If, moreover, U_P is simply connected, then

$$\Pi_1(H) \cong \mathbb{Z}_2 \oplus \Gamma_{\mathrm{ab}}.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma) \arrow[r, "i_*"] \arrow[d, "\alpha"', "\cong"] & \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \arrow[d, "\beta"] \\ \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma/\Gamma) \arrow[r, "i_*"'] & \Omega_1^{\mathrm{fr}}(U_P/\Gamma), \end{tikzcd}$$

where the vertical homomorphisms divide out the action of Γ. By the note above, α is an isomorphism, and both groups on the left are $\Omega_1^{\mathrm{fr}}(*) \cong \mathbb{Z}_2$. The homomorphism β is well defined, since the action of Γ on U_P is orientation-preserving and thus dividing out this action preserves the trivialization of the normal bundles and thus sends Γ-framed manifolds to framed manifolds. Hence, the diagram is equivalent to

$$\begin{tikzcd} \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma) \arrow[r, "i_*"] \arrow[d, "\alpha"', "\cong"] & \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \arrow[d, "\beta"] \\ \Omega_1^{\mathrm{fr}}(*) \arrow[r, "i_*"'] & \Omega_1^{\mathrm{fr}}(U_P/\Gamma). \end{tikzcd}$$

In order to produce a splitting of the top i_* we consider $\sigma' : \Omega_1^{\mathrm{fr}}(U_P/\Gamma) \longrightarrow \Omega_1^{\mathrm{fr}}(*)$ on the bottom, given by the obvious map $U_P/\Gamma \longrightarrow *$. This obviously splits the bottom row and thus $\sigma = \alpha^{-1} \circ \sigma' \circ \beta : \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \longrightarrow \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma)$ is well defined and provides the desired splitting. $\square$

Therefore, we have the following.

Theorem 2.19. *The short exact sequence*

$$0 \longrightarrow \Omega_1^{\Gamma\ \mathrm{fr}}(\Gamma x) \underset{i_*}{\overset{\sigma}{\rightleftarrows}} \Omega_1^{\Gamma\ \mathrm{fr}}(U_P) \xrightarrow{\Phi} H_1(\overline{U}_P;\mathbb{Z}) \longrightarrow 0.$$

splits.

Combining 2.12 and 2.13 with the previous theorem, we obtain our main Theorem 2.7.

3. Some applications of the decomposition theorem for the first *G*-stem

We start this section with a brief discussion of examples of the Decomposition Theorem 2.7 beginning with the simplest groups. We do this for the convenience of the reader, since they are all spread in the literature, mostly written in rather different ways.

Examples 3.1.

1. Let $G = \mathbf{1}$ be the trivial group. Then there is only one $H \subset G$ and $W(H) = G/H = \mathbf{1}$ has dimension 0. Thus $\pi_1^{\mathrm{st}} = \mathbb{Z}_2$.
2. Historically, the first case of $\pi_1^{G\ \mathrm{st}}$ described was for $G = \mathbb{S}^1$, when

$$\pi_1^{\mathbb{S}^1\ \mathrm{st}} \cong \mathbb{Z}_2 \oplus \bigoplus_{H \subset \mathbb{S}^1} \mathbb{Z}$$

and was given this way by Dylawerski [9].
3. Let G be a finite group. Then for every $H \subset G$, $\dim W(H) = 0$. Thus

$$\pi_1^{G\ \mathrm{st}} \cong \bigoplus_{(H) \in \mathrm{Or}(G)} (\mathbb{Z}_2 \oplus W(H)_{\mathrm{ab}}). \tag{3.2}$$

If G is abelian, then $W(H) = G/H$ and thus

$$\pi_1^{G\ \mathrm{st}} \cong \bigoplus_{H \subset G} (\mathbb{Z}_2 \oplus G/H). \tag{3.3}$$

Particular cases are $G = \mathbb{Z}_p$, where p is prime. Then

$$\pi_1^{\mathbb{Z}_p\ \mathrm{st}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p.$$

Description (3.3) agrees with the decomposition in terms of prime factors of G given by Ize and Vignoli [14–17].

4. Let G be either $O(2)$ or $SO(3)$. Then G has infinitely many conjugacy classes of closed subgroups H such that $W(H)$ is finite (see [3, IV.(4.10) Ex.9]). Thus $\pi_1^{G\,\mathrm{st}}$ has infinitely many $\mathbb{Z}_2$-summands and for each of them also a $W(H)_{\mathrm{ab}}$-summand (see [2] for further details on the case $G = SO(3)$).
5. Let $G = O(k)$. Then G has infinitely many finite conjugacy classes of subgroups H generated by reflections such that $W(H)$ is finite (see [3, V.(2.19) Ex.6]). Thus, as in the previous example, $\pi_1^{O(k)\,\mathrm{st}}$ has infinitely many $\mathbb{Z}_2$-summands and for each of them also a $W(H)_{\mathrm{ab}}$-summand.

Examples 2, 3, and 4 above show infinite compact Lie groups G, for which $\pi_1^{G\,\mathrm{st}}$ is also an infinite—and quite complicated—group. This is true in general, as we shall prove below. We have the following.

Theorem 3.4. *Let G be a compact Lie group. Then $\pi_1^{G\,\mathrm{st}}$ is finite if and only if G is finite.*

Remark 3.5. In fact, we prove that if G is not finite, then $\pi_1^{G\,\mathrm{st}}$ is not even finitely generated; i.e., $\pi_1^{G\,\mathrm{st}}$ is either finite or infinitely generated.

The proof of the theorem requires to construct an infinite collection of nonconjugate subgroups of G, such that their Weyl groups have dimension 0 or 1. In the first case, each of them contributes with at least a $\mathbb{Z}_2$-summand; while in the second, with a $\mathbb{Z}$- or $\mathbb{Z}_2$-summand, according to Theorem 2.7.

Before passing to the proof of the theorem, we shall state some results in this direction. The following is an immediate consequence of the structure theorem on compact abelian Lie groups and of the fact that the sequence of sets of roots of unity of growing order m builds up such a collection for the circle (see, for instance [17]).

Lemma 3.6. *Let G be an infinite compact abelian Lie group. Then the groups $\mathbb{Z}_m$, $m \in \mathbb{N}$, constitute an infinite sequence of nonconjugate subgroups of G that satisfy* $\dim W(\mathbb{Z}_m) = 1$.

The next result, whose proof was suggested to us by Bob Oliver, is the main result of this section. It is, clearly, the nonabelian counterpart to Lemma 3.6.

Proposition 3.7. *Let G be a nonabelian connected compact Lie group. Then there exists an infinite sequence of nonconjugate subgroups $\{H_m\}$ such that $W(H_m)$ is finite.*

Proof. We start with the further assumption that G is semisimple.

Let T be the maximal torus and let $k = |W(T)|$ be the order of the Weyl group. For simplicity denote $W(T)$ by W and $N(T)$ by N. We have $1 < k < \infty$. Let $T_m = \{g \in T \mid g^m = 1\}$, and let $\varphi = \varphi_k : T \longrightarrow T$ be given by $\varphi(g) = g^k$. Since the extension $T \hookrightarrow N \twoheadrightarrow W$ represents an element in the cohomology group $H^2(W; T)$, and since this group is anihilated by multiplication by k, then there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T & \longrightarrow & N & \longrightarrow & W & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle 1_W} & & \\
1 & \longrightarrow & T & \longrightarrow & T \rtimes W & \longrightarrow & W & \longrightarrow & 1,
\end{array}$$

where $T \rtimes W$ is the semidirect product given by the canonical action ρ of W on T by conjugation; that is, $\rho(q(y))(t) = yty^{-1}$ ($q : N \twoheadrightarrow W$ the quotient map, and $(t, w) \star (t', w') = (t\rho(w)(t'), ww')$). Moreover, $\pi(y) = (t, q(h))$ if $y = ht \in N$, $t \in T$. Recall that $t \mapsto (t, 1)$ embeds T as a subgroup of $T \rtimes W$, and $(t, w) \mapsto w$ is a quotient map of groups. (Remind that the bottom row with the semidirect product represents the 0 element in $H^2(W; T)$, see [12] for this and all related topics.)

Put $H_k = \pi^{-1}(W)$. This group has the following properties:

- $H_k \cap T = T_k$,
- $H_k \hookrightarrow N \twoheadrightarrow W$ is surjective.

Finally take for all $m \in k\mathbb{Z}$, $H_m = H_k \cdot T_m \subseteq N$. Then we have

- $H_m \cap T = T_m$.
- $T_m \subseteq H_m$ is a subgroup such that $H_m / T_m \cong H_k / T_k \cong W$.

Thus it follows $|H_m| = |T_m||W| = c(m)$. We have

$$m \neq m' \Longrightarrow c(m) \neq c(m'), \quad \text{thus} \quad m \neq m' \Longrightarrow (H_m) \neq (H'_m).$$

If G *is semisimple* and connected, then $Z(G)$ is finite (see [3, Chapter V.(3.13),(3.14)]). Thus

(a) T^W is finite, since $T^W = Z(G)$; and also
(b) the centralizer $C_G(T)$ is an abelian subgroup that contains T. Thus $C_G(T) = T$.

To see (a), observe that if $t \in T^W$, then, by [3, (2.6)], for all $g \in G$, $t = gtg^{-1}$, hence $t \in Z(G)$. So $T^W \subset Z(G)$. Conversely, since $Z(G) \subset T$, and since $Z(G)$ remains pointwise fixed under conjugation by any element, we have $Z(G) \subset T^W$.

Since $\bigcup_m T_m$ is dense in T, we have $C_G(\bigcup_m T_m) = T$. Hence

$$T_m \subset T_{m'} \implies C_G(T_m) \supset C_G(T_{m'})$$

and $\{C_G(T_m)\}$ is a decreasing sequence of subgroups. It has to be stationary and so there is an l with

$$C_G(T_l) = T.$$

We may assume that $k|l$. Then $C_G(H_l) \subset T^W$. Namely, observe first that $C_G(H_l) \subset C_G(T_l) = T$. Take $g \in C_G(H_l)$ and $\beta = \alpha t \in N(T)$, where $t \in T$ and $\alpha \in H_k$, which can be done by the choice of H_k . Then $(\alpha t)g(\alpha t)^{-1} = \alpha tgt^{-1}\alpha^{-1} = \alpha g\alpha^{-1} = g$, because $C_G(H_l) \subset C_G(H_k)$.

Since T^W is finite, so is $C_G(H_l)$, and since $N(H_l)/C_G(H_l)$ acts effectively on H_l, the quotient group is a subgroup of the symmetric group $\Sigma_{|H_l|}$, thus finite. This shows that also the normalizer $N(H_l)$ in G is finite, and with it $N(H_m)$ for all m divisible by l are also finite.

Thus $\{H_m \mid m \in l\mathbb{Z}\}$ is an infinite family of finite subgroups of G, no two conjugate (since they have different order), all of which have finite normalizers. Hence, all Weyl groups $W(H_m)$ are finite.

Finally, if G is connected but *not semisimple*, then the quotient group $\overline{G} = G/Z(G)$ is semisimple. As above, take $\overline{H}_1, \overline{H}_2, \ldots$ a family of nonconjugate (of different order) subgroups of $\overline{G}$ such that the normalizers $N_{\overline{G}}(\overline{H}_m)$ are finite for all m.

Let H_m be the inverse image of $\overline{H}_m$ in G under the quotient homomorphism. Then $H_m \cong Z(G) \times \overline{H}_m$ and $N_G(H_m) \cong Z(G) \times N_{\overline{G}}(\overline{H}_m)$. Thus the Weyl groups $W_G(H_m) = N_G(H_m)/H_i \cong N_{\overline{G}}(\overline{H}_m)$ are finite for all m. Clearly these groups H_m are not conjugate to each other, since otherwise the groups $\overline{H}_m$ would be conjugate to each other. □

Lemma 3.8. *Let G be a compact nonabelian Lie group and consider its extension $G_0 \hookrightarrow G \twoheadrightarrow \Gamma = G/G_0$, where G_0 is the connected component of the unit element. If there exists an infinite sequence of subgroups $\{H_m^0\}$, $H_m^0 \subset G_0$, such that* $\dim W(H_m^0) = 0$, *and such that they are not conjugate in G_0, then there exists an infinite sequence of subgroups $\{H_m\}$, nonconjugate in G, such that* $\dim W(H_m) = 0$.

Proof. Define $H_m = H_m^0 \subset G_0 \subset G$. We shall show that among the members of this family there are infinitely many conjugacy classes. Observe first that the relation to be conjugate in G_0 is finer than that of being conjugate in G. Moreover, each conjugacy class in G is a union of at most $|\Gamma|$ conjugacy classes in G_0. Thus, if we pick an element in each conjugacy class in G of the family $\{H_m\}$, we still obtain an infinite subfamily of nonconjugate subgroups.

According to the above, it is now enough to show that if $W_0(H_m) = N_0(H_m)/H_m$ is finite, then $W(H) = N(H)/H$ is also finite, where $N_0(H_m)$ is the normalizer of H_m in G_0. The proof of this fact is straightforward. □

Proof of the theorem. If G is finite, then clearly, by Theorem 2.7, $\pi_1^{G\,\mathrm{st}}$ is finite. Thus we assume that G is infinite and consider two cases.

Case 1: G_0 is nonabelian. Apply Proposition 3.7 and Lemma 3.8.

Case 2: G_0 is abelian. In this case, $G_0 = T$ is the torus. Then apply Lemmas 3.6 and 3.8

Thus we have in both cases that $\pi_1^{G\,\mathrm{st}}$ is infinitely generated. □

We obviously have the following consequence.

Corollary 3.9. *Let X be any finite G-CW-complex. Then $\pi_1^{G\,\mathrm{st}}(X)$ is finite if and only if G is finite.*

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