Topological groups and Mackey functors

Marcelo A. Aguilar & Carlos Prieto

Abstract Let M be a Mackey functor for a finite group G and let X be a pointed G-space. We define a topological group $\overline{F}^G(X,M)$, whose homotopy groups are isomorphic to the Bredon-Illman equivariant homology of X with coefficients in a coefficient system $\overline{M}_*$ associated to M. When M is a homological Mackey functor, we define another topological group $\mathbb{F}^G(X,M)$, whose homotopy groups are isomorphic to the Bredon-Illman equivariant homology of X with coefficients in the covariant part of M. These topological groups are defined using simplicial groups $\overline{F}^G(\mathcal{S}(X),M)$ and $F^G(\mathcal{S}(X),M)$, which have the same underlying groups, namely the groups of G-fixed points $F(\mathcal{S}_n(X),M)^G$, where $\mathcal{S}(X)$ is the singular simplicial set of X.

Furthermore, we study the transfer for finite covering G-maps and give its pullback property. We also analyze the composite of the transfer with the homomorphism induced by the projection map, in particular, in the case of (G,Γ)-bundles.

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1 Introduction

Let M be a Mackey functor for a finite group G and X a pointed G-space. In [2] we defined an abelian group $F^G(X,M)$ with a topology that made it into a topological group. This group is given as the geometric realization of a simplicial group $F^G(\mathcal{S}(X),M)$, where $\mathcal{S}(X)$ denotes the singular simplicial set of X. This simplicial group is a quotient of another simplicial group $F(\mathcal{S}(X),M)$, which has a simplicial action of G via isomorphisms. The nth group $F^G(\mathcal{S}(X),M)_n$ is the fixed-point subgroup $F(\mathcal{S}(X),M)_n^G$. We can also define another simplicial group, which is a simplicial subgroup of $F(\mathcal{S}(X),M)$, denoted by $\overline{F}^G(\mathcal{S}(X),M)$, whose nth group is also $F(\mathcal{S}(X),M)_n^G$.

Therefore, with the same groups of fixed points $F(\mathcal{S}(X),M)_n^G$ we have defined two different simplicial groups. Their geometric realizations, in turn, define two different topological groups $F^G(X,M)$ (as above) and $\overline{F}^G(X,M)$. In [2] we showed that the homotopy groups of $F^G(X,M)$ are isomorphic to the Bredon-Illman G-equivariant homology of X with coefficients in the covariant part of M. In this paper we show that the homotopy groups of $\overline{F}^G(X,M)$ are isomorphic to the Bredon-Illman G-equivariant homology of X with coefficients in a covariant coefficient system $\overline{M}_*$ associated to M.

In [2] we also introduced a continuous transfer $t_p^G : F^G(X,M) \longrightarrow F^G(E,M)$ for an n-fold covering G-map $p : E \longrightarrow X$. In this paper we prove that this transfer has the pullback property.

The elements of $F^G(X,M)$ are defined in terms of the singular simplexes of X. However, when M is a homological Mackey functor, we can define another topological abelian group $\mathbb{F}^G(X,M)$, whose elements are given directly in terms of the points of X. We prove that if X has the homotopy type of a G-CW-complex, then this group is homotopy equivalent to $F^G(X,M)$, and thus its homotopy groups also yield the same G-equivariant homology theory with coefficients in M. The homological Mackey functors are precisely those for which the composite of the transfer and the projection is given by the expected formula.

We also study the transfer for a class of covering G-maps, called (Γ, G)-bundles.

The paper is organized as follows. In Section 2, for any pointed G-set C, we recall the definition of the abelian group $F(C,M)$, which is indeed a functor on C. We show that G acts on this group by isomorphisms, and use it to define the subgroup $F(C,M)^G$ of G-fixed elements and the two different functorial structures on it. In Section 3, for any G-function $p : A \longrightarrow C$ with finite fibers, we define a transfer homomorphism $t_p^G : F(X,M)^G \longrightarrow F(E,M)^G$ and study its properties, especially the pullback property. In Section 4, if X is a pointed G-space, we define topological groups $F^G(X,M)$ and $\overline{F}^G(X,M)$ and we show that the functors $F(-,M)$ and $F^G(-,M)$ are characterized by certain universal properties. In Section 5, we construct a topological abelian group $\mathbb{F}^G(X,M)$, which has the abelian group $F^G(X^\delta,M)$ as underlying group, where X^δ denotes the underlying pointed G-set of X. We prove also a universal property that characterizes $\mathbb{F}^G(X,M)$ as a topological group. In Section 6, when $p : E \longrightarrow X$ is a covering G-map, we study the continuity of the transfers t_p^G for the groups $F^G(X,M)$ and $\mathbb{F}^G(X,M)$.

The main part of the paper is Section 7, where we prove that the homotopy groups of the (functorial) topological group $\overline{F}^G(X,M)$ are isomorphic to the (reduced) Bredon-Illman equivariant homology groups of X with coefficients in the coefficient system $\overline{M}_*$, given on orbits G/H by $\overline{M}_*(G/H) = M(G/H)$

and on quotient functions $q : G/H \longrightarrow G/K$ by $\overline{M}_*(q) = [K : H]M_*(q)$. We also prove that, if M is homological, the homotopy groups of $\mathbb{F}^G(X, M)$ realize the Bredon-Illman homology with coefficients in the covariant part M_* of M.

Finally, in Section 8 we study the transfers for some special examples of covering G-maps $p : E \longrightarrow X$, namely for (G, Γ)-bundles. We show that for a homological Mackey functor, the transfers have particularly nice properties.

The topological setting of this paper is the category of k-*spaces* (see e.g. [9, 11]).

2 The equivariant function-group functors

Throughout the paper G will denote a finite group and we shall write $H \subset G$ for a subgroup H of G. Let $G\text{-}\mathcal{S}\mathrm{et}_{\mathrm{fin}}$ denote the category of finite G-sets and G-equivariant functions (G*-functions*). Recall that a *Mackey functor* (see [4], for instance) consists of two functors, one covariant and one contravariant, both with the same object function $M : G\text{-}\mathcal{S}\mathrm{et}_{\mathrm{fin}} \longrightarrow \mathcal{A}\mathrm{b}$. If $\alpha : S \longrightarrow T$ is a G-function between G-sets, we denote the covariant part in morphisms by $M_*(\alpha) : M(S) \longrightarrow M(T)$ and the contravariant part by $M^*(\alpha) : M(T) \longrightarrow M(S)$. The functor has to be additive in the sense that the two embeddings $S \hookrightarrow S \sqcup T \hookleftarrow T$ into the disjoint union of G-sets define an isomorphism $M(S \sqcup T) \cong M(S) \oplus M(T)$ and if one has a pullback diagram of G-sets

$$\text{(2.1)} \qquad \begin{array}{ccc} U & \xrightarrow{\widetilde{\beta}} & S \\ {\scriptstyle\widetilde{\alpha}}\downarrow & & \downarrow{\scriptstyle\alpha} \\ T & \xrightarrow[\beta]{} & V\,, \end{array}$$

then

$$\text{(2.2)} \qquad M_*(\widetilde{\beta}) \circ M^*(\widetilde{\alpha}) = M^*(\alpha) \circ M_*(\beta)$$

(see [4] for details).

By the additivity property, the Mackey functor M is determined by its restriction $M : \mathcal{O}(G) \longrightarrow \mathcal{A}\mathrm{b}$, where $\mathcal{O}(G)$ is the full subcategory of G-orbits G/H, $H \subset G$. A particular role will be played by the G-function $R_{g^{-1}} : G/H \longrightarrow G/gHg^{-1}$, given by *right translation* by $g^{-1} \in G$, namely

$$R_{g^{-1}}(g'H) = g'Hg^{-1} = g'g^{-1}(gHg^{-1})\,.$$

We shall often denote the coset gH by $[g]_H$ or simply by $[g]$, if there is no danger of confusion. Observe that if C is a G-set and $x \in C$, then the canonical bijection $G/G_x \longrightarrow G/G_{gx}$ is precisely $R_{g^{-1}}$, where as usual G_x denotes the *isotropy subgroup* of x, namely the maximal subgroup of G that leaves x fixed.

Definition 2.3 Let M be a Mackey functor. Define the set $\widehat{M}$ as the union

$$\widehat{M} = \bigcup_{H \subset G} M(G/H) .$$

If C is any pointed G-set (where the base point x_0 is fixed under the action of G), then we define the set

$$F(C, M) = \{u : C \longrightarrow \widehat{M} \mid u(x) \in M(G/G_x), \;\; u(x_0) = 0, \;\text{ and }\; u(x) = 0$$
$$\text{for almost all } x \in C\} .$$

One may write the elements $u \in F(C, M)$ as $u = \sum_{x \in C} l_x x$, where $l_x = u(x) \in M(G/G_x)$ (the sum is obviously finite). $F(C, M)$ is again a G-set with the left action of G on $F(C, M)$ given by

$$(g \cdot u)(x) = M_*(R_{g^{-1}})(u(g^{-1}x)) .$$

For simplicity, if $l \in \widehat{M}$ and $g \in G$, we shall denote by gl the element $M_*(R_{g^{-1}})(l)$. Thus the action of G on $F(C, M)$ can be written as

$$g\left(\sum_x l_x x\right) = \sum_x (gl_x)(gx) = \sum_x (gl_{g^{-1}x})x .$$

The G-set $F(C, M)$ is indeed an abelian group with the sum $u + v$ for $u, v \in F(C, M)$ given by $(u + v)(x) = u(x) + v(x) \in M(G/G_x)$. We shall denote by $F(C, M)^G$ the subgroup of fixed points of $F(C, M)$ under the action of G.

In what follows, we shall define two functors from the category of arbitrary pointed G-sets $G\text{-}\mathcal{S}\mathrm{et}_*$ to the category of abelian groups $\mathcal{A}\mathrm{b}$

$$G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{F^G(-,M)} \mathcal{A}\mathrm{b} \qquad G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{\overline{F}^G(-,M)} \mathcal{A}\mathrm{b} .$$

These two functors have the same value on objects, namely

$$F^G(C, M) = \overline{F}^G(C, M) = F(C, M)^G ,$$

as defined above, but on morphisms, they are different. In order to define these functors on morphisms, we shall extend $F(C, M)$ to a functor $G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{A}\mathrm{b}$ as follows.

Let $\gamma_x : M(G/G_x) \longrightarrow F(C, M)$ be given by $\gamma_x(l) = lx$. Then we clearly have the following.

Proposition 2.4 *Let A be an abelian group and for each $x \in C$ let $\varphi_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi_{x_0} = 0$, where $x_0 \in X$ is the base point. Then there exists a unique homomorphism $\varphi : F(X, M) \longrightarrow A$ such that $\varphi \circ \gamma_x = \varphi_x$. In a diagram*

$$\begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x"] \arrow[dr, "\varphi_x"'] & F(C, M) \arrow[d, dotted, "\varphi"] \\ & A\,. \end{tikzcd}$$

□

The previous proposition allows us to define a covariant functor structure on $F(-, M)$ and the functor $\overline{F}(-, M)^G$.

Definition 2.5 For any G-function $f : C \longrightarrow D$, we shall denote by $\widehat{f_x} : G/G_x \longrightarrow G/G_{f(x)}$ the canonical quotient G-function. Let f be a pointed G-function. Define the family

$$f_x : M(G/G_x) \longrightarrow F(D, M) \quad \text{by} \quad f_x(l) = M_*(\widehat{f_x})(l)f(x)\,.$$

By Proposition (2.4) this family determines a homomorphism

$$f_* : F(C, M) \longrightarrow F(D, M)$$

given by

$$f_*\left(\sum_x l_x x\right) = \sum_x M_*(\widehat{f_x})(l_x)f(x)\,.$$

This turns $F(-, M)$ into a covariant functor. Moreover, since

$$gM_*(\widehat{f_x})(l) = M_*(\widehat{f_{gx}})(gl)\,,$$

f_* is G-equivariant, and so, by restriction, it defines a homomorphism

$$\overline{f}_*^G : F(C, M)^G \longrightarrow F(D, M)^G\,.$$

This defines the functor $\overline{F}^G(-, M)$.

REMARK 2.6 We denote by the category whose objects are abelian groups with a G-action by group isomorphisms, and whose morphisms are G-equivariant homomorphisms. Notice that the functor $F(-, M)$ is indeed a functor G-$\mathcal{S}\mathrm{et}_* \longrightarrow$.

To define the second covariant functor $F^G(-,M)$, take a pointed G-set C and consider the abelian group $F(C,M)^G$ once more. Let x_0 be the base point of the G-set C which remains fixed under the action of G and for each $x \in C$, let $\gamma_x^G : M(G/G_x) \longrightarrow F(C,M)^G$ be given by $\gamma_x^G(l) = \sum_{i=1}^n (g_i l)(g_i x)$, where $\{[g_1], \ldots [g_n]\} = G/G_x$. Then $\gamma_{x_0}^G = 0$ and $\gamma_x^G = \gamma_{gx}^G \circ M_*(R_{g^{-1}})$.

In order to define the functor $F^G(-M)$, we showed that the abelian group $F(X,M)^G$, together with the family $\{\gamma_x^G\}$, is characterized by the following property (see [2, 1.6]).

Proposition 2.7 *Let A be an abelian group and for each $x \in C$ let $\varphi'_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi'_{x_0} = 0$, where $x_0 \in C$ is the base point, and such that $\varphi'_x = \varphi'_{gx} \circ M_*(R_{g^{-1}})$. Then there exists a unique homomorphism $\varphi' : F(C,M)^G \longrightarrow A$ such that $\varphi' \circ \gamma_x^G = \varphi'_x$. In a diagram*

$$\begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x^G"] \arrow[dr, "\varphi'_x"'] & F(C,M)^G \arrow[d, dotted, "\varphi'"] \\ & A\,. \end{tikzcd}$$

□

Notice that this proposition is a "coordinate-free" description of the fact that algebraically

$$F(C,M)^G \cong \bigoplus_{[x]\in C/G-\{[x_0]\}} M(G/G_x)\,.$$

The previous proposition allows us to define the second covariant functor $F^G(-,M)$.

Definition 2.8 Let $f : C \longrightarrow D$ be a pointed G-function. Define the family

$$f'_x : M(G/G_x) \longrightarrow F(D,M)^G \quad \text{by} \quad f'_x(l) = \gamma_{f(x)}^G M_*(\widehat{f}_x)(l)\,.$$

By Proposition (2.7) this family determines a homomorphism

$$f_*^G : F(C,M)^G \longrightarrow F(D,M)^G\,.$$

Then, for any $u = \sum_{i=1}^k \gamma_{x_i}^G(l_i) \in F(C,M)^G$, one has

$$f_*^G(u) = \sum_{i=1}^k \gamma_{f(x_i)}^G M_*(\widehat{f}_{x_i})(l_i)\,.$$

We denote this functor by $F^G(-,M)$.

The following result puts the definition of the functor structures $\overline{f}_*^G$ and f_*^G in a diagram.

Proposition 2.9 *Let C be a pointed G-set and let $\beta_C : F(C,M) \longrightarrow F(C,M)^G$ be the surjective homomorphism given on generators by $\beta_C(lx) = \gamma_x^G(l)$. If $f : C \longrightarrow D$ is a pointed G-function, then one has the following commutative diagram.*

(2.9)
$$\begin{tikzcd}
\overline{F}^G(C,M) \arrow[r, hook, "i_C"] \arrow[d, "\overline{f}_*^G"'] & F(C,M) \arrow[r, two heads, "\beta_C"] \arrow[d, "f_*"'] & F^G(C,M) \arrow[d, "f_*^G"] \\
\overline{F}^G(D,M) \arrow[r, hook, "i_D"'] & F(D,M) \arrow[r, two heads, "\beta_D"'] & F^G(D,M) .
\end{tikzcd}$$

This means, in particular, that $\beta : F(-,M) \longrightarrow F^G(-,M)$ is a natural transformation. ◻

Notice that the horizontal composites in (2.9) are not the identity.

The following result measures the difference between f_*^G and $\overline{f}_*^G$ in the canonical generators $\gamma_x^G(l) \in F(C,M)^G$.

Proposition 2.10 *Let $f : C \longrightarrow D$ be a pointed G-function. Then*

$$\overline{f}_*^G(\gamma_x^G(l)) = [G_{f(x)} : G_x] f_*^G(\gamma_x^G(l)) \in F(C,M)^G .$$

Proof Let $G/G_{f(x)} = \{[g_1], \ldots, [g_m]\}$ and $G_{f(x)}/G_x = \{[h_1], \ldots, [h_k]\}$. Then $G/G_x = \{[g_1h_1], [g_1h_2], \ldots, [g_mh_{k-1}], [g_mh_k]\}$. First observe that by definition,

$f_*^G(\gamma_x^G(l)) = \gamma_{f(x)}^G(M_*(\widehat{f}_x)(l))$. Therefore,

$$
\begin{aligned}
\overline{f}_*^G(\gamma_x^G(l)) &= \overline{f}_*^G\left(\sum_{(i,j)=(1,1)}^{(m,k)} M_*(R_{(g_ih_j)^{-1}})(l)g_ih_jx\right) \\
&= \sum_{(i,j)=(1,1)}^{(m,k)} M_*(\widehat{f}_{g_ih_jx})M_*(R_{(g_ih_j)^{-1}})(l)g_ih_jf(x) \\
&= \sum_{(i,j)=(1,1)}^{(m,k)} M_*(\widehat{f}_{g_ix})M_*(R_{g_i^{-1}})(l)g_if(x) \\
&= \sum_{j=1}^{k}\sum_{i=1}^{m} M_*(R_{g_i^{-1}})M_*(\widehat{f}_x)(l)g_if(x) \\
&= [G_{f(x)}\!:\!G_x]\gamma_{f(x)}^G(M_*(\widehat{f}_x)(l)) \\
&= [G_{f(x)}\!:\!G_x]f_*^G(\gamma_x^G(l))\,.
\end{aligned}
$$

□

Remark 2.11 From the previous result it follows that both homomorphisms $\overline{f}_*^G$ and f_*^G coincide if the G-map f is isovariant (i.e. if $G_{f(x)} = G_x$ for all $x \in C$), for instance if D is G-free or if C and D are G-trivial.

Definition 2.12 Let M be a Mackey functor for the finite group G. We define the coefficient system $\overline{M}_* : \mathcal{O}(G) \longrightarrow \mathcal{A}\mathrm{b}$ as follows. Put $\overline{M}_*(G/H) = M(G/H)$. Moreover, let $f : G/H \longrightarrow G/K$ be a G-function. If $f = R_g : G/H \longrightarrow G/g^{-1}Hg$, then $\overline{M}_*(f) = M_*(f)$, and if $f = q : G/H \longrightarrow G/K$, where $H \subset K$, is the quotient function, then $\overline{M}_*(f) = [K\!:\!H]M_*(f)$.

Theorem 2.13 *The functors* $\overline{F}^G(-,M), F^G(-,M) : G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{A}\mathrm{b}$ *are characterized by properties* (a) *and* (b_1), *and* (a) *and* (b_2), *respectively, where:*

(a) *Let A be an abelian group and for each $x \in C$ let $\varphi'_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi'_{x_0} = 0$, where $x_0 \in C$ is the base point, and such that $\varphi'_x = \varphi'_{gx} \circ M_*(R_{g^{-1}})$. Then there exists unique homomorphism $\varphi' : F(C,M)^G \longrightarrow A$ such that $\varphi' \circ \gamma_x^G = \varphi'_x$. In a diagram*

$$
\begin{tikzcd}
M(G/G_x) \arrow[r, "\gamma_x^G"] \arrow[dr, "\varphi'_x"'] & F(C,M)^G \arrow[d, dotted, "\varphi'"] \\
 & A\,.
\end{tikzcd}
$$

Note here that $\overline{F}^G(C,M) = F(C,M)^G = F^G(C,M)$.

(b) *Given a pointed G-function $f : C \longrightarrow D$, the following diagrams commute:*

$$(\mathrm{b}_1)\qquad \begin{array}{ccc} M(G/G_x) & \xrightarrow{\gamma_x^G} & \overline{F}^G(C,M) \\ {\scriptstyle \overline{M}_*(\widehat{f}_x)}\downarrow & & \downarrow{\scriptstyle \overline{f}_*^G} \\ M(G/G_{f(x)}) & \xrightarrow[\gamma_{f(x)}^G]{} & \overline{F}^G(D,M)\,, \end{array}$$

$$(\mathrm{b}_2)\qquad \begin{array}{ccc} M(G/G_x) & \xrightarrow{\gamma_x^G} & F^G(C,M) \\ {\scriptstyle M_*(\widehat{f}_x)}\downarrow & & \downarrow{\scriptstyle f_*^G} \\ M(G/G_{f(x)}) & \xrightarrow[\gamma_{f(x)}^G]{} & F^G(D,M)\,. \end{array}$$

Proof Part (a) is Proposition (2.7). Part (b) follows from the definition and from Proposition (2.10).

To see that (a) and (b_1) characterize the functor $\overline{F}^G(-,M)$, assume that we have two functors $F(-)$ and $F'(-)$ that satisfy (a) and (b_1). Property (a) allows us to construct $\alpha_C : F(C) \longrightarrow F'(C)$ and $\alpha'_C : F'(C) \longrightarrow F(C)$ that are inverse to each other. Moreover, property (b_1) allows us to show that α and α' are natural transformations. Similarly, one proves that (a) and (b_2) characterize the functor $F^G(-,M)$. □

Remark 2.14 Notice that in the proof of the previous theorem one only needs the covariant part of M. Thus the result is equally valid for any covariant coefficient system.

3 The transfer for the functor $F^G(-;M)$

We use the property (2.4) to give the transfer. We start with the following definition, that was given in [2, 1.10]; we put it now in terms of the property (2.4).

Definition 3.1 Let M be a Mackey functor and $p : A \longrightarrow C$ a G-function with finite fibers, that is, a G-function such that for each $x \in C$, the fiber

$p^{-1}(x) \subset A$ is finite. For any $x \in C$, let $t_x : M(G/G_x) \longrightarrow F(A^+, M)$ be given by

$$t_x(l) = \sum_{a \in p^{-1}(x)} M^*(\widehat{p}_a)(l)a \,.$$

By (2.4) for $F(C^+, M)$, there is a unique homomorphism

$$t_p : F(C^+, M) \longrightarrow F(A^+, M)\,,$$

such that $t_p \circ \gamma_x = t_x$. Explicitly, on generators,

$$t_p\,(lx) = \sum_{a \in p^{-1}(x)} M^*(\widehat{p}_a)(l)a \,.$$

Since p is a G-function, t_p is also a G-function, as we show in the lemma below, and thus it determines, by restriction, the *transfer*

$$t_p^G : F(C^+, M)^G \longrightarrow F(A^+, M)^G \,.$$

REMARK 3.2 The homomorphism $t_p : F(C^+, M) \longrightarrow F(A^+, M)$ can also be described as follows:

$$t_p(u)(a) = M^*(\widehat{p}_a)(u(p(a)))$$

(and $t_p(u)(*) = 0$).

Lemma 3.3 *$t_p : F(C^+, M) \longrightarrow F(A^+, M)$ is a G-homomorphism.*

Proof We have on the one hand

$$t_p(g \cdot u)(a) = M^*(\widehat{p}_a)(g \cdot u(p(a))) = M^*(\widehat{p}_a)M_*(R_{g^{-1}})(u(g^{-1}p(a)))\,,$$

while on the other hand we have

$$(g \cdot t_p(u))(a) = M_*(R_{g^{-1}})(t_p(u)(g^{-1}a)) = M_*(R_{g^{-1}})M^*(\widehat{p}_{g^{-1}a})(u(g^{-1}p(a)))\,.$$

Both terms are equal, since $M^*(\widehat{p}_a) \circ M_*(R_{g^{-1}}) = M_*(R_{g^{-1}}) \circ M^*(\widehat{p}_{g^{-1}a})$, and this follows from the fact that the following square is clearly a pullback diagram of G-sets:

$$\begin{tikzcd}
G/G_{g^{-1}a} \arrow[r, "R_{g^{-1}}", "\cong"'] \arrow[d, "\widehat{p}_{g^{-1}a}"'] & G/G_a \arrow[d, "\widehat{p}_a"] \\
G/G_{g^{-1}p(a)} \arrow[r, "\cong", "R_{g^{-1}}"'] & G/G_{p(a)}\,.
\end{tikzcd}$$

□

Remark 3.4 Assume that $p : A \longrightarrow C$ and $q : C \longrightarrow D$ are G-functions with finite fibers. Then one has that $\widehat{(q \circ p)}_a = \widehat{q}_{p(a)} \circ \widehat{p}_a$. Using this, one easily verifies that the transfer is functorial in the sense that $t^G_{q \circ p} = t^G_p \circ t^G_q$.

Lemma 3.5 *Let $p : A \longrightarrow C$ be a G-function with finite fibers. Then*

$$t^G_p(\gamma^G_x(l)) = \sum_{[a] \in p^{-1}(x)/G_x} \gamma^G_a(M^*(\widehat{p}_a)(l)) \,, \tag{3.5}$$

Proof The isotropy group G_x acts on $p^{-1}(x)$ and the inclusion $j : p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $\overline{j} : p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$. Let $\gamma^G_x(l)$ be a generator of $F^G(C^+, M)$. Since the value of the function $\gamma^G_x(l)$ on points which do not belong to Gx is zero, and $\gamma^G_x(l)(x) = l$, we have that

$$t^G_p(\gamma^G_x(l)) = \sum_{[a] \in p^{-1}(x)/G_x} \gamma^G_a(M^*(\widehat{p}_a)(l)) \,.$$

$\square$

We shall now prove that the transfer t^G_p has the pullback property. We start with some preliminary results on groups. One can easily prove the following.

Lemma 3.6 *Let $H, H' \subset K \subset G$ be subgroups of G and consider the fibered product*

$$G/H \times_{G/K} G/H' = \{([g]_H, [g']_{H'}) \mid g, g' \in G \text{ and } g^{-1}g' \in K\} \,.$$

Consider the set of double cosets $H\backslash K/H' = \{{}_H[g_r]_{H'} \mid r = 1, \ldots, k\}$, where $g_1, \ldots, g_k \in K$ are fixed representatives. If $H''_r = H \cap g_r H' g_r^{-1}$, then there is an isomorphism of G-sets

$$\varphi : \sqcup_{r=1}^k G/H''_r \xrightarrow{\cong} G/H \times_{G/K} G/H' \,,$$

given by $\varphi[g]_{H''_r} = ([g]_H, [gg_r]_{H'})$. $\square$

Lemma 3.7 *Let $H, H' \subset K \subset G$ be subgroups of G and let M be a Mackey functor. Consider the isomorphism*

$$\bigoplus_{r=1}^{k} M(G/H''_r) \longrightarrow M(\sqcup_{r=1}^k G/H''_r)$$

given by the family $M_*(\kappa_r)$, where $\kappa_r : G/H''_r \hookrightarrow \sqcup_{r=1}^k G/H''_r$ is the inclusion. Then its inverse is given by the homomorphism induced by the family $M^*(\kappa_r)$.

Proof The following are pullback digrams:

$$\begin{tikzcd} G/H''_r \arrow[r, "="] \arrow[d, "="'] & G/H''_r \arrow[d, "\kappa_r"] \\ G/H''_r \arrow[r, "\kappa_r"'] & \sqcup G/H''_r \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & G/H''_s \arrow[d, "\kappa_s"] \\ G/H''_r \arrow[r, "\kappa_r"'] & \sqcup G/H''_r \end{tikzcd}$$

where $r \neq s$. Therefore

$$M^*(\kappa_r) \circ M_*(\kappa_r) = 1_{M(G/H''_r)} \quad \text{and} \quad M^*(\kappa_s) \circ M_*(\kappa_r) = 0\,.$$

Thus the result follows. □

Lemma 3.8 *Let $H, H' \subset K \subset G$ be subgroups of G and let M be a Mackey functor. Take $w \in M(G/H \times_{G/K} G/H')$; then*

$$w = \sum_{r=1}^{k} M_*(\varphi_r) M^*(\varphi_r)(w)\,,$$

where $\varphi_r = \varphi \circ \kappa_r$.

Proof By the previous lemma, for any $z \in M(\sqcup G/H''_r)$ we have

$$z = \sum_{r=1}^{k} M_*(\kappa_r) M^*(\kappa_r)(z)\,. \tag{3.4}$$

By Lemma (3.6), we have an isomorphism

$$M_*(\varphi) : M(\sqcup_{r=1}^{k} G/H''_r) \longrightarrow M(G/H \times_{G/K} G/H')\,.$$

Then for some $z \in M(\sqcup_{r=1}^{k} G/H''_r)$, $w = M_*(\varphi)(z)$. By (3.4), $M_*(\varphi)(z) = M_*(\varphi)(\sum_{r=1}^{k} M_*(\kappa_r) M^*(\kappa_r)(z)) = \sum_{r=1}^{k} M_*(\varphi_r) M^*(\varphi_r)(w)$. The last equality follows from the fact that $M_*(\varphi)^{-1} = M^*(\varphi)$, as one easily sees. □

Let $p : A \longrightarrow C$ be a G-function with finite fibers and let $f : D \longrightarrow C$ be any G-function. Consider the pullback diagram

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "p'"'] & A \arrow[d, "p"] \\ D \arrow[r, "f"'] & C\,, \end{tikzcd} \tag{3.5}$$

where $A' = D \times_C A = \{(y, a) \mid f(y) = p(a)\}$. Consider the restriction of f' from the fiber $(p')^{-1}(y)$ to the fiber $p^{-1}(f(y))$. This function induces a surjective function

$$q : (p')^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)}\,.$$

In what follows we analyze the fibers of q.

Lemma 3.6 *There is a bijection*

$$\overline{\delta} : G_y \backslash G_{f(y)} / G_{a_0} \longrightarrow q^{-1}(G_{f(y)} a_0)\,,$$

where $a_0 \in p^{-1}(f(y))$, given by $\overline{\delta}({}_{G_y}[g]_{G_{a_0}}) = G_y(y, ga_0)$.

Proof The function $\overline{\delta}$ is induced by the surjection $\delta : G_{f(y)} \longrightarrow q^{-1}(G_{f(y)} a_0)$ given by $\delta(g) = G_y(y, ga_0)$. One easily checks that δ factors through the set of double cosets and that $\overline{\delta}$ is injective. $\square$

Theorem 3.7 *Let $p : A \longrightarrow C$ be a G-function with finite fibers, and let $f : D \longrightarrow C$ be a G-function. Then*

$$t_p^G \circ f_*^G = (f')_*^G \circ t_{p'}^G : F^G(D^+, M) \longrightarrow F^G(A^+, M)\,,$$

where f' and p' are as in the pullback diagram (3.5).

Proof Take a generator $\gamma_y^G(l)$, $y \in D$ and $l \in M(G/G_y)$, and consider $q : (p')^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)}$ as in Lemma (3.6). Then, by Definition (2.8) and the formula (3.5), we have

$$t_p^G f_*^G(\gamma_y^G(l)) = \sum_{[a_\iota] \in p^{-1}(f(y))/G_{f(y)}} \gamma_{a_\iota}^G M^*(\widehat{p}_{a_\iota}) M_*(\widehat{f}_y)(l)\,. \tag{3.8}$$

On the other hand, we have

$$(f')_*^G t_{p'}^G(\gamma_y^G(l)) = \sum_{[y,a] \in (p')^{-1}(y)/G_y} \gamma_a^G M_*(\widehat{f'}_{(y,a)}) M^*(\widehat{p'}_{(y,a)})(l)\,. \tag{3.9}$$

We can write $(p')^{-1}(y)/G_y = \sqcup q^{-1}(G_{f(y)} a_\iota)$, where $G_{f(y)} a_\iota = [a_\iota]$. By Lemma (3.6), $p^{-1}(f(y))/G_{f(y)} = \{[y, g_r a_\iota]\}$, where the group-elements g_r are such that $\{{}_{G_y}[g_r]_{G_{a_\iota}}\}_{r=1}^k = G_y \backslash G_{f(y)} / G_{a_\iota}$ (notice that the set $\{g_r\}_{r=1}^k$ depends on each ι). Clearly we have

(3.10)

$$\gamma_{g_r a_\iota}^G M_*(\widehat{f'}_{(y,g_r a_\iota)}) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l) = \gamma_{a_\iota}^G M_*(R_{g_r} \circ \widehat{f'}_{(y,g_r a_\iota)}) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l)\,.$$

Consider the following pullback diagram

$$\begin{tikzcd}
G/G_y \cap G_{g_r a_\iota} \arrow[dr, dashed, "\varphi_r"] \arrow[drr, bend left, "R_{g_r} \circ \widehat{f'}_{(y,g_r a_\iota)}"] \arrow[ddr, bend right, "\widehat{p'}_{(y,g_r a_\iota)}"'] & & \\
& G_y \times_{G/G_{f(y)}} G/G_{a_\iota} \arrow[r, "\tau"] \arrow[d, "\pi"'] & G/G_{a_\iota} \arrow[d, "\widehat{p}_{a_\iota}"] \\
& G/G_y \arrow[r, "\widehat{f}_y"'] & G/G_{f(y)}\,.
\end{tikzcd}$$

Hence, $M^*(\widehat{p}_{a_\iota}) \circ M_*(\widehat{f}_y) = M_*(\tau) \circ M^*(\pi)$. Using Lemma (3.8), we can write

$$M^*(\pi)(l) = \sum_{r=1}^{k} M_*(\varphi_r) M^*(\varphi_r) M^*(\pi)(l) = \sum_{r=1}^{k} M_*(\varphi_r) M^*(\widehat{p'}_{(y, g_r a_\iota)})(l)\,.$$

Composing with $M_*(\tau)$ on the left, we obtain

$$\begin{aligned} M_*(\tau) M^*(\pi)(l) &= \sum_{r=1}^{k} M_*(\tau) M_*(\varphi_r) M^*(\widehat{p'}_{(y, g_r a_\iota)})(l) \\ &= \sum_{r=1}^{k} M_*(R_{g_r} \circ \widehat{f'}_{(y, g_r a_\iota)}) M^*(\widehat{p'}_{(y, g_r a_\iota)})(l)\,. \end{aligned}$$

Hence

$$M^*(\widehat{p}_{a_\iota}) M_*(\widehat{f}_y)(l) = \sum_{r=1}^{k} M_*(R_{g_r} \circ \widehat{f'}_{(y, g_r a_\iota)}) M^*(\widehat{p'}_{(y, g_r a_\iota)})(l)\,,$$

and the result follows. □

4 The topological function groups

We start this section extending the definitions given in the previous sections in the case of G-sets to the case of simplicial G-sets. We denote by Δ the category whose objects are the ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$ and whose morphisms are order-preserving functions between them. A *simplicial pointed G-set* is thus a contravariant functor $K : \Delta \longrightarrow G\text{-}\mathcal{S}\mathrm{et}_*$. We denote by K_n the value of K in $\mathbf{n}$, and given a morphism $\mu : \mathbf{m} \longrightarrow \mathbf{n}$, we denote by $\mu^K : K_n \longrightarrow K_m$ the corresponding pointed G-function.

Definition 4.1 Let K be a simplicial pointed G-set and M a Mackey funtor for G. We define the simplicial abelian groups $F^G(K, M)$ and $\overline{F}^G(K, M)$ as the following composites:

$$\Delta \xrightarrow{K} G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{F^G(-,M)} \mathcal{A}\mathrm{b}\,, \qquad \Delta \xrightarrow{K} G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{\overline{F}^G(-,M)} \mathcal{A}\mathrm{b}\,.$$

Therefore, for each n, the value of the functors $F^G(K, M)$ and $\overline{F}^G(K, M)$ at n are given by $F^G(K_n, M)$ and $\overline{F}^G(K_n, M)$, respectively.

Notice that by Remark (2.6), there is also a simplicial abelian G-group defined by the composite

$$\Delta \xrightarrow{K} G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{F(-,M)} \cdot$$

Proposition 4.2 *Let K be a simplicial pointed G-set. Then*

(a) $\overline{F}^G(K,M)$ *is a simplicial subgroup of* $F(K,M)$*, and*

(b) $F^G(K,M)$ *is a simplicial quotient group of* $F(K,M)$.

Proof This follows by applying Proposition (2.9) to $\mu^K : K_n \longrightarrow K_m$, where $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ. The inclusion of (a) is given by the natural transformation $i : \overline{F}^G(-,M) \hookrightarrow F(-,M)$, and the surjection of (b) is given by the natural transformation $\beta : F(-,M) \twoheadrightarrow F^G(-,M)$. $\square$

In what follows, we shall use the previous definitions to associate topological abelian groups $F^G(X,M)$ and $\overline{F}^G(X,M)$ to a pointed G-space X. We shall work in the category of k-spaces. We understand by a k-*space* a topological space X with the property that a set $W \subset X$ is closed if and only if $f^{-1}W \subset Z$ is closed for any continuous map $f : Z \longrightarrow X$, where Z is any compact Hausdorff space (see [9, 11]).

If S is a simplicial set (G-set, group, etc.), we denote by $|S|$ its *geometric realization.* This is a quotient space of

$$\sqcup_n S_n \times \Delta^n$$

(see [8] for details).

Lemma 4.3 *Let S be a simplicial pointed G-set. Then there is a canonical homeomorphism $|S^G| \longrightarrow |S|^G$.*

Proof Let $i : S^G \hookrightarrow S$ be the inclusion. This morphism induces an embedding $|i| : |S^G| \longrightarrow |S|$. One easily sees that the image of $|i|$ is a subset of $|S|^G$. In order to see that $|S|^G$ is indeed the image of $|i|$, let $[\sigma,t] \in |S|^G$ be represented by a nondegenerate element (σ,t). Then $g[\sigma,t] = [g\sigma,t]$ coincides with $[\sigma,t]$. Since σ is nondegenerate, so is also $g\sigma$. Therefore, $g\sigma = \sigma$ and so $[\sigma,t]$ is in the image of $|i|$. $\square$

Definition 4.4 Let X be a pointed G-space and let $\mathcal{S}(X)$ be the associated singular simplicial pointed G-set, where the base point in each $\mathcal{S}_n(X)$ is the constant n-simplex with value x_0. We define the following topological spaces:

$$F^G(X,M) = |F^G(\mathcal{S}(X),M)|\,, \quad \overline{F}^G(X,M) = |\overline{F}^G(\mathcal{S}(X),M)|\,.$$

Notice that these two spaces have the structure of regular CW-complexes.

Remark 4.5 One may also define $F(X, M) = |F(\mathcal{S}(X), M)|$ and by Lemma (4.3), $\overline{F}^G(X, M) = |F(\mathcal{S}(X), M)|^G = F(X, M)^G$.

If X is a G-space, then the underlying groups of $F^G(X, M)$ and $\overline{F}^G(X, M)$ differ from the (discrete) group $F(X^\delta, M)^G$, as defined in section 2, where X^δ denotes the underlying G-set of X. However, we have the following.

Proposition 4.6 *If X is a discrete pointed G-space, then the topological abelian groups $F^G(X, M)$ and $\overline{F}^G(X, M)$ are discrete and both are isomorphic to the abelian group $F(X^\delta, M)^G$.*

Proof Notice that if K is a simplicial set such that $K_n = C$ for all n, and $f^K = \mathrm{id}_C$ for all f in Δ, then $|K|$ is a discrete space homeomorphic to C, because $|K|$ is a CW-complex with one n-cell for each nondegenerate n-simplex of K. We call such a simplicial set *trivial*.

Now, if X is discrete, then $\mathcal{S}_n(X) = X^\delta$ for all n and $f^{\mathcal{S}(X)} = \mathrm{id}_X$ for all f, thus it is trivial. Therefore, the simplicial groups $F^G(\mathcal{S}(X), M)$ and $\overline{F}^G(\mathcal{S}(X), M)$ are trivial too. Hence

$$|F^G(\mathcal{S}(X), M)| \cong F(X^\delta, M)^G \cong |\overline{F}^G(\mathcal{S}(X), M)|\,.$$

□

Remark 4.7 The functors $F^G(-, M)$ and $\overline{F}^G(-, M)$, restricted to the category of discrete pointed G-spaces, are indeed naturally isomorphic to the functors $F^G((-)^\delta, M)$ and $\overline{F}^G((-)^\delta, M)$, respectively.

Proposition 4.8 *Let X be a pointed G-space. Then the spaces $F^G(X, M)$ and $\overline{F}^G(X, M)$ are topological abelian groups (in the category of* k*-spaces).*

Proof Since $F^G(\mathcal{S}(X), M)$ and $\overline{F}^G(\mathcal{S}(X), M)$ are simplicial abelian groups, their geometric realizations $|F^G(\mathcal{S}(X), M)|$ and $|\overline{F}^G(\mathcal{S}(X), M)|$ are topological groups (in the category of k-spaces, see [9, 11]). □

Remark 4.9 In a similar way to the previous proposition, we have that $F(X, M)$ is a topological abelian G-group. By Proposition (4.2) and [5], we have that

(a) $\overline{F}^G(X, M)$ is a topological subgroup of $F(X, M)$, and

(b) $F^G(X, M)$ is a topological quotient group of $F(X, M)$.

We have the following.

Definition 4.10 Let K be a simplicial pointed G-set and M a Mackey functor for G. Let Λ be any simplicial abelian group. We shall say that a family of homomorphisms $\{\varphi_\sigma : M(G/G_\sigma) \longrightarrow \Lambda_n \mid \sigma \in K_n, n \geq 0\}$ is *simplicial* if the following conditions are satisfied:

(a) If $\sigma_0 \in K_n$ is the base point, then $\varphi_{\sigma_0} = 0$, and

(b) for each morphism $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ in Δ, the following diagram commutes:

$$\begin{tikzcd}
M(G/G_\sigma) \arrow[r, "\varphi_\sigma"] \arrow[d, "M_*(\widehat{\mu^K}_\sigma)"'] & \Lambda_n \arrow[d, "\mu^\Lambda"] \\
M(G/G_{\mu^K(\sigma)}) \arrow[r, "\varphi_{\mu^K(\sigma)}"'] & \Lambda_m\,.
\end{tikzcd}$$

We say that the simplicial family is G*-invariant* if for all $\sigma \in K$ and all $g \in G$,

$$\varphi_{g\sigma} = \varphi_\sigma \circ M_*(R_g)\,,$$

Corresponding to the property (2.4), we have the following.

Proposition 4.11 *Let K be a simplicial pointed G-set and M a Mackey functor for G. Then*

(i) *the family $\{\gamma_\sigma : M(G/G_\sigma) \longrightarrow F(K_n, M) \mid \sigma \in K_n, n \geq 0\}$ is simplicial. Moreover*

(ii) *if Λ is any simplicial abelian group and $\{\varphi_\sigma\ M(G/G_\sigma) \longrightarrow \Lambda_n \mid \sigma \in K_n, n \geq 0\}$ is a simplicial family of homomorphisms, then there is a unique simplicial homomorphism $\varphi : F(K, M) \longrightarrow \Lambda$, such that $\varphi_n \circ \gamma_\sigma = \varphi_\sigma$, where $\sigma \in K_n$, $n \geq 0$.*

Proof Let $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ be a morphism in Δ. To see (i), take $l \in M(G/G_\sigma)$. Then

$$\mu_*^K \gamma_\sigma(l) = \mu_*^K(l\sigma) = M_*(\widehat{\mu^K}_\sigma)(l)\mu^K(\sigma) = \gamma_{\mu^K(\sigma)} M_*(\widehat{\mu^K}_\sigma)(l)\,.$$

We now prove (ii). By Proposition (2.4), for each n there is a unique homomorphism $\varphi_n : F(K_n, M) \longrightarrow \Lambda_n$ such that $\varphi_n \circ \gamma_\sigma = \varphi_\sigma$. To check that the family $\{\varphi_n\}$ is a morphism of simplicial groups, take a generator $l\sigma \in F(K_n, M)$. Then

$$\varphi_m \mu_*^K(l\sigma) = \varphi_m(M_*(\widehat{\mu^K}_\sigma)(l)\mu^K(\sigma)) = \varphi_{\mu^K(\sigma)}(M_*(\widehat{\mu^K}_\sigma)(l)) =$$
$$= \mu^\Lambda \varphi_\sigma(l) = \mu^\Lambda \varphi_n(l\sigma)\,.$$

□

We now have the following result, which is similar to the previous proposition.

Proposition 4.12 *Let K be a simplicial pointed G-set and M a Mackey functor for G. Then*

(i) *the family $\{\gamma_\sigma^G : M(G/G_\sigma) \longrightarrow F^G(K_n, M) \mid \sigma \in K_n, n \geq 0\}$ is simplicial and G-invariant. Moreover*

(ii) *if Λ is any simplicial abelian group and $\{\varphi_\sigma\ M(G/G_\sigma) \longrightarrow \Lambda_n \mid \sigma \in K_n, n \geq 0\}$ is a simplicial G-invariant family of homomorphisms, then there is a unique simplicial homomorphism $\varphi^G : F^G(K, M) \longrightarrow \Lambda$, such that $\varphi_n^G \circ \gamma_\sigma^G = \varphi_\sigma$, where $\sigma \in K_n$, $n \geq 0$.* □

Before passing to the definition of the functorial structures of $F(X;M)$, $F^G(X;M)$, and $\overline{F}^G(X;M)$, recall that a morphism of simplicial pointed G-sets $\alpha : K \longrightarrow Q$ consists of a family of pointed G-functions $\alpha_n : K_n \longrightarrow Q_n$ such that, if $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in Δ, then one has a commutative diagram

$$\begin{tikzcd} K_n \arrow[r, "\alpha_n"] \arrow[d, "\mu^K"'] & Q_n \arrow[d, "\mu^Q"] \\ K_m \arrow[r, "\alpha_m"'] & Q_m\,. \end{tikzcd}$$

Since we have functors $F(-,M), F^G(-,M), \overline{F}^G(-,M) : G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{A}\mathrm{b}$, they yield commutative diagrams

$$\begin{tikzcd} F(K_n,M) \arrow[r, "\alpha_{n*}"] \arrow[d, "\mu_*^K"'] & F(Q_n,M) \arrow[d, "\mu_*^Q"] \\ F(K_m,M) \arrow[r, "\alpha_{m*}"'] & F(Q_m,M)\,, \end{tikzcd} \qquad \begin{tikzcd} F^G(K_n,M) \arrow[r, "{\alpha_n}_*^G"] \arrow[d, "{\mu^K}_*^G"'] & F^G(Q_n,M) \arrow[d, "{\mu^Q}_*^G"] \\ F^G(K_m,M) \arrow[r, "{\alpha_m}_*^G"'] & F^G(Q_m,M)\,, \end{tikzcd}$$

$$\begin{tikzcd} \overline{F}^G(K_n,M) \arrow[r, "\overline{\alpha_n}_*^G"] \arrow[d, "\overline{\mu^K}_*^G"'] & \overline{F}^G(Q_n,M) \arrow[d, "\overline{\mu^Q}_*^G"] \\ \overline{F}^G(K_m,M) \arrow[r, "\overline{\alpha_m}_*^G"'] & \overline{F}^G(Q_m,M)\,. \end{tikzcd}$$

Hence the functors $F(-,M)$, $F^G(-,M)$, and $\overline{F}^G(-,M)$ extend to functors of simplicial pointed G-sets.

Definition 4.13 Let $f : X \longrightarrow Y$ be a continuous pointed G-map. The map f induces a morphism of simplicial pointed G-sets $\mathcal{S}(f) : \mathcal{S}(X) \longrightarrow \mathcal{S}(Y)$, which defines homomorphisms of simplicial groups

$$\begin{aligned}
\mathcal{S}(f)_* &: F(\mathcal{S}(X), M) \longrightarrow F(\mathcal{S}(Y), M)\,,\\
\mathcal{S}(f)_*^G &: F^G(\mathcal{S}(X), M) \longrightarrow F^G(\mathcal{S}(Y), M)\,,\\
\overline{\mathcal{S}(f)}_*^G &: \overline{F}^G(\mathcal{S}(X), M) \longrightarrow \overline{F}^G(\mathcal{S}(Y), M)\,.
\end{aligned}$$

Define the homomorphisms

$$\begin{aligned}
f_* &: F(X, M) \longrightarrow F(Y, M)\,,\\
f_*^G &: F^G(X, M) \longrightarrow F^G(Y, M)\,,\\
\overline{f}_*^G &: \overline{F}^G(X, M) \longrightarrow \overline{F}^G(Y, M)\,,
\end{aligned}$$

by $f_* = |\mathcal{S}(f)_*|$, $f_*^G = |\mathcal{S}(f)_*^G|$, and $\overline{f}_*^G = |\overline{\mathcal{S}(f)}_*^G|$, respectively.

Remark 4.14 One may obtain the simplicial homomorphisms

$$\begin{aligned}
\mathcal{S}(f)_* &: F(\mathcal{S}(X), M) \longrightarrow F(\mathcal{S}(Y), M)\,,\\
\mathcal{S}(f)_*^G &: F^G(\mathcal{S}(X), M) \longrightarrow F^G(\mathcal{S}(Y), M)\,,
\end{aligned}$$

using the properties (4.11) and (4.12) for the families $\{\varphi_\sigma\}$ and $\{\varphi_\sigma^G\}$ given by

$$\begin{aligned}
\varphi_\sigma(l) &= \gamma_{\mathcal{S}(f)(\sigma)}(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l)) \in F(\mathcal{S}_n(X), M)\,,\\
\varphi_\sigma^G(l) &= \gamma_{\mathcal{S}(f)(\sigma)}^G(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l)) \in F^G(\mathcal{S}_n(X), M)\,.
\end{aligned}$$

They provide the following explicit expressions for them on generators:

$$\begin{aligned}
\mathcal{S}(f)_*(\gamma_\sigma(l)) &= \gamma_{\mathcal{S}(f)(\sigma)}(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l))\,,\\
\mathcal{S}(f)_*^G(\gamma_\sigma^G(l)) &= \gamma_{\mathcal{S}(f)(\sigma)}^G(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l))\,.
\end{aligned}$$

Since $\overline{\mathcal{S}(f)}_*^G$ is the restriction of $\mathcal{S}(f)_*$, the first gives also an explicit expression in this case.

Clearly, we have the following result.

Proposition 4.15 *If $f : X \longrightarrow Y$ is a continuous pointed G-map, then $f_* : F(X, M) \longrightarrow F(Y, M)$, $f_*^G : F^G(X, M) \longrightarrow F^G(Y, M)$, and $\overline{f}_*^G : \overline{F}^G(X, M) \longrightarrow \overline{F}^G(Y, M)$ are continuous homomorphisms. Thus $F(-, M)$, $F^G(-, M)$, and $\overline{F}^G(-, M)$ are covariant functors from the category of pointed G-spaces to the category of topological abelian groups. In particular, $F(X, M)$ is a topological abelian G-group.* ◻

Remark 4.16 Let $f : X \longrightarrow Y$ be a pointed G-map. By (2.9), it follows that one has an epimorphism of simplicial groups $\beta_{\mathcal{S}(X)} : F(\mathcal{S}(X), M) \longrightarrow F^G(\mathcal{S}(X), M)$. Thus, by [5], its geometric realization

$$\beta_X : F(X, M) \longrightarrow F^G(X, M)$$

is an identification for any pointed G-space X. One can visualize both functor structures in an analogous way to the commutative diagram (2.9), namely,

$$\begin{tikzcd} \overline{F}^G(X,M) \arrow[r, hook] \arrow[d, "\overline{f}^G_*"'] & F(X,M) \arrow[r, two heads, "\beta_X"] \arrow[d, "f_*"'] & F^G(X,M) \arrow[d, "f^G_*"] \\ \overline{F}^G(Y,M) \arrow[r, hook] & F(Y,M) \arrow[r, two heads, "\beta_Y"'] & F^G(Y,M)\,, \end{tikzcd} \tag{4.16}$$

where the groups are now topological and all the homomorphisms are continuous.

To finish this section, we prove that the functors $F(X, M)$, $F^G(X, M)$, and $\overline{F}^G(X, M)$ are homotopy invariant. For that, we need the following.

Lemma 4.17 *Let K and Q be simplicial pointed G-sets and be $\alpha_0, \alpha_1 : K \longrightarrow Q$ be morphisms. If α_0 and α_1 are G-homotopic, then*

(a) $\alpha_{0*}, \alpha_{1*} : F(K, M) \longrightarrow F(Q, M)$ *are G-homotopic homomorphisms;*

(b) $\alpha^G_{0*}, \alpha^G_{1*} : F^G(K, M) \longrightarrow F^G(Q, M)$ *are homotopic homomorphisms;*

(c) $\overline{\alpha}^G_{0*}, \overline{\alpha}^G_{1*} : \overline{F}^G(K, M) \longrightarrow \overline{F}^G(Q, M)$ *are homotopic homomorphisms.*

Proof Let $\mathcal{H} : K \times \Delta[1] \longrightarrow Q$ be a G-homotopy between α_0 and α_1, since $\mathcal{H}$ is G-equivariant (where $\Delta[1]$ has the trival action), it induces homomorphisms

$$\mathcal{H}_* : F(K \times \Delta[1], M) \longrightarrow F(Q, M)\,,$$
$$\mathcal{H}^G_* : F^G(K \times \Delta[1], M) \longrightarrow F^G(Q, M)\,,$$
$$\overline{\mathcal{H}}^G_* : \overline{F}^G(K \times \Delta[1], M) \longrightarrow \overline{F}^G(Q, M)\,.$$

Let $\iota : F(K, M) \times \Delta[1] \longrightarrow F(K \times \Delta[1], M)$ be given by

$$\iota_n(u, a)(\sigma, b) = \begin{cases} u(\sigma) & \text{if } b = a\,, \\ 0 & \text{if } b \neq a\,, \end{cases}$$

where $(u, a) \in F(K_n, M) \times \Delta[1]_n$ and $(\sigma, b) \in K_n \times \Delta[1]_n$. We have that $\iota_n(u + u', a) = \iota_n(u, a) + \iota_n(u', a)$. Therefore

$$\iota_n(\sum_\sigma l_\sigma \sigma, a) = \sum_\sigma l_\sigma(\sigma, a)\,.$$

One can easily check that ι is a morphism of simplicial pointed sets, (where the base point in $\Delta[1]_n$ is the constant function with value 0). Then $\mathcal{H}_* \circ \iota$ is a homotopy between α_{0*} and α_{1*}.

Since ι and $\mathcal{H}_*$ are G-equivariant, the restriction of $\mathcal{H}_* \circ \iota$ to $\overline{F}^G(K, M) \times \Delta[1]$ is a homotopy between $\overline{\alpha}^G_{0*}$ and $\overline{\alpha}^G_{1*}$.

Now let $\iota^G : F^G(K, M) \times \Delta[1] \longrightarrow F^G(K \times \Delta[1], M)$ be given by

$$\iota^G_n(u, a)(\sigma, b) = \begin{cases} u(\sigma) & \text{if } b = a\,, \\ 0 & \text{if } b \neq a\,, \end{cases}$$

where $(u, a) \in F^G(K_n, M) \times \Delta[1]_n$ and $(\sigma, b) \in K_n \times \Delta[1]_n$. Since u is a G-invariant element, it follows that $\iota^G_n(u, a)$ is also G-invariant. We also have that $\iota^G_n(u + u', a) = \iota^G_n(u, a) + \iota^G_n(u', a)$. Therefore $\iota^G_n(\sum_\sigma \gamma^G_\sigma(l_\sigma), a) = \sum_\sigma \gamma^G_{(\sigma,a)}(l_\sigma)$. One can easily see that ι^G is a morphism of simplicial pointed sets. The composite $\mathcal{H}^G_* \circ \iota^G$ is a homotopy between α^G_{0*} and α^G_{1*}. □

Proposition 4.18 *If $f_0, f_1 : X \longrightarrow Y$ are G-homotopic pointed maps, then*

(a) $f_{0*}, f_{1*} : F(X, M) \longrightarrow F(Y, M)$ *are G-homotopic homomorphisms,*

(b) $\overline{f}^G_{0*}, \overline{f}^G_{1*} : \overline{F}^G(X, M) \longrightarrow \overline{F}^G(Y, M)$ *are homotopic homomorphisms, and*

(c) $f^G_{0*}, f^G_{1*} : F^G(X, M) \longrightarrow F^G(Y, M)$ *are homotopic homomorphisms.*

Proof For convenience, we shall take the standard 1-simplex Δ^1 instead of the unit interval I. Thus let $H : X \times \Delta^1 \longrightarrow Y$ be a pointed G-homotopy from f_0 to f_1. Consider the morphism of simplicial G-sets $R : \mathcal{S}(X) \times \Delta[1] \longrightarrow \mathcal{S}(Y)$ given as follows. If $s \in \Delta^n$, define $R_n : \mathcal{S}_n(X) \times \Delta[1]_n \longrightarrow \mathcal{S}_n(Y)$ by

$$R_n(\sigma, a)(s) = H(\sigma(s), a_\#(s))\,,$$

where $a_\# : \Delta^n \longrightarrow \Delta^1$ is the affine map determined by a. Then R is a G-equivariant homotopy between $\mathcal{S}(f_0)$ and $\mathcal{S}(f_1)$. Thus, by the previous lemma, there is a homotopy T between the morphisms $\mathcal{S}(f_0)_*$ and $\mathcal{S}(f_1)_*$. Then

$$H' : |F(\mathcal{S}(X), M)| \times |\Delta[1]| \xleftarrow{\approx} |F(\mathcal{S}(X), M) \times \Delta[1]| \xrightarrow{|T|} |F(\mathcal{S}(Y), M)|\,,$$

where the homeomorphism is canonical, is a homotopy between $f_{0*} = |\mathcal{S}(f_0)_*|$ and $f_{1*} = |\mathcal{S}(f_1)_*|$, and thus we have (a). Similarly, also using the previous lemma, we obtain (b) and (c). □

5 The topological function group $\mathbb{F}^G(X,M)$

In this section we shall define a new topological abelian group $\mathbb{F}^G(X,M)$, whose description is simpler than that of $F^G(X,M)$. Here our pointed G-spaces will be pointed k-spaces.

Let X be a pointed G-space and let $\mathcal{S}(X)$ be the associated singular simplicial pointed G-set, where the base point in each $\mathcal{S}_n(X)$ is the constant n-simplex with value x_0. Denote by X^δ the underlying pointed G-set of X. We shall define a topology on the abelian group $F(X^\delta,M)^G$ as follows. Take the surjective homomorphism

$$\pi_X^G : |F^G(\mathcal{S}(X),M)| \twoheadrightarrow F(X^\delta,M)^G$$

defined by

$$\pi_X^G\left(\left[\sum_\sigma \gamma_\sigma^G(l_\sigma),\ t\right]\right) = \sum_\sigma \gamma_{\sigma(t)}^G M_*(p_{\sigma,t})(l_\sigma)\,.$$

We give $F(X^\delta,M)^G$ the identification topology, where $p_{\sigma,t} : G/G_\sigma \longrightarrow G/G_{\sigma(t)}$ is the quotient map. We denote the resulting space by $\mathbb{F}^G(X,M)$.

Proposition 5.1 *Let X be a pointed G-space. Then $\mathbb{F}^G(X,M)$ is a topological group (in the category of* k*-spaces).*

Proof Consider the following commutative diagram:

$$\begin{tikzcd}
|F^G(\mathcal{S}(X),M)| \times |F^G(\mathcal{S}(X),M)| \arrow[r] \arrow[d, "\pi_X^G\times\pi_X^G"'] & |F^G(\mathcal{S}(X),M)| \arrow[d, "\pi_X^G"] \\
\mathbb{F}^G(X,M)\times\mathbb{F}^G(X,M) \arrow[r] & \mathbb{F}^G(X,M)\,,
\end{tikzcd}$$

since the product $\pi_X^G\times\pi_X^G$ in the category of k-spaces is an identification, the result follows. □

Let $f : X \longrightarrow Y$ be a continuous pointed G-map. It induces a pointed G-function $f : X^\delta \longrightarrow Y^\delta$ which defines a homomorphism $f_*^G : F(X^\delta,M)^G \longrightarrow F(Y^\delta,M)^G$. We have the following result.

Proposition 5.2 *If $f : X \longrightarrow Y$ is a continuous pointed G-map, then*

$$f_*^G : \mathbb{F}^G(X,M) \longrightarrow \mathbb{F}^G(Y,M)$$

is a continuous homomorphism. Thus $\mathbb{F}^G(-,M)$ is a covariant functor from the category of pointed G-spaces to the category of topological abelian groups.

Proof The G-map f induces a morphism of simplicial G-sets $\mathcal{S}(f) : \mathcal{S}(X) \longrightarrow \mathcal{S}(Y)$ which in turn defines a homomorphism of simplicial groups

$$\mathcal{S}(f)_*^G : F^G(\mathcal{S}(X), M) \longrightarrow F^G(\mathcal{S}(Y), M)\,.$$

Consider the following diagram, where the top map is continuous:

$$\begin{tikzcd} {|F^G(\mathcal{S}(X), M)|} \arrow[r, "{|\mathcal{S}(f)_*^G|}"] \arrow[d, "\pi_X^G"'] & {|F^G(\mathcal{S}(Y), M)|} \arrow[d, "\pi_Y^G"] \\ \mathbb{F}^G(X, M) \arrow[r, "f_*^G"'] & \mathbb{F}^G(Y, M)\,. \end{tikzcd}$$

It is a straightforward verification that it is commutative. Therefore, f_*^G is continuous. $\square$

Remark 5.3 Notice that in (4.13) we defined a continuous homomorphism

$$f_*^G : F^G(X, M) \longrightarrow F^G(Y, M)\,,$$

which should not be confused with

$$f_*^G : \mathbb{F}^G(X, M) \longrightarrow \mathbb{F}^G(Y, M)\,.$$

They are related by the commutativity of the diagram

$$\begin{tikzcd} F^G(X, M) \arrow[r, "f_*^G"] \arrow[d, "\pi_X^G"'] & F^G(Y, M) \arrow[d, "\pi_Y^G"] \\ \mathbb{F}^G(X, M) \arrow[r, "f_*^G"'] & \mathbb{F}^G(Y, M)\,, \end{tikzcd}$$

which is just the diagram in the proof of (5.2).

We shall now give a topological characterization of the group $\mathbb{F}^G(X, M)$, similar to Proposition (2.4). In order to do this, we need the following.

Definition 5.4 Let X be a pointed G-space. Let A be a topological abelian group in the category of k-spaces, and for each $x \in X$ let $\varphi_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi_{x_0} = 0$, where $x_0 \in X$ is the base point. We say that $\{\varphi_x\}$ is a *continuous family* if the homomorphism

$$\widetilde{\varphi} : |F(\mathcal{S}(X), M)| \longrightarrow A$$

given by

$$\widetilde{\varphi}\left[\sum_{\sigma \in \mathcal{S}_n(X)} l_\sigma \sigma, t\right] = \sum_{\sigma \in \mathcal{S}_n(X)} \varphi_{\sigma(t)} M_*(p_{\sigma,t})(l_\sigma)\,,$$

is continuous, where $p_{\sigma,t} : G/G_\sigma = G/G_{(\sigma,t)} \twoheadrightarrow G/G_{\sigma(t)}$ is the quotient map. We say that the family is G-*invariant*, if $\varphi_x = \varphi_{gx} \circ M_*(R_{g^{-1}})$ for all $g \in G$.

The universal property that characterizes the topological abelian group $\mathbb{F}^G(X, M)$, together with the family $\{\gamma_x^G\}$, is the following.

Proposition 5.5 (i) $\{\gamma_x^G\}$ *is an equivariant continuous family.*

(ii) *Let A be a topological abelian group and let $\{\varphi_x\}$ be an equivariant continuous family. Then there exists a unique continuous homomorphism $\varphi : \mathbb{F}^G(X, M) \longrightarrow A$ such that $\varphi \circ \gamma_x^G = \varphi_x$.*

Proof By definition, the family $\{\varphi_x\}$ induces a continuous homomorphism $\widetilde{\varphi} : |F(\mathcal{S}(X), M)| \longrightarrow A$ and since the family is G-invariant, then by (2.7) there exists a unique homomorphism $\varphi : \mathbb{F}^G(X, M) \longrightarrow A$ such that $\varphi \circ \gamma_x^G = \varphi_x$ which satisfies $\varphi \circ \pi_X^G \circ |\beta_{\mathcal{S}(X)}| = \widetilde{\varphi}$. The simplicial homomorphism $\beta_{\mathcal{S}(X)}$ is surjective, hence by [5], $|\beta_{\mathcal{S}(X)}|$ is an identification, and since π_X^G is also an identification, φ is continuous. □

Observe that the continuity of f_*^G shown above follows also from this universal property in a similar manner as that of (4.15).

We now show that the functor $\mathbb{F}^G(-, M)$ is homotopy invariant.

Proposition 5.6 *If $f_0, f_1 : X \longrightarrow Y$ are G-homotopic pointed maps, then*

$$f_{0*}^G, f_{1*}^G : \mathbb{F}^G(X, M) \longrightarrow \mathbb{F}^G(Y, M)$$

are homotopic homomorphisms.

Proof By (4.18), we have a homotopy $H' : F^G(X, M) \times \Delta^1 \longrightarrow F^G(Y, M)$. It is straightforward to verify that the map $\pi_Y^G \circ H'$ is compatible with the identification $\pi_X^G \times 1$, so that the following diagram commutes:

$$\begin{tikzcd}
F^G(X, M) \times \Delta^1 \arrow[r, "H'"] \arrow[d, two heads, "\pi_X^G \times 1"'] & F^G(Y, M) \arrow[d, two heads, "\pi_Y^G"] \\
\mathbb{F}^G(X, M) \times \Delta^1 \arrow[r, "H''"'] & \mathbb{F}^G(Y, M) .
\end{tikzcd}$$

Then H'' is the desired homotopy. □

To finish this section we shall show that the group-functor $\mathbb{F}^G(-,M)$ has the same properties of $F^G(-,M)$, when M is a homological Mackey functor. Recall the following.

Definition 5.7 A Mackey functor M for G is said to be *homological* if whenever $K \subset H \subset G$ and $q : G/H \longrightarrow G/K$ is the quotient function, one has $M_*(q)M^*(q) = [H:K]$, that is, multiplication by the index of K in H.

EXAMPLE 5.8 Given a G-module L, one defines a homological Mackey functor M_L as follows. Put $M_L(G/H) = L^H$ and define

$$M_{L*}(R_{g^{-1}}) : L^H \longrightarrow L^{gHg^{-1}}\,, \quad l \longmapsto gl\,,$$

$$M_L^*(R_{g^{-1}}) : L^{gHg^{-1}} \longrightarrow L^H\,, \quad l \longmapsto g^{-1}l\,,$$

and if $H \subset K$, $K/H = \{[k_i]_H\}$, and $q : G/H \longrightarrow G/K$ is the quotient function, then

$$M_{L*}(q) : L^H \longrightarrow L^K\,, \quad l \longmapsto \sum k_i l\,,$$

$$M_L^*(q) : L^K \longrightarrow L^H \quad \text{is the inclusion.}$$

Definition 5.9 Given a G-module L, we define the functors $F(-,L)$ and $F^G(-,L)$ form the category of pointed G-sets to the category of abelian groups as follows:

$$F(C,L) = \{u : C \longrightarrow L \mid u(*) = 0 \text{ and } u(x) = 0 \text{ for almost all } x \in C\}\,,$$
$$F^G(C,L) = \{u \in F(C,L) \mid u(gx) = gu(x) \text{ for all } x \in X,\ g \in G\}\,,$$

(see [1, Def. 1.1]). Moreover, if X is a topological pointed G-space, then we can define a topology on $F(X,L)$ and on $F^G(X,L)$ as follows. Take the surjection

$$\mu : \sqcup_q (L \times X)^q \twoheadrightarrow F(X,L)\,,$$

where $\mu(l_1, x_1, \ldots, l_q, x_q) = l_1x_1 + \cdots + l_qx_q$, and give $F(X,L)$ the identification topology, then give $F^G(X,L)$ the relative topology. We now have that $F(-,L)$ and $F^G(-,L)$ are functors from the category of pointed G-spaces to the category of abelian topological groups.

Lemma 5.10 *The functors $F^G(-,L)$ and $F^G(-,M_L)$ form the category of pointed G-sets to the category of abelian groups are equal.*

Proof Notice first that $\widehat{M_L} = L$ and if $u \in F^G(C, L)$, then $u(x) \in L^{G_x} = M_L(G/G_x)$. Let $f : C \longrightarrow D$ be a pointed G-function. Consider the projection $G/G_x \twoheadrightarrow G/G_{f(x)}$ with fiber $G_{f(x)}/G_x$. One can describe the cosets in G/G_x as products of the cosets in $G/G_{f(x)}$ and those in $G_{f(x)}/G_x$, in a similar way as in the proof of Lemma (5.16), below. Then we can write a generator $\gamma_x^G(l)$ as $\sum(g_i h_j l)(g_i h_j x)$. Now we can easily check that the value of the homomorphisms induced by the functors $F^G(-, L)$ and $F^G(-, M_L)$ are equal on this generator. □

Remark 5.11 Observe that when X is a topological pointed G-space and L is a G-module, we have two different abelian groups, namely, $F^G(X, L)$ as defined above, and $F^G(X, M_L) = |F^G(\mathcal{S}(X), M_L)|$ as defined in (4.4). However, $F^G(X, L)$ and $\mathbb{F}^G(X, M_L)$ are equal as abelian groups. Furthermore, the identity $\mathbb{F}^G(X, M_L) \longrightarrow F^G(X, L)$ is always continuous, as proved in [3]. We prove below (5.17) that it is a homeomorphism if $X = |K|$.

The following result of Thevenaz and Webb [10, Thm. (16.5)(i)] will be used in what follows.

Theorem 5.12 *Given a homological Mackey functor M, there exists a G-module L and an epimorphism of Mackey functors $\xi : M_L \twoheadrightarrow M$.*

Definition 5.13 We shall denote by $\xi_\diamondsuit : F^G(-, M_L) \longrightarrow F^G(-, M)$ the natural transformation determined by $\xi : M_L \twoheadrightarrow M$, namely, if $u \in F^G(C, M_L)$, then $\xi_\diamondsuit(u)(x) = \xi_{G/G_x}(u(x))$, where $x \in C$.

Notice that for each C, $\xi_\diamondsuit$ is surjective, because if $\gamma_x^G(l')$ is a generator of $F^G(C, M)$ and $\xi_{G/G_x}(l) = l'$, then $\xi_\diamondsuit(\gamma_x^G(l)) = \gamma_x^G(l')$.

Definition 5.14 For a simplicial pointed G-set K and a G-module L, we gave in [1, Prop. 2.3] a G-isomorphism of topological groups $\psi : |F(K, L)| \longrightarrow F(|K|, L)$ given on generators by $\psi([l\sigma, t]) = l[\sigma, t]$. We shall denote its restriction to the fixed-point subgroup by

$$\psi_L^G : |F^G(K, L)| \longrightarrow F^G(|K|, L)\,.$$

On the other hand, for any Mackey functor M for G we defined in [2, Prop. 2.6] an isomorphism

$$\psi_M^G : |F^G(K, M)| \longrightarrow \mathbb{F}^G(|K|, M)$$

as discrete groups, given by

$$\psi_M^G([\gamma_\sigma^G(l), t]) = \gamma_{[\sigma,t]}^G M_*(q_{\sigma,t})(l)\,,$$

where $q_{\sigma,t} : G/G_\sigma \longrightarrow G/G_{[\sigma,t]}$ is the quotient function.

REMARK 5.15 Notice that the identification

$$\pi_X^G : |F^G(\mathcal{S}(X), M)| \twoheadrightarrow \mathbb{F}^G(X, M)$$

factors as the composite

$$\rho_{X*}^G \circ \psi_M^G : |F^G(\mathcal{S}(X), M)| \longrightarrow \mathbb{F}^G(|\mathcal{S}(X)|, M) \longrightarrow \mathbb{F}^G(X, M)\,.$$

Lemma 5.16 *The following is a commutative diagram*

$$\begin{array}{ccc} |F^G(K,L)| & \xrightarrow{\mathrm{id}} & |F^G(K,M_L)| \\ {\scriptstyle \psi_L^G}\downarrow & & \downarrow{\scriptstyle \psi_{M_L}^G} \\ F^G(|K|,L) & \xrightarrow[\mathrm{id}]{} & \mathbb{F}^G(|K|,M_L)\,. \end{array}$$

Proof If we assume that $G/G_{[\sigma,t]} = \{[g_i] \mid, i = 1, \ldots, r\}$ and $G_{[\sigma,t]}/G_\sigma = \{[h_j] \mid j = 1, \ldots, s\}$, then $G/G_\sigma = \{[g_i h_j] \mid (i,j) = (1,1), \ldots, (r,s)\}$. Thus we can write

$$\gamma_\sigma^G(l) = \sum_{(i,j)=(1,1)}^{(r,s)} (g_i h_j l)(g_i h_j \sigma) \in F^G(K, L)\,.$$

Therefore, $\psi_L^G([\gamma_\sigma^G(l), t]) = \sum_{(i,j)=(1,1)}^{(r,s)} (g_i h_j l)[g_i h_j \sigma, t]$.

On the other hand, $M_{L*}(q_{\sigma,t})(l) = \sum_{j=1}^{s} h_j l$, hence

$$\begin{aligned} \psi_{M_L}^G([\gamma_\sigma^G(l), t]) &= \gamma_{[\sigma,t]}^G\left(\sum_{j=1}^{s} h_j l\right) \\ &= \sum_{i=1}^{r} g_i \left(\sum_{j=1}^{s} (h_j l) g_i [\sigma, t]\right) \\ &= \sum_{i=1}^{r} g_i \left(\sum_{j=1}^{s} (h_j l) g_i h_j [\sigma, t]\right) \\ &= \psi_L^G([\gamma_\sigma^G(l), t])\,, \quad \text{since} \quad h_j \in G_{[\sigma,t]}\,. \end{aligned}$$

□

Proposition 5.17 *If K is a simplicial pointed G-set, then*

$$\mathrm{id} : \mathbb{F}^G(|K|, M_L) \longrightarrow F^G(|K|, L)$$

is a homeomorphism.

Proof To simplify the notation we put $Y = |K|$. Consider the following diagram.

$$\begin{tikzcd}
\mathbb{F}^G(|\mathbb{S}(Y)|, M_L) \arrow[dr, two heads, "\rho^G_{Y*}"'] & {|F^G(\mathbb{S}(Y), M_L)|} \arrow[l, "\psi^G_{M_L}"'] \arrow[r, equal] \arrow[d, two heads, "\pi^G_Y"] & {|F^G(\mathbb{S}(Y), L)|} \arrow[r, "\psi^G_L", "\cong"'] \arrow[d, two heads, "\widetilde{\pi}^G_Y"] & F^G(|\mathbb{S}(Y)|, L) \arrow[dl, two heads, "\widetilde{\rho}^G_{Y*}"] \\
 & \mathbb{F}^G(Y, M_L) \arrow[r, "\mathrm{id}"'] & F^G(Y, L)\,. &
\end{tikzcd}$$

The triangles commute by Remark (5.15) and the commutativity of the square follows from Lemma (5.10) and Lemma (5.16). On the other hand, by [2, 3.5], $\rho_Y : |\mathbb{S}(Y)| \longrightarrow Y$ is a G-retraction and, therefore, $\widetilde{\rho}^G_{Y*}$ is a retraction too, moreover ψ^G_L is a homeomorphism (see [1, Prop. 2.3]) and hence $\widetilde{\pi}^G_Y$ is an identification. Since by definition π^G_Y is an identification, it follows that the identity on the bottom is a homeomorphism. $\square$

As a consequence, we have the following.

Corollary 5.18 *For any pointed G-space X,*

$$\mathrm{id} : \mathbb{F}^G(|\mathbb{S}(X)|, M_L) \longrightarrow F^G(|\mathbb{S}(X)|, L)$$

is a homeomorphism. $\square$

We have the next.

Proposition 5.19 *Let M be a homological Mackey functor. Then*

$$\psi^G_M : |F^G(\mathbb{S}(X), M)| \longrightarrow \mathbb{F}^G(|\mathbb{S}(X)|, M)$$

is an isomorphism of topological groups.

Proof Consider the following diagram

$$\begin{tikzcd}
 & {|F^G(\mathbb{S}(|\mathbb{S}(X)|), M_L)|} \arrow[r, two heads, "|\xi_\diamond|"] \arrow[d, two heads, "\pi^G_{|\mathbb{S}(X)|}"'] & {|F^G(\mathbb{S}(|\mathbb{S}(X)|), M)|} \arrow[d, two heads, "\pi^G_{|\mathbb{S}(X)|}"] \\
F^G(|\mathbb{S}(X)|, L) & \mathbb{F}^G(|\mathbb{S}(X)|, M_L) \arrow[l, "\mathrm{id}"', "\cong"] \arrow[r, "\xi_\diamond"] & \mathbb{F}^G(|\mathbb{S}(X)|, M) \\
{|F^G(\mathbb{S}(X), L)|} \arrow[u, "\psi^G_L", "\cong"'] \arrow[r, leftrightarrow, "\cong", "\mathrm{id}"'] & {|F^G(\mathbb{S}(X), M_L)|} \arrow[u, "\psi^G_{M_L}"] \arrow[r, two heads, "|\xi_\diamond|"'] & {|F^G(\mathbb{S}(X), M)|}\,. \arrow[u, dashed, "\psi^G_M"']
\end{tikzcd}$$

The subdiagram on the left commutes by Lemma (5.16), and the identity on the top of it is a homeomorphism by Corollary (5.18). One easily verifies that the other two subdiagrams commute too. Since $\xi_\Diamond$ is surjective, $|\xi_\Diamond|$ on the top is an identification (see [5]), hence $\xi_\Diamond$ in the middle is also an identification. Since $|\xi_\Diamond|$ on the bottom is an identification too and ψ^G_L is a homeomorphism, as mentioned in (5.14), ψ^G_M is a homeomorphism as well. $\square$

Proposition 5.20 *Let X be a pointed G-space of the homotopy type of a G-CW-complex, and let M be a homological Mackey functor. Then $\pi^G_X : F^G(X;M) \longrightarrow \mathbb{F}^G(X,M)$ is a natural homotopy equivalence of topological groups.*

Proof By [1, Prop. 2.12], $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is a G-homotopy equivalence. On the other hand, by Proposition (5.19), ψ^G_M is an isomorphism of topological groups, and by (5.6) the functor $\mathbb{F}^G(-,M)$ is homotopy invariant. Therefore, by Remark (5.15),

$$\pi^G_X : F^G(X,M) = |F^G(\mathcal{S}(X),M)| \xrightarrow{\psi^G_M} \mathbb{F}^G(|\mathcal{S}(X)|,M) \xrightarrow{\rho^G_{X*}} \mathbb{F}^G(X,M)$$

is a homotopy equivalence of topological groups.

It is easy to see that the homormorphisms π^G_X are natural, namely that if $f : X \longrightarrow Y$ is a pointed G-map, then the following diagram commutes:

$$\begin{tikzcd}
|F^G(\mathcal{S}(X),M)| \arrow[r, "|\mathcal{S}(f)^G_*|"] \arrow[d, "\pi^G_X"'] & |F^G(\mathcal{S}(Y),M)| \arrow[d, "\pi^G_Y"] \\
\mathbb{F}^G(X,M) \arrow[r, "f^G_*"'] & \mathbb{F}^G(Y,M)\,.
\end{tikzcd}$$

$\square$

6 Continuity of the transfers

In this section we study the continuity of the transfer for the topological-group functors $F^G(-,M)$ and $\mathbb{F}^G(-,M)$. The following is the topological counterpart of Definition (3.1). Let $p : E \longrightarrow X$ be an *n-fold covering G-map*, i.e., an ordinary n-fold covering map, such that E and X are G-spaces and p is equivariant. Hence $\mathcal{S}(p) : \mathcal{S}(E) \longrightarrow \mathcal{S}(X)$ has finite fibers. We have the following.

Proposition 6.1 *The transfers*

$$t^G_{\mathcal{S}_q(p)} : F^G(\mathcal{S}_q(X)^+,M) \longrightarrow F^G(\mathcal{S}_q(E)^+,M)$$

determine a homomorphism of simplicial abelian groups

$$t^G_{\mathcal{S}(p)} : F^G(\mathcal{S}(X)^+, M) \longrightarrow F^G(\mathcal{S}(E)^+, M)\,.$$

Proof Let $f : \mathbf{r} \longrightarrow \mathbf{q}$ be a morphism in Δ and consider the diagram

$$\begin{tikzcd} \mathcal{S}_q(E) \arrow[r, "f^{\mathcal{S}(E)}"] \arrow[d, "\mathcal{S}_q(p)"'] & \mathcal{S}_r(E) \arrow[d, "\mathcal{S}_r(p)"] \\ \mathcal{S}_q(X) \arrow[r, "f^{\mathcal{S}(X)}"'] & \mathcal{S}_r(X)\,. \end{tikzcd}$$

Take $\sigma \in \mathcal{S}_q(X)$. If $\mathcal{S}_q(p)^{-1}(\sigma) = \{\widetilde{\sigma}_i \mid i = 1, \ldots, n\}$, then $\mathcal{S}_r(p)^{-1}(f^{\mathcal{S}(X)}(\sigma)) = \{\widetilde{\sigma}_i \circ f_\# \mid i = 1, \ldots, n\}$. Therefore this is a pullback diagram. By Theorem (3.7), the following is a commutative diagram:

$$\begin{tikzcd} F^G(\mathcal{S}_q(E)^+, M) \arrow[r, "(f^{\mathcal{S}(E)^+})^G_*"] & F^G(\mathcal{S}_r(E)^+, M) \\ F^G(\mathcal{S}_q(X)^+, M) \arrow[r, "(f^{\mathcal{S}(X)^+})^G_*"'] \arrow[u, "t^G_{\mathcal{S}_q(p)}"] & F^G(\mathcal{S}_r(X)^+, M) \arrow[u, "t^G_{\mathcal{S}_r(p)}"'] \end{tikzcd}$$

and thus $t^G_{\mathcal{S}(p)} : F^G(\mathcal{S}(X)^+, M) \longrightarrow F^G(\mathcal{S}(E)^+, M)$ is a homomorphism of simplicial groups. □

Hence we have the following.

Definition 6.2 Let $p : E \longrightarrow X$ be an n-fold covering G-map. Define the transfer $t^G_p : F^G(X^+, M) \longrightarrow F^G(E^+, M)$ by

$$t^G_p = |t^G_{\mathcal{S}(p)}|\,.$$

(Notice that for any space X, one has $\mathcal{S}_n(X^+) = \mathcal{S}_n(X)^+$.)

Thus we have the next result.

Theorem 6.3 *The transfer $t^G_p : F^G(X^+, M) \longrightarrow F^G(E^+, M)$ is a continuous homomorphism.* □

Let now M be a homological Mackey functor. We shall now give a description of the transfer for the functor $\mathbb{F}^G(-, M)$.

Let $p : E \longrightarrow X$ be an n-fold covering G-map. By (3.1), we have a transfer $t^G_p : F^G((X^\delta)^+, M) \longrightarrow F^G((E^\delta)^+, M)$, which is a homomorphism $t^G_p : \mathbb{F}^G(X^+, M) \longrightarrow \mathbb{F}^G(E^+, M)$.

Theorem 6.4 *The transfer $t_p^G : \mathbb{F}^G(X^+, M) \longrightarrow \mathbb{F}^G(E^+, M)$ is continuous.*

Proof The continuity of t_p^G follows from the commutativity of the next diagram:

$$\begin{tikzcd}
{|F^G(\mathcal{S}(X^+), M)|} \arrow[r, "|t^G_{\mathcal{S}(p)}|"] \arrow[d, "\psi^G_M"'] \arrow[dd, bend right=60, "\pi^G_X"'] & {|F^G(\mathcal{S}(E^+), M)|} \arrow[d, "\psi^G_M"] \arrow[dd, bend left=60, "\pi^G_E"] \\
F^G(|\mathcal{S}(X^+)|, M) \arrow[r, "t^G_{|\mathcal{S}(p|)}"] \arrow[d, "\rho^G_{X^+*}"'] & F^G(|\mathcal{S}(E^+)|, M) \arrow[d, "\rho^G_{E^+*}"] \\
\mathbb{F}^G(X^+, M) \arrow[r, "t^G_p"'] & \mathbb{F}^G(E^+, M)\,.
\end{tikzcd}$$

The square at the bottom commutes by the pullback property (3.7) applied to the pullback diagram

$$\begin{tikzcd}
{|\mathcal{S}(E)|} \arrow[r, "\rho_E"] \arrow[d, "\mathcal{S}(p)"'] & E \arrow[d, "p"] \\
{|\mathcal{S}(X)|} \arrow[r, "\rho_X"'] & X\,.
\end{tikzcd}$$

To see that this is indeed a pullback square, we shall show that for each $[\tau, t] \in |\mathcal{S}(X)|$, the fiber $|\mathcal{S}(p)|^{-1}([\tau, t])$ is mapped bijectively by ρ_E onto the fiber $p^{-1}(\tau(t))$. So, assume first that (σ, t) is a nondegenerate representative of $[\sigma, t]$. Since p is an n-fold covering map, the fiber $\mathcal{S}(p)^{-1}(\tau)$ has n elements, namely $\{\widetilde{\tau}_1, \ldots, \widetilde{\tau}_n\}$. We have a bijection $\mathcal{S}(p)^{-1}(\tau) \approx |\mathcal{S}(p)|^{-1}([\tau, t])$ given by $\widetilde{\tau}_j \leftrightarrow [\widetilde{\tau}_j, t]$. On the other hand, since p is a covering map, there is a bijection $\mathcal{S}(p)^{-1}(\tau) \approx p^{-1}(\tau(t))$ given by $\widetilde{\tau}_j \leftrightarrow \widetilde{\tau}_j(t)$.

To prove that the diagram at the top commutes, we consider the inverse isomorphisms φ^G_M of ψ^G_M, given by $\varphi^G_M(\gamma^G_{[\sigma,t]}(l)) = [\gamma^G_\sigma(l), t]$ provided that (σ, t) is a nondegenerate representative. We shall show that

$$|t^G_{\mathcal{S}(p)}| \circ \varphi^G_M = \varphi^G_M \circ t^G_{|\mathcal{S}(p)|}\,.$$

Take $\gamma^G_{[\sigma,t]}(l) \in F^G(|\mathcal{S}(X^+)|, M)$. Then

$$|t^G_{\mathcal{S}(p)}|\varphi^G_M(\gamma^G_{[\sigma,t]}(l)) = \left[t^G_{\mathcal{S}(p)}(\gamma^G_\sigma(l)), t\right] = \left[\sum_{i=1}^{k} \gamma^G_{\widetilde{\sigma}_i} M^*(\widehat{\mathcal{S}(p)}_{\widetilde{\sigma}_i})(l), t\right]$$

and

$$\varphi_M^G t^G_{|\mathcal{S}(p)|}(\gamma^G_{[\sigma,t]}(l)) = \varphi_M^G \left(\sum_{i=1}^{k} \gamma^G_{[\widetilde{\sigma}_i,t]} M^*(\widehat{|\mathcal{S}(p)|}_{[\widetilde{\sigma}_i,t]})(l) \right)$$
$$= \left[\sum_{i=1}^{k} \gamma^G_{\widetilde{\sigma}_i} M^*(\widehat{|\mathcal{S}(p)|}_{[\widetilde{\sigma}_i,t]})(l), t \right] ,$$

where $\{\widetilde{\sigma}_i \mid i = 1, \ldots, k\}$ is a set of representatives of $\mathcal{S}(p)^{-1}(\sigma)/G_\sigma$. To prove that the sums are equal, observe that, as we already mentioned above, there is a bijection between $\mathcal{S}(p)^{-1}(\sigma)$ and $|\mathcal{S}(p)|^{-1}([\sigma,t])$. Since (σ,t) is nondegenerate, by [2, Prop. 2.4], the isotropy groups G_σ and $G_{[\sigma,t]}$ are equal. Hence, $\{[\widetilde{\sigma}_i,t] \mid i = 1, \ldots, k\}$ is a set of representatives of $|\mathcal{S}(p)|^{-1}([\sigma,t])/G_{[\sigma,t]}$. Moreover, since $(\widetilde{\sigma}_i,t)$ is also nondegenerate, then $G_{[\widetilde{\sigma}_i,t]} = G_{\widetilde{\sigma}_i}$, and therefore $\widehat{|\mathcal{S}(p)|}_{[\widetilde{\sigma}_i,t]} = \widehat{\mathcal{S}(p)}_{\widetilde{\sigma}_i}$. □

7 Homotopical homology theories

In the definition of the functors $F(-,M)$, $F^G(-,M)$, and $\overline{F}^G(-,M)$, given in Section 2, the contravariant structure of the Mackey functor M was not used. Therefore the same definitions are valid if instead of M, we take a covariant coefficient system N_* for the finite group G. Hence we have functors $F(-,N_*)$, $F^G(-,N_*)$, and $\overline{F}^G(-,N_*)$. We shall prove the following.

Theorem 7.1 *Let N_* be a covariant coefficient system for G and let X be a pointed G-space. Then the homotopy groups $\pi_q\big(F^G(X,N_*)\big)$ are naturally isomorphic to the (reduced) Bredon-Illman G-equivariant homology groups $\widetilde{H}^G_q(X;N_*)$.*

For the proof of this theorem we need the following result.

Theorem 7.2 ([2, Thm. 4.5]) *There is an isomorphism between Illman's chain complex $S^G(X,*;N_*)$ (cf. [6, p. 15]) and the chain complex $F^G(\mathcal{S}(X),N_*)$.* □

Proof of Theorem (7.1). We shall give an isomorphism

$$\widetilde{H}^G_q(X;N_*) \cong H_q\big(F^G(\mathcal{S}(X),N_*)\big) \longrightarrow \pi_q\big(F^G(X,N_*)\big) .$$

Here the left-hand side is the Bredon-Illman (reduced) homology of X with coefficients in N_*, which by definition is the homology of the chain complex $S^G(X,*;N_*)$, and the first isomorphism follows from the natural isomorphism of Theorem (7.2).

To construct the arrow, we shall give several isomorphisms as depicted in the following diagram.

$$\begin{tikzcd} H_q\big(F^G(\mathcal{S}(X),N_*)\big) \arrow[dr, dotted] & \pi_q\big(F^G(\mathcal{S}(X),N_*)\big) \arrow[l, "i_*"', "\cong"] \arrow[r, "\Psi", "\cong"'] & \pi_q\big(\mathcal{S}(|F^G(\mathcal{S}(X),N_*)|)\big) \arrow[d, "\Phi", "\cong"'] \\ & \pi_q\big(F^G(X,N_*)\big) \arrow[r, equal] & \pi_q\big(|F^G(\mathcal{S}(X),N_*)|\big) \end{tikzcd}$$

By [2, Prop. 4.2], i_* is an isomorphism. In particular, this shows that every cycle in $\widetilde{H}^G(X;N_*)$ is represented by a chain u, all of whose faces are zero. We call this a *special chain.*

The homomorphism Ψ, which is given by $\Psi(u)[t]=[u,t]$, where u is a special q-chain and $t\in\Delta^q$, is an isomorphism, as follows from [8, 16.6].

In order to define Φ, we must express $\Psi(u)$ as a map $\gamma:(\Delta[q],\dot{\Delta}[q])\longrightarrow\big(\mathcal{S}\big(|F^G(\mathcal{S}(X),N_*)|\big),*\big)$. By the Yoneda lemma, γ is the unique map such that $\gamma(\delta_q)=\Psi(u)$, where $\delta_q=\mathrm{id}:\mathbf{q}\longrightarrow\mathbf{q}$. The homomorphism Φ, defined by $\Phi[\gamma][f,s]=\gamma(f)(s)$, for $f\in\Delta[q]_n$ and $s\in\Delta^n$, is given by the adjunction between the realization functor and the singular complex functor (see [8, 16.1]). $\square$

Proposition 7.3 *The functors $\overline{F}^G(-,M)$ and $F^G(-,\overline{M}_*)$ from G-$\mathcal{S}\mathrm{et}_*$ to $\mathcal{A}\mathrm{b}$ are the same.*

Proof Since the covariant functors M_* and $\overline{M}_*$ are equal in objects, then the groups $\overline{F}^G(C,M)$ and $F^G(C,\overline{M}_*)$ are equal. We shall see that on morphisms, these functors are also equal. For this, let $f:C\longrightarrow D$ be a pointed G-function and take $x\in C$. Consider the canonical projection $G/G_x\longrightarrow G/G_{f(x)}$ with fiber $G_{f(x)}/G_x$. Let us write $G/G_{f(x)}=\{[g_i]\mid i=1,\dots,r\}$ and $G_{f(x)}/G_x=\{[h_j]\mid j=1,\dots,k\}$. Therefore, $G/G_x=\{[g_ih_j]\mid i=1,\dots,r, j=1,\dots,k\}$. Take a generator $\gamma_x^G(l)\in\overline{F}^G(C,M)=F^G(C,\overline{M}_*)$. Then on the one hand,

$$\begin{aligned} f_*^G(\gamma_x^G(l)) &= \gamma_{f(x)}^G(\overline{M}_*(\widehat{f}_x)(l)) \\ &= \sum_i g_i\overline{M}_*(\widehat{f}_x)(l)(g_if(x)) \\ &= \sum_i [G_{f(x)}:G_x]g_iM_*(\widehat{f}_x)(l)(g_if(x))\,. \end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{f}_*^G(\gamma_x^G(l)) &= f_*(\gamma_x^G(l))\\ &= f_*\left(\sum_{i,j}(g_ih_jl)(g_ih_jx)\right)\\ &= \sum_{i,j} M_*(\widehat{f}_{g_ih_jx})(g_ih_jl)f(g_ih_jx)\,.\end{aligned}$$

Since $h_j \in G_{f(x)}$ and by the formula $gM_*(\widehat{f}_x)(l) = M_*(\widehat{f}_{gx})(gl)$ given in Definition (2.5), we have

$$\overline{f}_*^G(\gamma_x^G(l)) = \sum_{i,j} g_iM_*(\widehat{f}_x)(l)g_if(x) = \sum_i [G_{f(x)}\!:\!G_x]g_iM_*(\widehat{f}_x)(l)(g_if(x))\,.$$

□

Corollary 7.4 $\overline{F}^G(X,M) = F^G(X,\overline{M}_*)$ *when* X *is a pointed* G*-space.*

Proof By the previous proposition, the simplicial groups $\overline{F}^G(\mathcal{S}(X),M)$ and $F^G(\mathcal{S}(X),\overline{M}_*)$ are equal. Therefore their geometric realizations are equal as topological groups, and thus the result follows. □

By Theorem (7.1) and the previous proposition, we have the following result.

Theorem 7.5 *Let* M *be a Mackey functor and* X *a pointed* G*-space. Then the homotopy groups* $\pi_q\big(\overline{F}^G(X,M)\big)$ *are naturally isomorphic to the (reduced) Bredon-Illman* G*-equivariant homology groups* $\widetilde{H}_q^G(X;\overline{M}_*)$ *with coefficients in the coefficient system* $\overline{M}_*$*.* □

As a consequence of Proposition (5.20), the homotopy invariance (5.6), and Theorem (7.1), we have the following.

Theorem 7.6 *Let* M *be a homological Mackey functor and* X *a pointed* G*-space of the homotopy type of a* G*-CW-complex. Then the homotopy groups* $\pi_q\big(\mathbb{F}^G(X,M)\big)$ *are naturally isomorphic to the (reduced) Bredon-Illman* G*-equivariant homology groups* $\widetilde{H}_q^G(X;M_*)$ *with coefficients in the coefficient system* M_**.* □

8 Some applications

We shall consider in this section a special family of finite covering G-maps and study the transfer homomorphism for this family.

Definition 8.1 Let G and Γ be two finite groups. A (G,Γ)-*bundle* is a principal Γ-bundle $p: E \longrightarrow X$, such that E and X are G-spaces, p is equivariant, and the actions satisfy

$$g(a\gamma) = (ga)\gamma \quad \text{for all} \quad g \in G\,,\ a \in E\,,\ \gamma \in \Gamma\,. \tag{8.2}$$

Two (G,Γ)-bundles over X are (G,Γ)-*equivalent* if they are Γ-equivalent via a G-equivariant bundle map.

Example 8.3 Let G and Γ be two finite groups, let $\xi: G \longrightarrow \Gamma$ be a homomorphism, and let X be a G-space. Then we may consider the first projection $X \times \Gamma \longrightarrow X$. Define a G-action on $X \times \Gamma$ by $g(x,\gamma) = (gx, \xi(g)\gamma)$. Then we obtain a (G,Γ)-bundle, which we denote by p_ξ.

Observe that in this case the isotropy group $G_{(x,\gamma)} = G_x \cap \ker\xi$ for all $\gamma \in \Gamma$. Note that for any finite covering G-map $p: E \longrightarrow X$, the inclusion $j: p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $\overline{j}: p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$.

Lemma 8.4 *Let N_x be the cardinality of $p_\xi^{-1}(Gx)/G \approx p^{-1}(x)/G_x$. Then the index $[G_x : G_x \cap \ker\xi] = |\Gamma|/N_x$.*

Proof There is a G-bijection between $p_\xi^{-1}(Gx)$ and $G/G_x \times \Gamma$ given by the correspondence $(gx,\gamma) \leftrightarrow ([g],\gamma)$, where G acts on $p_\xi^{-1}(Gx)$ by $g'(gx,\gamma) = (g'gx, \xi(g')\gamma)$ and on $G/G_x \times \Gamma$ by $g'([g],\gamma) = ([g'g], \xi(g')\gamma)$. Thus the orbit of $([g],\gamma)$ has $[G : G_x \cap \ker\xi]$ elements. Hence, the cardinality of $G/G_x \times \Gamma$ is

$$[G:G_x]|\Gamma| = N_x[G:G_x \cap \ker\xi]\,.$$

Therefore,

$$[G_x : G_x \cap \ker\xi] = [G : G_x \cap \ker\xi]/[G:G_x] = |\Gamma|/N_x\,.$$

□

Definition 8.5 Let G and Γ be two finite groups. A (G,Γ)-bundle $p: E \longrightarrow X$ is said to be a (G,Γ)-*locally trivial bundle* if for each $x_0 \in X$ there is a G_{x_0}-invariant neighborhood U_{x_0}, such that the restricted bundle $p^{-1}U_{x_0} \longrightarrow U_{x_0}$ is (G_{x_0},Γ)-equivalent to $p_{\xi_{x_0}} : U_{x_0} \times \Gamma \longrightarrow U_{x_0}$, for some homomorphism $\xi_{x_0} : G_{x_0} \longrightarrow \Gamma$, as in Example (8.3).

Remark 8.6 Lashof [7] gave a different condition for (G,Γ)-local triviality. However, he showed that his condition implies the definition above. He also constructed a universal (G,Γ)-bundle to classify numerable (G,Γ)-locally trivial bundles.

On the other hand, any principal (G,Γ)-bundle over a completely regular base space is a (G,Γ)-locally trivial bundle (see [7]).

Example 8.7 Let G be a finite group and let X be a *bi-G-space*, namely a space with a left and a right G-action such that for any $x \in X$ and $g, g' \in G$, $(gx)g' = g(xg')$. Let $K \subset H \subset G$ be subgroups such that K is normal in H, and assume that the right action of H on X is free. Put $\Gamma = H/K$. Then we can define a principal (G,Γ)-bundle as follows. Let $p : X/K \longrightarrow X/H$ be the canonical projection. One can easily verify that G acts on the left on both X/K and X/H in the obvious way, and that there is a free right Γ-action on X/K using the right action of G.

The bi-G-action on X implies that condition (8.2) is satisfied. Assume now that X is completely regular (and Hausdorff). One can show that X/H is also completely regular. Therefore we have that $p : X/K \longrightarrow X/H$ is a (G,Γ)-locally trivial bundle.

Lemma 8.8 *Let $p : E \longrightarrow X$ be a (G,Γ)-locally trivial bundle and take $x_0 \in X$. Then the index $[G_{x_0} : G_{x_0} \cap \ker \xi_{x_0}] = |\Gamma|/N_{x_0}$, where N_{x_0} is the cardinality of $p^{-1}(x_0)/G_{x_0}$, as in Lemma (8.4).*

Proof Let U_{x_0} be a neighborhood of x_0 as in Definition (8.5). Then the restricted bundle $p^{-1}U_{x_0} \longrightarrow U_{x_0}$ is (G_{x_0},Γ)-equivalent to $p_{\xi_{x_0}} : U_{x_0} \times \Gamma \longrightarrow U_{x_0}$. Thus the desired formula follows from Lemma (8.4). □

Theorem 8.9 *For any finite covering G-map $p : E \longrightarrow X$ and a homological Mackey functor M one has the following formula*

$$p_*^G t_p^G(\gamma_x^G(l)) = \sum_{\kappa \in K} [G_x : G_{a_\kappa}] \gamma_x^G(l) \in \mathbb{F}^G(X, M)\,, \tag{8.10}$$

where $p^{-1}(x)/G_x = \{[a_\kappa] \mid \kappa \in K\}$.

Proof By equation (3.5), we can write

$$p_*^G t_p^G(\gamma_x^G(l)) = \sum_{\kappa \in K} p_*^G \gamma_{a_\kappa}^G M^*(\widehat{p}_{a_\kappa})(l) = \sum_{\kappa \in K} \gamma_x^G M_*(\widehat{p}_{a_\kappa}) M^*(\widehat{p}_{a_\kappa})(l)\,.$$

Since the composite $M_*(\widehat{p}_{a_\kappa}) \circ M^*(\widehat{p}_{a_\kappa})$ is multiplication by $[G_x : G_{a_\kappa}]$, the result follows. □

We now have the following consequence of Theorem (8.9) and Lemma (8.8).

Theorem 8.11 *Let $p : E \longrightarrow X$ be a (G,Γ)-locally trivial bundle and let M be a homological Mackey functor. Then one has that each of the composites*

$$p_*^G \circ t_p^G : \mathbb{F}^G(X^+, M) \longrightarrow \mathbb{F}^G(X^+, M) \quad \text{and}$$

$$p_*^G \circ t_p^G : H_*^G(X, M) \cong \pi_q\big(\mathbb{F}^G(X^+, M)\big) \longrightarrow \pi_q\big(\mathbb{F}^G(X^+, M)\big) \cong H_*^G(X, M)\,,$$

is multiplication by $|\Gamma|$.

Proof We only have to prove the result for the composite on the top. By (8.10), if $v = \gamma_{x_0}^G(l) \in \mathbb{F}^G(X^+, M)$, then

$$p_*^G t_p^G(\gamma_{x_0}^G(l)) = \sum_{\kappa \in K} [G_{x_0} : G_{x_0} \cap \ker \xi_{x_0}] \gamma_x^G(l)\,,$$

where $\{[a_\kappa] \mid \kappa \in K\} = p^{-1}(x_0)/G_{x_0}$. By Lemma (8.8), $[G_{x_0} : G_{x_0} \cap \ker \xi_{x_0}] = |\Gamma|/N_{x_0}$, and since N_{x_0} is the cardinality of K, $p_*^G t_p^G(\gamma_{x_0}^G(l)) = |\Gamma| \gamma_{x_0}^G(l)$. Since any element $v \in \mathbb{F}^G(X^+, M)$ is a sum of terms of the form $\gamma_{x_0}^G(l)$, the result follows. □

Instituto de Matemáticas
Universidad Nacional Autónoma de México
04510 México, D.F.
Mexico
marcelo@math.unam.mx, cprieto@math.unam.mx

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