Simplicial ramified covering maps

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Abstract In this paper we define the concept of a ramified covering map in the category of simplicial sets and we show that it has properties analogous to those of the topological ramified covering maps. We show that the geometric realization of a simplicial ramified covering map is a topological ramified covering map, and we also consider the relation with ramified covering maps in the category of simplicial complexes.

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0 Introduction

In [9], L. Smith defined an *n-fold ramified covering map* in the category of topological spaces as a finite-to-one map $p : E \longrightarrow X$ together with a multiplicity function $\mu : E \longrightarrow \mathbb{N}$ which have the following two properties:

1. For each $x \in X$, $\sum_{a\in p^{-1}(x)} \mu(a) = n$.
2. The map $\varphi_p : X \longrightarrow \mathrm{SP}^n E$ given by

$$\varphi_p(x) = \langle \underbrace{a_1, \dots, a_1}_{\mu(a_1)}, \dots, \underbrace{a_r, \dots, a_r}_{\mu(a_r)} \rangle \,,$$

   is continuous, where $p^{-1}(x) = \{a_1, \dots, a_r\}$, and $\mathrm{SP}^n E$ denotes the nth symmetric product defined as the quotient of the product E^n by the action of the symmetric group in n letters.

Examples of ramified covering maps include branched coverings ([1]) and orbit maps of the action of a finite group on spaces.

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In this paper the word simplicial will have two meanings. In the first four sections it will refer to simplicial sets, and in Section 5 it will refer to simplicial complexes.

In Section 1, we introduce the concept of an n-fold ramified covering map in the category of simplicial sets and we show that it has properties analogous to those proved by Smith in the topological case. In Section 2 we give an alternative characterization of an ordinary covering map in the category of simplicial sets $\mathcal{S}$impset (see Corollary 2.4). In Section 3 we show that the simplicial ramified covering maps have properties analogous to those proved by Dold [2] for topological ramified covering maps. In Section 4 we show that the geometric realization of a simplicial ramified covering map is a topological ramified covering map.

The geometric realization functor $|\cdot| : \mathcal{S}\mathrm{impset} \longrightarrow \mathcal{T}\mathrm{op}$ has a right adjoint functor $\mathcal{S} : \mathcal{T}\mathrm{op} \longrightarrow \mathcal{S}\mathrm{impset}$ given by the singular simplicial set associated to a space. The simplicial map $\mathcal{S}(p) : \mathcal{S}(E) \longrightarrow \mathcal{S}(X)$ associated to a topological ramified covering map $p : E \longrightarrow X$ is not in general a simplicial ramified covering map, because it is not always finite-to-one (see Example 5.1). However, if we work in the category of simplicial complexes, then there is another functor K from this category to $\mathcal{S}\mathrm{impset}$. In Section 5 we define the concept of an n-fold ramified covering map $p : C \longrightarrow D$ in the category of simplicial complexes and we show that the associated map of simplicial sets $\widehat{p} : K(C) \longrightarrow K(D)$ is a ramified covering map. Furthermore, we show that the map of triangulated spaces $|p| : |C| \longrightarrow |D|$ is a topological ramified covering map.

1 Simplicial ramified covering maps

We refer the reader to [7], [4], or [5] for the theory of simplicial sets. Given a simplicial set S, we shall denote by $d_i : S_m \longrightarrow S_{m-1}$ the face operators, and by $s_i : S_m \longrightarrow S_{m+1}$ the degeneracy operators.

Definition 1.1 Let $p : K \longrightarrow Q$ be a map of simplicial sets. We say that p is a *simplicial n-fold ramified covering map*, if the following hold:

1. For each m, $p_m : K_m \longrightarrow Q_m$ has finite fibers.
2. The restricted function $d_i|_{p_m^{-1}(x)} : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i(x))$ is surjective for all i.
3. There is a family of *multiplicity functions* $\mu_m : K_m \longrightarrow \mathbb{N}$, such that:
   (a) For all $x \in Q_m$, one has $\sum_{a\in p_m^{-1}(x)} \mu_m(a) = n$.

(b) $\mu_{m+1} \circ s_i = \mu_m : K_m \longrightarrow \mathbb{N}$.

(c) For all x and $a \in p_m^{-1}(x)$ one has $\mu_{m-1}(d_i(a)) = \sum_{\alpha=1}^{l} \mu_m(a_\alpha)$, where $\{a_1, \ldots, a_l\} = (d_i)^{-1}(d_i(a)) \cap p_m^{-1}(x)$.

Remark 1.2 Properties 3 (a) and (b) imply that the restrictions $s_i| : p_m^{-1}(x) \longrightarrow p_{m+1}^{-1}(s_i(x))$ are bijective.

Remark 1.3 Definition 1.1 is based on the concept of weighted map given in [3].

The proof of the following result is straightforward.

Proposition 1.4 *Let $p : K \longrightarrow Q$ be a simplicial n-fold ramified covering map, and let $f : Q' \longrightarrow Q$ be a map of simplicial sets. Then the pullback of p over f, $p' : K' = Q' \times_Q K \longrightarrow Q'$, is a simplicial n-fold ramified covering map.* ■

We shall need the following result, which we state for an action of a finite group G on a set X on the left.

Lemma 1.5 *Consider a left action $G \times X \longrightarrow X$, and let $H \subset G$ be a subgroup. If $q : H\backslash X \longrightarrow G\backslash X$ is the canonical surjection of the orbit sets, then there is a bijection*

$$\overline{\delta} : H\backslash G/G_{x_0} \longrightarrow q^{-1}([x_0]_G)\,,$$

where $x_0 \in X$, given by $\overline{\delta}({}_H[g]_{G_{x_0}}) = [gx_0]_H$.

Proof: The function $\overline{\delta}$ is induced by the surjection $\delta : G \longrightarrow q^{-1}([x_0]_G)$ given by $\delta(g) = [gx_0]_H$. One easily checks that δ factors through the set of double cosets and that $\overline{\delta}$ is injective. ■

Proposition 1.6 *Let T be a simplicial set. Then the map of simplicial sets $\pi : T^n \times_{\Sigma_n} \overline{n} \longrightarrow T^n/\Sigma_n$, where $\overline{n} = \{1, 2, \ldots, n\}$, is a simplicial n-fold ramified covering map.*

Proof: T is a contravariant functor $\Delta \longrightarrow \mathcal{S}\mathrm{et}$. Consider the functors

$$E_n : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{S}\mathrm{et} \quad \text{and} \quad B_n : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{S}\mathrm{et}$$

given by $S \longmapsto S^n \times_{\Sigma_n} \overline{n}$ and $S \longmapsto S^n/\Sigma_n$, respectively. Then $T^n \times_{\Sigma_n} \overline{n} = E_n \circ T$ and $T^n/\Sigma_n = B_n \circ T$. The natural transformation $E_n \longrightarrow B_n$ that maps $\langle s_1, \ldots, s_n; j\rangle$ to $\langle s_1, \ldots, s_n\rangle$ determines the map of simplicial sets

$$\pi : T^n \times_{\Sigma_n} \overline{n} \longrightarrow T^n/\Sigma_n\,.$$

Since $\pi_m^{-1}\langle t_1, \ldots, t_n\rangle = \{\langle t_1, \ldots, t_n; j\rangle \mid j \in \overline{n}\}$ for all m and the face functions are the identity on the j-coordinate, conditions 1 and 2 are clearly satisfied.

In what follows, we shall write $t = (t_1, \ldots, t_n)$ and $d_i(t) = (d_i(t_1), \ldots, d_i(t_n))$.

To verify conditions 3 on μ, we define

$$\mu_m : T_m^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{N} \quad \text{by} \quad \mu_m\langle t; j\rangle = \#t^{-1}t(j)\,,$$

where t is seen as a function $t : \overline{n} \longrightarrow T_m$. The different sets $t^{-1}t(j)$ form a partition of the set $\overline{n}$; therefore, we have condition 3 (a).

Condition 3 (b) follows from the fact that the degeneracy functions s_i are always injective.

To see condition 3 (c), take $\langle t\rangle \in T_m^n/\Sigma_n$ and any $\langle t; j\rangle \in T_m^n \times_{\Sigma_n} \overline{n}$, with $t = (t_1, \ldots, t_n)$ a specific representative. We have to prove

$$\mu_{m-1}\langle d_i \circ t; j\rangle = \sum_{r=1}^{l} \mu_m\langle t; \sigma_r(j)\rangle\,.$$

For this purpose, observe first that one has bijections

$$\begin{aligned}\pi_m^{-1}\langle t\rangle &\cong \{(t_1, \ldots, t_n; j) \mid j = 1, \ldots, n\}/(\Sigma_n)_{(t_1,\ldots,t_n)}\\ &\cong (\Sigma_n)_{(t_1,\ldots,t_n)}\backslash\overline{n}\end{aligned}$$

and analogously

$$\begin{aligned}\pi_{m-1}^{-1}\langle d_i \circ t\rangle &\cong \{(d_i(t_1), \ldots, d_i(t_n); j) \mid j = 1, \ldots, n\}/(\Sigma_n)_{(d_i(t_1),\ldots,d_i(t_n))}\\ &\cong (\Sigma_n)_{(d_i(t_1),\ldots,d_i(t_n))}\backslash\overline{n}\end{aligned}$$

where $(\Sigma_n)_{(t_1,\ldots,t_n)}$ and $(\Sigma_n)_{(d_i(t_1),\ldots,d_i(t_n))}$ denote the corresponding isotropy groups. The restriction of d_i to the fiber $\pi_m^{-1}\langle t\rangle$ corresponds to the quotient function $(\Sigma_n)_{(t_1,\ldots,t_n)}\backslash\overline{n} \twoheadrightarrow (\Sigma_n)_{(d_i(t_1),\ldots,d_i(t_n))}\backslash\overline{n}$, whose fiber over a class $[j]$ is, by Lemma 1.5,

$$(\Sigma_n)_j \cap (\Sigma_n)_{(d_i(t_1),\ldots,d_i(t_n))}\backslash(\Sigma_n)_{(d_i(t_1),\ldots,d_i(t_n))}/(\Sigma_n)_{(t_1,\ldots,t_n)}\,.$$

Let the elements of this set be the double cosets $[\sigma_r]$, $r = 1, \ldots, l$. Therefore,

$$(d_i|_{\pi_m^{-1}\langle t\rangle})^{-1}\langle d_i \circ t; j\rangle = \{\langle t; \sigma_r(j)\rangle \mid r = 1, \ldots, l\}\,.$$

Since

$$(d_i \circ t)^{-1}(d_i(t_j)) = t^{-1}(d_i)^{-1}(d_i(t_j)) = t^{-1}\{t_{\sigma_r(j)} \mid r = 1, \ldots, l\}\,,$$

we have

$$\mu_{m-1}\langle d_i \circ t; j\rangle = \#(d_i \circ t)^{-1}(d_i(t_j)) = \sum_{r=1}^{l} \#t^{-1}(t_{\sigma_r(j)}) = \sum_{r=1}^{l} \mu_m\langle t; \sigma_r(j)\rangle\,.$$

■

Definition 1.7 Let $p : K \longrightarrow Q$ be a simplicial ramified covering map with multiplicity functions μ_m. For each m define $\varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n K_m$ by

$$\varphi_{p_m}(x) = \langle \underbrace{a_1, \dots, a_1}_{\mu_m(a_1)}, \dots, \underbrace{a_r, \dots, a_r}_{\mu_m(a_r)} \rangle ,$$

where $p_m^{-1}(x) = \{a_1, \dots, a_r\}$.

The proof of the following result is straightforward.

Proposition 1.8 *The functions $\varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n K_m$ determine a map of simplicial sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$.* ∎

Theorem 1.9 *Let $p : K \longrightarrow Q$ be a simplicial ramified covering map, and take $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$. Then p is the pullback over φ_p of the simplicial ramified covering map $\pi : K^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n K$.*

Proof: Since for each m, $p_m : K_m \longrightarrow Q_m$ together with μ_m is a (discrete) ramified covering map p_m is the pullback of $\pi_m : K_m^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n K_m$ over φ_{p_m}, as shown in [9]. Thus, for each m, consider the pullback diagram

$$\begin{tikzcd} K_m \arrow[r, "\widetilde{\varphi}_{p_m}"] \arrow[d, "p_m"'] & K_m^n \times_{\Sigma_n} \overline{n} \arrow[d, "\pi_m"] \\ Q_m \arrow[r, "\varphi_{p_m}"'] & \mathrm{SP}^n K_m . \end{tikzcd}$$

We shall see that the functions $\widetilde{\varphi}_{p_m} : Q_m \longrightarrow K_m^n \times_{\Sigma_n} \overline{n}$ determine a function of simplicial sets. These functions are explicitly given as follows. Take $a \in K_m$. Write $p_m^{-1}(p_m(a)) = \{a = a_1, a_2, \dots, a_r\}$. Then

$$\widetilde{\varphi}_{p_m}(a) = \langle \underbrace{a_1, \dots, a_1}_{\mu_m(a_1)}, \dots, \underbrace{a_r, \dots, a_r}_{\mu_m(a_r)}; 1 \rangle .$$

Now we can proceed exactly as in the proof of 1.8.

We shall now see that

$$\begin{tikzcd} K \arrow[r, "\widetilde{\varphi}_p"] \arrow[d, "p"'] & K^n \times_{\Sigma_n} \overline{n} \arrow[d, "\pi"] \\ Q \arrow[r, "\varphi_p"'] & \mathrm{SP}^n K \end{tikzcd}$$

is a pullback diagram in the category of simplicial sets. For that purpose, let $\alpha : S \longrightarrow Q$ and $\beta : S \longrightarrow K^n \times_{\Sigma_n} \overline{n}$ be maps of simplicial sets such that

$$\varphi_p \circ \alpha = \pi \circ \beta : S \longrightarrow \mathrm{SP}^n K .$$

Since the diagram is a pullback diagram for each m, there exists a unique $\gamma_m : S_m \longrightarrow K_m$ such that $p_m \circ \gamma_m = \alpha_m$ and $\widetilde{\varphi}_{p_m} \circ \gamma_m = \beta_m$. It is enough to check that the functions γ_m determine a map of simplicial sets $\gamma : S \longrightarrow K$. Given an order-preserving function $f : \mathbf{k} \longrightarrow \mathbf{m}$, in what follows we denote by F the value of any of the functors S, K, or Q. Hence we have to prove that $F \circ \gamma_m = \gamma_k \circ F$. Consider the diagram

$$\begin{tikzcd}
S_m \arrow[r, "\gamma_m"] \arrow[d, "F"'] & K_m \arrow[d, "F"] & \\
S_k \arrow[r, "\gamma_k"'] & K_k \arrow[r, "\widetilde{\varphi}_{p_k}"'] \arrow[d, "p_k"'] & K_k^n \times_{\Sigma_n} \overline{n} \arrow[d, "\pi_k"] \\
& Q_k \arrow[r, "\varphi_{p_k}"'] & \mathrm{SP}^n K_k \,.
\end{tikzcd}$$

Since the square at the bottom right is a pullback diagram, it is enough to show that

$$\varphi_{p_k} \circ p_k \circ (F \circ \gamma_m) = \varphi_{p_k} \circ p_k \circ (\gamma_k \circ F)\,.$$

For the left-hand side we have:

$$\varphi_{p_k} \circ p_k \circ (F \circ \gamma_m) = \varphi_{p_k} \circ F \circ p_m \circ \gamma_m = \varphi_{p_k} \circ F \circ \alpha_m\,.$$

while for the right-hand side we have:

$$\varphi_{p_k} \circ p_k \circ (\gamma_k \circ F) = \varphi_{p_k} \circ \alpha_k \circ F = \varphi_{p_k} \circ F \circ \alpha_m\,,$$

hence the assertion. ■

2 Ordinary covering maps of simplicial sets

Let S be a simplicial set and take $\sigma \in S_m$, then there is a unique map of simplicial sets $f^\sigma : \Delta[m] = \mathrm{mor}_\Delta(-, m) \longrightarrow S$, such that $f^\sigma(\mathrm{id}) = \sigma$.

Definition 2.1 ([4]) Let $p : K \longrightarrow Q$ be a map of simplicial sets. Then p is called a *simplicial n-fold (ordinary) covering map* if for each m and any $x \in Q_m$, there is a pullback diagram in the category of simpicial sets

(2.2)
$$\begin{tikzcd}
\Delta[m] \times \overline{n} \arrow[r] \arrow[d, "\pi"'] & K \arrow[d, "p"] \\
\Delta[m] \arrow[r, "f^x"'] & Q\,,
\end{tikzcd}$$

where f^x is as above and π is the projection.

Theorem 2.3 *Let $p : K \longrightarrow Q$ be a simplicial n-fold ramified covering map, such that each $\mu_m : K_m \longrightarrow \mathbb{N}$ is constant with value 1. Then p is a simplicial n-fold (ordinary) covering map.*

Proof: First note that since μ_m is constant with value 1, all fibers $p_m^{-1}(x)$ have exactly n elements. Define the simplicial set $L(p)$ as follows. Set $L(p)_m = \{(a_1, a_2, \dots, a_n) \in p_m^{-1}(x)^n \mid i \neq j \Longrightarrow a_i \neq a_j,\ x \in Q_m\}$ and define

$$s_i : L(p)_m \longrightarrow L(p)_{m+1} \quad \text{and} \quad d_i : L(p)_m \longrightarrow L(p)_{m-1}$$

by

$$s_i(a_1, \dots, a_n) = (s_i(a_1), \dots, s_i(a_n))$$

and

$$d_i(a_1, \dots, a_n) = (d_i(a_1), \dots, d_i(a_n)) \,.$$

They are well defined. Namely, since p is a simplicial function, both s_i and d_i send fibers into fibers. Moreover, since s_i is always injective and d_i, being surjective in fibers, is also injective, the images of s_i and d_i lie indeed in $L(p)_{m+1}$ and $L(p)_{m-1}$, respectively.

The symmetric group Σ_n acts freely on $L(p)_m$ for all m, therefore, if we look at Σ_n as a simplicial group, we have a principal action of it on $L(p)$. Hence, by [7, Cor. 20.5], we have a simplicial fiber bundle

$$\pi(p) : L(p) \times_{\Sigma_n} \overline{n} \longrightarrow L(p)/\Sigma_n \,.$$

Since its fibers are discrete (finite), it is thus a simplicial (ordinary) covering map. It is easy to check that the simplicial mapping $\langle a_1, \dots, a_n; j\rangle \mapsto a_j$, for each m, determines a simplicial isomorphism

$$\begin{tikzcd} L(p) \times_{\Sigma_n} \overline{n} \arrow[r, "\cong"] \arrow[d, "\pi(p)"'] & K \arrow[d, "p"] \\ L(p)/\Sigma_n \arrow[r, "\cong"'] & Q \,, \end{tikzcd}$$

where the bottom map is given by $\langle a_1, \dots, a_n\rangle \mapsto p_m(a_j)$, for any j. Since $\pi(p)$ is an ordinary covering map, so is p. ■

Corollary 2.4 *Let $p : K \longrightarrow Q$ be a map of simplicial sets, such that for each m, p_m is an n-to-1 function. Then p is a simplicial n-fold (ordinary) covering map if and only if, for each $x \in Q_m$ $(m \geq 0)$, the restrictions of the face operators*

$$d_i| : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i(x))$$

are bijective for all i.

Proof: First note that since the restrictions $s_i : p_m^{-1}(x) \longrightarrow p_{m+1}^{-1}(s_i(x))$ are always injective, under the assumptions, they are bijective.

By hypothesis, the restrictions of d_i to the fibers are also injective; thus, by definition, p is a ramified covering map with multiplicity function μ constant with value 1. Hence, by Theorem 2.3, p is a simplicial n-fold (ordinary) covering map.

Conversely, let p be a simplicial n-fold (ordinary) covering map. Since the face functions of $\Delta[m] \times \overline{n}$ map the fibers of π bijectively, then by diagram (2.2), also the restrictions $d_i| : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i(x))$ are bijective. ■

3 A simplicial analog of Dold's ramified covering maps

In this section we show that the simplicial ramified covering maps have properties analogous to those proved by Dold [2] for topological ramified covering maps.

Theorem 3.1 *Let $p : K \longrightarrow Q$ be a map of simplicial sets. Then p is an n-fold simplicial ramified covering map if and only if there is a map of simplicial sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ such that for each m the following hold:*

1. *If $a \in K_m$, then $a \in \varphi_{p_m}(p_m(a))$.*
2. *The composition $\mathrm{SP}^n p_m \circ \varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n Q_m$ is the diagonal map.*

Proof: Assume first that $p : K \longrightarrow Q$ is an n-fold simplicial ramified covering map. We have to check that φ_p, as given in Definition 1.7 (which by Proposition 1.8 is a map of simplicial sets), satisfies 1 and 2. To see 1, take $a \in K_m$. Since $a \in p_m^{-1}(p_m(a))$, by the definition of φ_{p_m}, $a \in \varphi_{p_m}(p_m(a))$. In order to see 2, take $x \in Q_m$; since

$$\varphi_{p_m}(x) = \langle \underbrace{a_1, \ldots, a_1}_{\mu_m(a_1)}, \ldots, \underbrace{a_r, \ldots, a_r}_{\mu_m(a_r)} \rangle \,,$$

where $p_m^{-1}(x) = \{a_1, \ldots, a_r\}$, then

$$\begin{aligned} \mathrm{SP}^n p_m \varphi_{p_m}(x) &= \mathrm{SP}^n p_m(\langle \underbrace{a_1, \ldots, a_1}_{\mu_m(a_1)}, \ldots, \underbrace{a_r, \ldots, a_r}_{\mu_m(a_r)} \rangle) \\ &= \langle \underbrace{p_m(a_1), \ldots, p_m(a_1)}_{\mu_m(a_1)}, \ldots, \underbrace{p_m(a_r), \ldots, p_m(a_r)}_{\mu_m(a_r)} \rangle \\ &= \langle x, \ldots, x \rangle \in \mathrm{SP}^n Q_m \,. \end{aligned}$$

Conversely, suppose that there is a map simplicial sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ satisfying conditions 1 and 2. Condition 1 implies condition 1 in Definition 1.1. Now, since φ_p is a map of simplicial sets, in particular, the equality $\varphi_{p_{m-1}} \circ d_i = d_i \circ \varphi_{p_m}$ holds. Therefore, for every $x \in Q$, if $\{a_1, \ldots, a_r\} = p_m^{-1}(x)$, then one has, on the one hand,

$$
\begin{aligned}
(3.2) \qquad d_i\varphi_{p_m}(x) &= d_i(\langle a_1, \ldots, a_1, \ldots, a_r, \ldots, a_r\rangle) \\
&= \langle d_i(a_1), \ldots, d_i(a_1), \ldots, d_i(a_r), \ldots, d_i(a_r)\rangle \,,
\end{aligned}
$$

while, on the other hand,

$$
(3.3) \qquad \varphi_{p_{m-1}}(d_i(x)) = \langle b_1, \ldots, b_1, \ldots, b_s, \ldots, b_s\rangle \,,
$$

where $p_{m-1}^{-1}(d(x)) = \{b_1, \ldots, b_s\}$. Hence, every b_k is the image of some a_j under d_i. Therefore $d_i : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i(x))$ is surjective, an thus condition 2 of Definition 1.1 holds.

In order to check condition 3 in Definition 1.1, define $\mu_m : K_m \longrightarrow \mathbb{N}$ as follows. Take $a \in K_m$ and consider $\varphi_{p_m}(p_m(a)) \in \mathrm{SP}^n K_m$. If $(a_1, \ldots, a_n) \in K_m^n$ is a representative of $\varphi_{p_m}(p_m(a))$, define

$$
\mu_m(a) = \#\{i \in \overline{n} \mid a_i = a\} \,.
$$

Clearly 3 (a) holds. Since s_i is always injective, also 3 (b) holds. Finally, 3 (c) follows from the equality of the expressions (3.2) and (3.3). ∎

In the following results, we use the equivalence proved in the previous theorem.

Proposition 3.4 *Let Γ be a finite group and $\Lambda \subset \Gamma$ be a subgroup of index n. If Γ acts simplicially on the right on a simplicial set W, then the orbit map of simplicial sets*

$$
\pi : W/\Lambda \longrightarrow W/\Gamma
$$

is an n-fold simplicial ramified covering map.

Proof: We shall prove that p satisfies conditions 1 and 2 of Theorem 3.1. Let $\varphi_\pi : W/\Gamma \longrightarrow \mathrm{SP}^n W/\Lambda$ be given for $[w]_\Gamma \in W_m/\Gamma$ by

$$
\varphi_{\pi_m}([w]_\Gamma) = \langle [w\gamma_1]_\Lambda, \ldots, [w\gamma_n]_\Lambda\rangle \in \mathrm{SP}^n W_m/\Lambda \,,
$$

where $\Gamma/\Lambda = \{[\gamma_1], \ldots, [\gamma_n]\}$ $(\gamma_1 = e \in \Gamma)$.

Since the action of Γ on W is simplicial, one easily verifies that φ_π is a map of simplicial sets. To see condition 1, take $a = [w]_\Lambda \in W_m/\Lambda$; since $\gamma_1 = e$, $a = [w\gamma_1]_\Lambda \in \varphi_{\pi_m}\pi_m(a) = \varphi_{\pi_m}([w]_\Gamma)$. To see condition 2, take $x = [w]_\Gamma \in W_m/\Gamma$. Then

$$
\mathrm{SP}^n\pi_m\varphi_{\pi_m}(x) = \mathrm{SP}^n\pi_m(\langle [w\gamma_1]_\Lambda, \ldots, [w\gamma_n]_\Lambda\rangle) = \langle [w\gamma_1]_\Gamma, \ldots, [w\gamma_n]_\Gamma\rangle =
$$

$$= \langle [w]_\Gamma, \dots, [w]_\Gamma \rangle = \langle x, \dots, x \rangle \,.$$

■

Conversely, we have the following.

Theorem 3.5 *Let $p : K \longrightarrow Q$ be an n-fold simplicial ramified covering map. Then there exists a simplicial set W with a simplicial action of the symmetric group Σ_n, such that the n-fold simplicial ramified covering map $\pi : W/\Sigma_{n-1} \longrightarrow W/\Sigma_n$ is isomorphic to p.*

Proof: Define the simplicial set W as follows. Take

$$W_m = \{(x; a_1, \dots, a_n) \in Q_m \times K_m^n \mid \varphi_{p_m}(x) = \langle a_1, \dots, a_n \rangle\} \,.$$

Define $d_i : W_m \longrightarrow W_{m-1}$ by $d_i(x; a_1, \dots, a_n) = (d_i(x); d_i(a_1), \dots, d_i(a_n))$, and $s_i : W_m \longrightarrow W_{m+1}$ in the same way.

This is well defined, since φ_p is a map of simplicial sets.

We define a right action of Σ_n on W_m by

$$(x; a_1, \dots, a_n)\sigma = (x; a_{\sigma(1)}, \dots, a_{\sigma(n)}) \,.$$

We consider Σ_{n-1} as the subgroup of those permutations that leave the first coordinate fixed. Let $\alpha_m : W_m/\Sigma_{n-1} \longrightarrow K_m$ and $\beta_m : W_m/\Sigma_n \longrightarrow Q_m$ be given by

$$\alpha_m([x; a_1, \dots, a_n]_{\Sigma_{n-1}}) = a_1 \quad \text{and} \quad \beta_m([x; a_1, \dots, a_n]_{\Sigma_n}) = x \,.$$

Let $\pi_m : W_m/\Sigma_{n-1} \longrightarrow W_m/\Sigma_n$ be the canonical surjection. Clearly $p_m \circ \alpha_m = \beta_m \circ \pi_m$. One can easily check that both α_m and β_m are bijective and determine maps α and β of simplicial sets. ■

4 Geometric realization of simplicial ramified covering maps

We recall that given a simplicial set S, the geometric realization $|S|$ is a CW-complex with one m-cell $\varphi_\sigma : \Delta^m \longrightarrow |S|$ for each nondegenerate simplex $\sigma \in S_m$ given by $\varphi_\sigma(t) = [\sigma, t]$.

Lemma 4.1 *Let $\gamma : S \longrightarrow T$ be a map of simplicial sets, with the property that for each $\tau \in T_m$ and any $s_i : T_m \longrightarrow T_{m+1}$, the restriction of s_i, $\gamma_m^{-1}(\tau) \longrightarrow \gamma_{m+1}^{-1}(s_i(\tau))$, is surjective. Then there is a bijection $\beta : \gamma_m^{-1}(\tau_0) \longrightarrow |\gamma|^{-1}([\tau_0, t_0])$, where $(\tau_0, t_0) \in T_m \times \mathring{\Delta}^m$ is a nondegenerate representative.*

Proof: Let β be given by $\beta(\sigma) = [\sigma, t]$, where $\gamma_m(\sigma) = \tau$.

Clearly, since τ_0 is nondegenerate, so is any $\sigma \in \gamma_m^{-1}(\tau_0)$. Since

$$|S| = \bigsqcup_{\sigma \in S'_m,\, m \geq 0} \varphi_\sigma(\overset{\circ}{\Delta}{}^m)\,,$$

where S'_m is the subset of S_m of nondegenerate elements, then β is injective.

Assume that (ρ, t) is a nondegenerate element such that $|\gamma|([\rho, t]) = [\gamma_l(\rho), t] = [\tau_0, t_0]$. If $\gamma_l(\rho)$ is also nondegenerate, then $l = m$, $t = t_0$, and $\gamma_m(\rho) = \tau_0$. If we now assume that $\gamma_l(\rho)$ is degenerate, then there is a unique nondegenerate element $\tau' \in T_r$ and an iterated degeneracy operator $s : T_r \longrightarrow T_l$ such that $s(\tau') = \gamma_l(\rho)$. By hypothesis, there is an element $\sigma' \in S_r$ such that $s(\sigma') = \rho$. Therefore $[\tau_0, t_0] = [\gamma_l(\rho), t] = [s(\tau'), t] = [\tau', s_\#(t)]$. Since $(\tau', s_\#(t))$ is nondegenerate, $\tau_0 = \tau'$ and $t_0 = s_\#(t)$, so that $r = m$, $\sigma' \in S_m$, and $\gamma_m(\sigma') = \tau_0$. Hence $\beta(\sigma') = [\sigma', t_0] = [\sigma', s_\#(t)] = [s(\sigma'), t] = [\rho, t]$, and thus β is surjective. ∎

Theorem 4.2 *Let $: K \longrightarrow Q$ be a simplicial n-fold ramified covering map. Then $|p| : |K| \longrightarrow |Q|$ is a topological n-fold ramified covering map.*

Proof: By the previous lemma, $|p|$ has finite fibers.

We shall now see that $|p| : |K| \longrightarrow |Q|$ satisfies Dold's definition, namely, the topological version of conditions 1 and 2 of Theorem 3.1.

Consider the map of simplicial sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$. There is a natural homeomorphism $|\mathrm{SP}^n Q| \longrightarrow \mathrm{SP}^n|Q|$ induced by the natural homeomorphism

$$|Q^n| \longrightarrow |Q|^n \quad \text{given by} \quad [(x_1, \ldots, x_n), t] \longmapsto ([x_1, t], \ldots, [x_n, t])\,.$$

Then, the composite

$$|Q| \xrightarrow{|\varphi_p|} |\mathrm{SP}^n K| \xrightarrow{\approx} \mathrm{SP}^n|K|$$

coincides with $\varphi_{|p|}$, as one easily shows. Therefore, $\varphi_{|p|}$ is continuous.

In order to see condition 1, take $[a, t] \in |K|$, $a \in K_m$. Then

$$\varphi_{|p|}|p|([a, t]) = \varphi_{|p|}([p_m(a), t]) = \langle [a_1, t], \ldots, [a_n, t] \rangle\,,$$

where $\varphi_{p_m} p_m(a) = \langle a_1, \ldots, a_n \rangle$. Since by condition 1 in the simplicial case, $a = a_i$ for some i, we have $[a, t] \in \varphi_{|p|}|p|([a, t])$.

Condition 2 follows from the simplicial condition 2, since

$$\mathrm{SP}^n|p| \circ \varphi_{|p|} : |Q| \xrightarrow{|\mathrm{SP}^n p \circ \varphi_p|} |\mathrm{SP}^n Q| \xrightarrow{\approx} \mathrm{SP}^n|Q|\,.$$

Hence $|p|$ is an n-fold ramified covering map. ∎

Remark 4.3 The multiplicity function $\mu : |K| \longrightarrow \mathbb{N}$ of the simplicial ramified covering map $|p| : |K| \longrightarrow |Q|$ is defined by $\mu([a,t]) = \mu_m(a)$, where $(a,t) \in K_m \times \Delta^m$ is the nondegenerate representative. One can verify that it satisfies the conditions given in Smith's definition of a ramified covering map.

5 Ramified covering maps in the category of simplicial complexes

The geometric realization functor $|\cdot| : \mathcal{S}\mathrm{impset} \longrightarrow \mathcal{T}\mathrm{op}$ has a right adjoint functor $\mathcal{S} : \mathcal{T}\mathrm{op} \longrightarrow \mathcal{S}\mathrm{impset}$ given by the singular simplicial set associated to a space. The simplicial map $\mathcal{S}(p) : \mathcal{S}(E) \longrightarrow \mathcal{S}(X)$ associated to a topological ramified covering map $p : E \longrightarrow X$ is not in general a simplicial ramified covering map, because it is not always finite-to-one as the example below shows. However, if we work in the category of simplicial complexes, then there is another functor K from this category to $\mathcal{S}\mathrm{impset}$. In this section we define the concept of an n-fold ramified covering map $p : C \longrightarrow D$ in the category of simplicial complexes and we show that the associated map of simplicial sets $\widehat{p} : K(C) \longrightarrow K(D)$ is a ramified covering map. Furthermore, we show that the map of triangulated spaces $|p| : |C| \longrightarrow |D|$ is a topological ramified covering map.

Example 5.1 Consider the topological ramified covering map $p : E \longrightarrow X$ given by $X = \mathbb{R}$ and $E \subset \mathbb{R}^2$ such that

$$E = \{(s,t) \mid t = \sin \pi s \text{ or } t = -\sin \pi s\}\,,$$

and p the projection to the first coordinate. Let moreover $\mu : E \longrightarrow \mathbb{N}$ be given by

$$\mu(s,t) = \begin{cases} 1 & \text{if } s \notin \mathbb{Z}\,, \\ 2 & \text{if } s \in \mathbb{Z}\,. \end{cases}$$

Since p is the orbit map of the action of $\mathbb{Z}_2$ on E given by $(s,t) \mapsto (s,-t)$, p is a ramified covering map (see Figure 1). Observe first that p has infinitely many sections. Indeed, each interval $[k, k+1]$ can be mapped by the section either with $s \mapsto (s, \sin \pi s)$ or with $s \mapsto (s, -\sin \pi s)$, so that for each real number (written binarily) there is a different section.

Consider now the map $\lambda : [0,1] \longrightarrow \mathbb{R}$ given as follows. First subdivide the interval by the points $1 - \frac{1}{2^k}$, $k \in \mathbb{N}$ and map the interval $[0, \frac{1}{2}]$ by

$$\lambda(s) = \begin{cases} 2s & \text{if } s \le \frac{1}{4}\,, \\ 1 - 2s & \text{if } s \ge \frac{1}{4}\,; \end{cases}$$

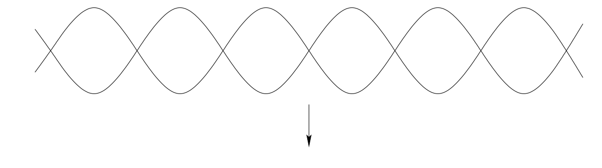

Figure 1: Topological ramified covering map of multiplicity 2

then map the interval $[\frac{1}{2}, \frac{3}{4}]$ by

$$\lambda(s) = \begin{cases} 2s - 1 & \text{if } s \leq \frac{5}{8}\,, \\ \frac{3}{2} - 2s & \text{if } s \geq \frac{5}{8}\,; \end{cases}$$

and so on. This map is continuous and has the property of mapping each point of the subdivision to 0. Thus, the restriction of λ to each of the subintervals can be continuously lifted by composing it with either $s \mapsto (s, \sin \pi s)$ or with $s \mapsto (s, -\sin \pi s)$. Hence, at each point $1 - \frac{1}{2^k}$, one has two different possibilities of continuing the lifting of λ and so one has that λ has infinitely many liftings.

This shows that for $\lambda \in \mathcal{S}_1(\mathbb{R})$, $\mathcal{S}p_1^{-1}(\lambda)$ has infinitely many elements.

Note that this kind of examples can easily be given for many other ramified covering maps.

Recall ([10]) that a *simplicial complex* C is a family of nonempty finite subsets of a set V_C, whose elements are the *vertices* of C and which have the following two properties:

(i) For each $v \in V_C$, the set $\{v\} \in C$.

(ii) Given $\sigma \in C$ and $\sigma' \subset \sigma$, then $\sigma' \in C$.

A map $f : C \longrightarrow D$ of simplicial complexes is given by a function $f : V_C \longrightarrow V_D$ such that if $\{v_0, \ldots, v_q\} \in C$, then $\{f(v_0), \ldots, f(v_q)\} \in D$.

In what follows, we shall assume that the vertices of any simplicial complex considered have a partial order such that each simplex is totally ordered. Moreover, we can also assume that any simplicial map preserves the order. This can always be achieved as follows. Let $f : C \longrightarrow D$ be any simplicial map. Put a total order on V_D and define a partial order on V_C such that $v < v' \iff f(v) < f(v')$. Then extend this partial order to a total order on V_C. Alternatively, one can consider the barycentric subdivision of each of the simplicial complexes. We denote by $\sigma^{(i)}$ the ith face of any ordered m-simplex $\sigma = (v_0 < \cdots < v_m)$ in a simplicial complex, which is defined by $\sigma^{(i)} = (v_0 < \cdots < \widehat{v_i} < \cdots < v_m)$, where we omit the ith vertex.

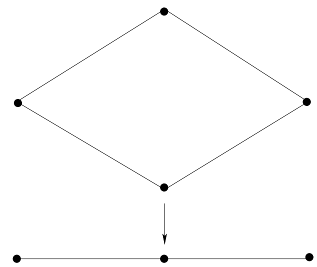

Figure 2: Ramified covering map of simplicial complexes

Definition 5.2 Let $p : C \longrightarrow D$ be a simplicial map between simplicial complexes. We say that p is an *n-fold ramified covering map of simplicial complexes* if there exists a *multiplicity function* $\mu : C \longrightarrow \mathbb{N}$ such that the following conditions hold:

1. For each vertex w of D, the fiber $p^{-1}(w)$ is a finite nonempty set and if $\sigma \in p^{-1}(\tau)$, then σ and τ have the same dimension.
2. For each simplex $\tau \in D$ and each simplex $\sigma' \in C$, such that $p(\sigma') = \tau^{(i)}$, there is a simplex $\sigma \in D$ such that $p(\sigma) = \tau$ and $\sigma^{(i)} = \sigma'$.
3. For each simplex τ in D,
$$\sum_{p(\sigma)=\tau} \mu(\sigma) = n\,.$$
4. For each simplex $\sigma \in C$,
$$\mu(\sigma^{(i)}) = \sum_{\substack{p(\sigma)=p(\sigma') \\ \sigma^{(i)}=\sigma'^{(i)}}} \mu(\sigma')\,.$$

EXAMPLE 5.3 Consider the simplicial complexes C with vertex set $V_C = \{v_0 < v_1 < v_2 < v_3\}$ and 1-simplexes $\{v_0, v_1\}$, $\{v_1, v_3\}$, $\{v_0, v_2\}$, $\{v_2, v_3\}$, and D with $V_D = \{w_0 < w_1 < w_2\}$ and 1-simplexes $\{w_0, w_1\}$, $\{w_1, w_2\}$, and the simplicial map $p : C \longrightarrow D$ such that $p(v_0) = w_0$, $p(v_1) = p(v_2) = w_1$, and $p(v_3) = w_2$. If one defines $\mu(v_0) = 2 = \mu(v_3)$ and $\mu(\sigma) = 1$ for any other simplex $\sigma \in C$, then p is a 2-fold ramified covering map of simplicial complexes (see Figure 2).

Definition 5.4 Given any simplicial complex C, one can associate to it a simplicial set $K(C)$ as follows. Define
$$K(C)_m = \{(v_0, \dots, v_m) \mid \{v_0, \dots, v_m\} \in C,\ v_0 \leq \cdots \leq v_m\}\,,$$

$d_i : K(C)_m \longrightarrow K(C)_{m-1}$ is given by

$$d_i(v_0, \ldots, v_m) = (v_0, \ldots, \widehat{v_i}, \ldots, v_m)\,,$$

and $s_i : K(C)_m \longrightarrow K(C)_{m+1}$ is given by

$$s_i(v_0, \ldots, v_m) = (v_0, \ldots, v_i, v_i, \ldots, v_m)\,.$$

If $p : C \longrightarrow D$ is an n-fold ramified covering map of simplicial complexes, call $\widehat{p}_m : K(C)_m \longrightarrow K(D)_m$ the induced map of simplicial sets, given by $\widehat{p}(v_0, \ldots, v_m) = (p(v_0), \ldots, p(v_m))$. Define $\mu_m : K(C)_m \longrightarrow \mathbb{N}$ by $\mu_m(\sigma) = \mu(\sigma')$, where $\sigma' \in K(C)_l$, $l \leq m$, is the unique nondegenerate simplex such that $s(\sigma') = \sigma$.

Lemma 5.5 *Let $p : C \longrightarrow D$ be an n-fold ramified covering map of simplicial complexes. Then, for any i, the restriction $s_i| : \widehat{p}_{m-1}^{-1}(\tau) \longrightarrow \widehat{p}_m^{-1}(s_i(\tau))$ is bijective.*

Proof: Take $\tau = (w_0, \ldots, w_{m-1})$. Then $s_i(\tau) = (w_0, \ldots, w_i, w_i, \ldots, w_{m-1})$. If $\widehat{p}_m(v_0, \ldots, v_m) = s_i(\tau)$, then $v_i = v_{i+1}$. Otherwise, the dimension of $\sigma' = (v_0, \ldots, v_m)$ would be higher than the dimension of $s_i(\tau)$, contradicting condition 1 in Definition 5.2. Hence $\sigma' = s_i(\sigma)$ for some $\sigma \in \widehat{p}_{m-1}^{-1}(\tau)$. Since s_i is always injective, the result follows. ■

We have the following result that relates both the concept of simplicial n-fold ramified covering map defined in Section 1 and the concept of n-fold ramified covering map of simplicial complexes just defined.

Theorem 5.6 *Let $p : C \longrightarrow D$ be an n-fold ramified covering map of simplicial complexes. Then $\widehat{p} : K(C) \longrightarrow K(D)$ is a simplicial n-fold ramified covering map with $\{\mu_m\}$ defined as above.*

Proof: Condition 1 in 5.2 implies that the fibers of $\widehat{p}_m$ are finite for each m. Thus condition 1 in Definition 1.1 holds.

We shall prove condition 2 in three steps. Take $\tau \in K(D)_m$ and $\sigma' \in K(C)_{m-1}$ such that $\widehat{p}_{m-1}(\sigma') = d_i(\tau)$.

First step: Assume that τ is nondegenerate. Therefore, $d_i(\tau) \in K(D)_{m-1}$ is nondegenerate, and hence σ' is also nondegenerate. By condition 2 in Definition 5.2, there exists a simplex $\sigma \in K(C)_m$ such that $\widehat{p}_m(\sigma) = \tau$ and $d_i(\sigma) = \sigma'$.

Second step: Assume that τ is degenerate of the form $\tau = s_j(\tau')$, where τ' is nondegenerate. Using the relations in a simplicial set, we have the following.

$$d_i(\tau) = d_i(s_j(\tau')) = \begin{cases} s_{j-1}d_i(\tau') & \text{if } j > i \text{ (1)}, \\ \tau' & \text{if } j = i-1 \text{ or } j = i \quad \text{(2)}, \\ s_j d_{i-1}(\tau') & \text{if } j < i-1 \text{ (3)}. \end{cases}$$

In case (2) we have that since $d_i(\tau) = \tau'$ is nondegenerate, σ' is also nondegenerate. Define $\sigma = s_j(\sigma')$. Then

$$\widehat{p}_m(\sigma) = \widehat{p}_m(s_j(\sigma')) = s_j(\widehat{p}_{m-1}(\sigma')) = s_j d_i(\tau) = s_j d_i s_j(\tau') = s_j(\tau') = \tau\,.$$

On the other hand,

$$d_i(\sigma) = d_i s_j(\sigma') = \sigma'\,.$$

In case (1), we claim that $\sigma' = s_{j-1}(\sigma'')$, where $\sigma'' \in K(C)_{m-2}$ is nondegenerate. To see this, notice that $s_{j-1}d_i(\tau') = \widehat{p}_{m-1}(\sigma')$. If we assume that σ' is nondegenerate, then σ' is a geometric simplex of dimension $m-1$, which is mapped by p to a simplex of dimension $m-2$ contradicting condition 1 of Definition 5.4.

Hence

$$\widehat{p}_{m-1}(\sigma') = \widehat{p}_{m-1}(s_{j-1}(\sigma'')) = s_{j-1}(\widehat{p}_{m-2}(\sigma''))\,;$$

while on the other hand

$$\widehat{p}_{m-1}(\sigma') = d_i(\tau) = s_{j-1}d_i(\tau')\,.$$

Since the degenerate operators are injective, we have that $\widehat{p}_{m-2}(\sigma'') = d_i(\tau')$. Because σ'' and τ' are nondegenerate, by the geometric condition there is a geometric simplex $\sigma' \in K(C)_{m-1}$ such that $\widehat{p}_{m-1}(\sigma') = \tau'$ and $d_i(\sigma') = \sigma''$.

Now define $\sigma = s_j(\sigma')$. Then

$$\widehat{p}_m(\sigma) = \widehat{p}_m(s_j(\sigma')) = s_j(\widehat{p}_{m-1}(\sigma')) = s_j(\tau') = \tau\,.$$

Moreover,

$$d_i(\sigma) = d_i s_j(\sigma') = s_{j-1}d_i(\sigma') = s_{j-1}(\sigma'') = \sigma'\,,$$

where the last equality follows from the claim above.

In case (3) the proof is analogous to case (1).

Third step: When $\tau = s(\tau')$, where s is a composite of degeneracy operators, the proof is done by induction on the number of degeneracy operators using the same kind of arguments as above.

We now pass to condition 3. Take

$$\tau = (w_0, \ldots, w_i, \ldots, w_m) \in K(D)_m$$

and let $\sigma = (v_0, \ldots, v_m)$ be any element in the fiber of τ. If τ is nondegenerate, then so is σ. Then, by condition 3 in 5.4,

$$\sum_{p(\sigma)=\tau} \mu_m(\sigma) = \sum_{p(\sigma)=\tau} \mu(\sigma) = n\,.$$

Thus 3 (a) in Definition 1.1 holds in this case. If τ is degenerate, then there is a nondegenerate $\tau' \in K(D)_{m-r}$ and a composite of degeneracy operators $s : K(D)_{m-r} \longrightarrow K(D)_m$ such that $s(\tau') = \tau$. Since by the previous lemma the corresponding composite of degeneracy operators $s : K(C)_{m-r} \longrightarrow K(C)_m$ maps the fiber $\widehat{p}_{m-r}^{-1}(\tau')$ bijectively onto the fiber $\widehat{p}_m^{-1}(\tau)$, and since $\mu_m \circ s = \mu_{m-r}$, condition 3 (a) in Definition 1.1 holds in general, as follows from the case when τ is nondegenerate.

Clearly 3 (b) in Definition 1.1 also holds.

Finally, to prove 3 (c), assume first that $\sigma \in K(C)_m$ is nondegenerate. Then

$$\mu_{m-1} d_i(\sigma) = \mu(\sigma^{(i)}) = \sum_{\substack{p(\sigma)=p(\sigma') \\ \sigma'^{(i)}=\sigma^{(i)}}} \mu(\sigma') = \sum_{\substack{\widehat{p}_m(\sigma)=\widehat{p}_m(\sigma') \\ d_i(\sigma')=d_i(\sigma)}} \mu_m(\sigma')\,.$$

If $\sigma = s(\sigma_1)$, then we have the following three cases. Either $d_i \circ s = \mathrm{id}$, $d_i \circ s = s$", or $d_i \circ s = s' \circ d_j$. In the first two cases, the assertion follows from the fact that d_i restricted to the fibers is bijective, because it is the inverse of s_j, which is bijective on fibers by Lemma 5.5. In the third case,

$$\begin{aligned}
\mu_{m-1} d_i(\sigma) &= \mu_{m-1} s' d_j(\sigma_1) \\
&= \mu_{m-k-1} d_j(\sigma_1) \\
&= \sum_{\substack{\widehat{p}_{m-k}(\sigma_1')=\widehat{p}_{m-k}(\sigma_1) \\ d_j(\sigma_1')=d_j(\sigma_1)}} \mu_{m-k}(\sigma_1') \\
&= \sum_{\substack{\widehat{p}_{m-k}(\sigma_1')=\widehat{p}_{m-k}(\sigma_1) \\ d_j(\sigma_1')=d_j(\sigma_1)}} \mu_m s(\sigma_1') \\
&= \sum_{\substack{\widehat{p}_m(\sigma')=\widehat{p}_m(\sigma) \\ d_i(\sigma')=d_i(\sigma)}} \mu_m(\sigma')\,,
\end{aligned}$$

where the third equality follows from the nondegenerate case and the last equality follows from the fact that the degeneracy operators induce bijections on the fibers, and because

$$d_i(\sigma') = d_i s(\sigma_1') = s' d_j(\sigma_1')\,.$$

■

The following result is proved in [6].

Proposition 5.7 *Given a simplicial complex C, there is a natural homeomorphism between the geometric realizations $\varphi_C : |C| \longrightarrow |K(C)|$.* ■

From the previous proposition and Theorem 4.2 we obtain the following.

Theorem 5.8 *Let $p : C \longrightarrow D$ be an n-fold ramified covering map of simplicial complexes. Then $|p| : |C| \longrightarrow |D|$ is a topological n-fold ramified covering map.* ■

Corollary 2.4 suggests the following.

Definition 5.9 Let $p : C \longrightarrow D$ be a simplicial map between simplicial complexes. We say that p is an *n-fold covering map of simplicial complexes* if the following conditions hold:

1. For each simplex τ of D, the fiber $p^{-1}(\tau)$ is a set of cardinality n and if $\sigma \in p^{-1}(\tau)$, then σ and τ have the same dimension.
2. For each simplex $\tau \in D$ and each simplex $\sigma' \in C$, such that $p(\sigma') = \tau^{(i)}$, there is a unique simplex $\sigma \in D$ such that $p(\sigma) = \tau$ and $\sigma^{(i)} = \sigma'$.

Remark 5.10 Notice that if we define $\mu : D \longrightarrow \mathbb{N}$ by $\mu(\sigma) = 1$ for all $\sigma \in D$, then p is an n-fold ramified covering map.

Rotman [8] has another definition of a covering map of simplicial complexes. It is equivalent to ours. Namely, we have the following.

Proposition 5.11 *Let $p : C \longrightarrow D$ be a simplicial map between simplicial complexes. Then p is an n-fold covering map if and only if for each simplex $\tau \in D$, $p^{-1}(\tau) = \bigsqcup_{k=1}^{n} \sigma_k$, so that $p|_{\sigma_k} : \sigma_k \longrightarrow \tau \in D$ is bijective.*

Proof: Assume first that p is an n-fold covering map. Let σ_k, σ_l be such that $p(\sigma_k) = p(\sigma_l) = \tau$. Suppose first that $\sigma_k \cap \sigma_l \neq \emptyset$ and $p(\sigma_k \cap \sigma_l) = \tau^{(i)}$. Then by condition 2 in the definition, $\sigma_k = \sigma_l$. The general case follows by induction on the number of missing vertices.

Conversely, assume that p satisfies Rotman's definition. Then condition 1 follows immediately. To see condition 2, we use the unique lifting property of p (see [8, Thms. 3.1 and 3.2]). Define the simplicial complex Δ_m whose vertex set is $\mathbf{m}$ and whose simplexes are all its subsets. Then any m-simplex $\tau \in D$ can be seen as an injective simplicial map $\tau : \Delta_m \longrightarrow D$. Consider the following lifting problem:

$$\begin{tikzcd}
(\Delta_{m-1}, 0) \arrow[r, "\sigma'"] \arrow[d, hook, "d_i"'] & (C, \sigma'(0)) \arrow[d, "p"] \\
(\Delta_m, d_i(0)) \arrow[r, "\tau"'] \arrow[ur, dashed, "\sigma"] & (D, p\sigma'(0))\,.
\end{tikzcd}$$

Since Δ_m is contractible, by the unique lifting property, there exists $\sigma : \Delta_m \longrightarrow C$ making the lower triangle commute. To see that the upper triangle also commutes, just observe that $\sigma \circ d_i$ and σ' are both liftings of $\tau \circ d_i$ that coincide in 0. By the uniqueness of the liftings, they are equal. ■

As a consequence of 5.8, 4.2, 5.7, and 4.3, we have the following.

Corollary 5.12 *Let $p : C \longrightarrow D$ be an n-fold covering map of simplicial complexes. Then $|p| : |C| \longrightarrow |D|$ is a topological n-fold covering map.* ■

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