Bredon homology and ramified covering G-maps

Marcelo A. Aguilar & Carlos Prieto*

Instituto de Matemáticas,
Universidad Nacional Autónoma de México,
04510 México, D.F., Mexico

Email: marcelo@math.unam.mx, cprieto@math.unam.mx

Abstract Let G be a finite group. The objective of this paper is twofold. First we prove that the cellular Bredon homology groups with coefficients in an arbitrary coefficient system M are isomorphic to the homotopy groups of certain topological abelian group. And second, we study ramified covering G-maps of simplicial sets and of simplicial complexes. As an application, we construct a transfer for them in Bredon homology, when M is a Mackey functor. We also show that the Bredon-Illman homology with coefficients in M satisfies the equivariant weak homotopy equivalence axiom in the category of G-spaces.

AMS Classification 55R12,57M12,55N91,55U10; 55P91,14F43,57M10

Keywords Equivariant ramified covering maps, simplicial sets, simplicial complexes, transfer, equivariant Bredon homology, homotopy groups, Mackey functors, equivariant weak homotopy equivalence axiom

Contents

*Corresponding author, Phone: ++5255-56224489, Fax ++5255-56160348. This author was partially supported by PAPIIT grant IN105106-3 and by CONACYT grant 58049. Webpage: http://www.matem.unam.mx/cprieto

0 Introduction

Let G be a finite group, M a covariant coefficient system for G, and X a G-space. In [2] we defined topological groups $F^G(X,M)$ whose homotopy groups are isomorphic to the Bredon-Illman equivariant homology of X with coefficients in M. The construction uses the singular simplicial G-set $\mathcal{S}(X)$, and thus the topological group is quite large. In this paper we consider regular G-CW-complexes X, i.e., G-CW-complexes whose characteristic maps are embeddings. In this case, we define another topological group for X. To do this we use the simplicial G-set $K(X)$ associated to X and constructed with the cells of X, whose geometric realization is homeomorphic to X. With this we obtain smaller topological groups $|F^G(K(X),M)|$ whose homotopy groups are isomorphic to the Bredon (cellular) homology groups. We use some of the techniques introduced in the first part, together with a result on model categories, to prove that an equivariant weak homotopy equivalence between arbitrary G-spaces induces an isomorphism in the Bredon-Illman homology groups with coefficients in any coefficient system.

On the other hand, in [4] we introduced the concepts of ramified covering maps in the categories of simplicial sets and of simplicial complexes. In this paper we generalize these concepts to the equivariant setting and we present some results on their structure. They have the property that their geometric realizations are topological ramified covering G-maps.

In [5] we constructed a transfer for ramified covering G-maps in Bredon-Illman equivariant homology with coefficients in a homological Mackey functor M. In this case one can define other topological groups $\mathbb{F}^G(X,M)$ which are given directly in terms of the points of X and whose homotopy groups are again isomorphic to the Bredon-Illman homology groups of X with coefficients in M. Using these groups one defines the transfer. For an arbitrary coefficient system M, the singular simplicial set of X cannot be used to construct the transfer, because if $p : E \longrightarrow X$ is a ramified covering map, then the fibers of $\mathcal{S}(p) : \mathcal{S}(E) \longrightarrow \mathcal{S}(X)$ are not finite in general (see the counterexample in [4]). Thus the homotopical approach to the Bredon homology given in this paper allows us to construct a transfer for G-equivariant ramified covering maps of simplicial sets and of simplicial complexes in Bredon homology with coefficients in an arbitrary Mackey functor M. We show that this transfer has the usual properties of a transfer.

The paper is organized as follows. In Section 1, given a simplicial G-set K, we construct the topological group $|F^G(K,M)|$ and we prove that its homotopy groups are isomorphic to the Bredon homology groups of $|K|$. From this, we obtain the same result for the case of regular G-CW-complexes. We finish the section with an application of these homotopy-theoretic methods. Namely, we prove that if $f : X \longrightarrow Y$ is an equivariant weak homotopy equivalence between arbitrary G-spaces, and M is any coefficient system, then $f_* : H^G(X; M) \longrightarrow H^G(Y; M)$ is an isomorphism of the Bredon-Illman homology groups. In Section 2, we recall the definition of an n-fold G-function with multiplicity $p : E \longrightarrow S$ (in the category of G-sets) given in [5], and we define the transfer $t_p^G : F^G(S, M) \longrightarrow F^G(E, M)$ for it. In order to deal with the topological case, one needs to study G-equivariant ramified covering maps in the category of simplicial sets. This we do in Section 3, where we introduce the concept of a G-equivariant simplicial (special) ramified covering map p (3.1) and we prove several properties of the class of those p, like the fact that it is closed under pullbacks, that their geometric realization is a topological ramified covering G-map, etc. Further on in this section, we define the transfer in the simplicial context. We finish the section by proving that if the Mackey functor M is homological, then the transfer constructed here coincides with the transfer constructed in [5] (3.22). Next, in Section 4, we give the definition of a (special) G-equivariant ramified covering map of simplicial complexes p (4.1), and in a natural way, we associate a ramified covering map of simplicial sets $K(p)$ (4.4) in such a way that the geometric realizations $|p|$ and $|K(p)|$ coincide. At the end of this section, we pass to the Bredon homology (applying the homotopy-group functors) and we give the transfer and its properties in homology.

1 Homotopical Bredon homology

Given a simplicial pointed G-set K (or equivalently, a pointed G-simplicial set) and a covariant coefficient system M, we shall show that the homotopy groups of the topological abelian group $|F^G(K, M)|$ are isomorphic to the Bredon equivariant homology groups $H_*^G(|K|, M)$. In order to do this, we need the following results.

We begin by recalling the definition of the simplicial abelian group $F^G(K, M)$ given in [2].

Definition 1.1 First consider any pointed G-set S with base point x_0, and a covariant coefficient system M for G. Take the set $\widehat{M} = \cup_{H \subset G} M(G/H)$.

Define now the abelian group

$$\begin{aligned} F^G(S, M) = \{u : S \longrightarrow \widehat{M} \mid\ & u(x) \in M(G/G_x), u(x_0) = 0, u(x) = 0 \\ & \text{for almost all } x \in S, \text{ and } u(gx) = M_*(R_{g^{-1}})(u(x))\}, \end{aligned}$$

where $R_{g^{-1}} : G/G_x \longrightarrow G/G_{gx}$ is given by right-translation with g^{-1}. For simplicity, we put $gl = M_*(R_{g^{-1}})(l) \in M(G/gHg^{-1})$ if $l \in M(G/H)$. The sum is given by $(u+v)(x) = u(x) + v(x) \in M(G/G_x)$. This group has canonical generators $\gamma_x^G(l)$ for $x \neq x_0$ and $l \neq 0$, given by

$$\gamma_x^G(l) = \sum_{i=1}^{r} (g_i l)(g_i x)\,,$$

where $G/G_x = \{[g_i] \mid i = 1, \ldots, r\}$ and lx denotes the function with value l at x and zero elsewhere (which is not an element of $F^G(S,M)$, but the sum is).

Given a pointed G-function $f : S \longrightarrow T$, define $f_*^G : F^G(S,M) \longrightarrow F^G(T,M)$ by $f_*^G(\gamma_x^G(l)) = \gamma_{f(x)}^G M_*(\widehat{f}_x)(l)$, where $\widehat{f}_x : G/G_x \twoheadrightarrow G/G_{f(x)}$ is the quotient G-function. Then $F^G(-,M)$ is a covariant functor from the category of pointed G-sets $G\text{-}\mathcal{S}\mathrm{et}_*$ to the category of abelian groups.

Now, if K is a simplicial pointed G-set, we shall use the following notations. Let $\alpha : \mathbf{m} \longrightarrow \mathbf{n}$ be a morphism in Δ, where $\mathbf{k} = \{0, 1, 2, \ldots, k\}$ for all $k \geq 0$. We denote by $\alpha^K : K_n \longrightarrow K_m$ the value of K in α, and by $\alpha_\# : \Delta^m \longrightarrow \Delta^n$ the affine map induced by α. We have a simplicial abelian group $F^G(K,M)$ given as the composite of the functors $K : \Delta \longrightarrow G\text{-}\mathcal{S}\mathrm{et}_*$ with $F^G(-,M)$.

Remark 1.2 If G is the trivial group, S is a pointed set and $M = \mathbb{Z}$, then we denote the abelian group $F^G(S,M)$ by $F(S,\mathbb{Z})$, which is the free abelian group generated by the elements of $S - \{x_0\}$.

Let K be a simplicial pointed G-set. Since $F^G(K,M)$ is a simplicial abelian group, we have a canonical chain complex $\{F^G(K_n,M), \partial_n\}$, where

$$\partial_n = \sum_{i=0}^{n} (-1)^i (d_i^K)_*^G\,,$$

and $d_i^K : K_n \longrightarrow K_{n-1}$ is the ith face operator of K. Moreover, if D_n denotes the subgroup generated by the degenerate elements of $F^G(K_n,M)$, then D_* is a chain subcomplex and therefore we have a quotient complex $\{F^G(K_n,M)/D_n, \overline{\partial}_n\}$.

On the other hand, we denote by $\nu_n(K) \subset K_n$ the set of nondegenerate elements. We have another chain complex $\{F^G(\nu_n(K)^+,M), \delta_n^G\}$, as we shall see below, where

$$\delta_n^G(\gamma_a^G(l)) = \sum_{i=0}^{n} (-1)^i \gamma_{d_i^K(a)}^G M_*(\widehat{d_i^K}_a)(l)\,,$$

with the convention that if some $d_i^K(a)$ is degenerate, then we put $\gamma^G_{d_i^K(a)} = 0$. Notice that since $\nu_n(K)$ has no base point if $n > 0$, we have to consider $\nu_n(K)$ with an extra base point. For $n = 0$ we shall take the base point in K_0. Clearly, $\nu_n(K)$ as well as $K_n - \nu_n(K)$ are G-invariant. We have the following.

Proposition 1.3 *The chain complexes*

$$\{F^G(K_n, M)/D_n, \overline{\partial}_n\} \quad \text{and} \quad \{F^G(\nu_n(K)^+, M), \delta_n^G\}$$

are isomorphic.

Proof: We first claim that $w \in D_n$ if and only if $w(b) = 0$ for all $b \in \nu_n(K)$.

Indeed, take $w \in D_n$. Then $w = \sum_{r=1}^m (s_{j_r}^K)_*^G(v_r)$, where $v_r = \sum_{a \in K_{n-1}} \gamma_a^G(l_a) \in F^G(K_{n-1}, M)$. Now

$$(s_{j_r}^K)_*^G \left(\sum_{a \in K_{n-1}} \gamma_a^G(l_a) \right) = \sum_{a \in K_{n-1}} \gamma^G_{s_{j_r}^K(a)}(l_a) \, .$$

Since $K_n - \nu_n(K)$ is G-invariant, then each $\gamma^G_{s_{j_r}^K(a)}(l_a)$ is zero at all elements in $\nu_n(K)$. Therefore $w(b) = 0$ for all $b \in \nu_n(K)$.

Conversely assume that $w(b) = 0$ for all $b \in \nu_n(K)$, then $w \neq 0$ only at degenearte elements. Take one representative $s_{i_1}^K(a_1), \ldots, s_{i_m}^K(a_m)$ of each orbit where w is nonzero. Hence

$$w = \sum_{r=1}^m \gamma^G_{s_{i_r}^K(a_r)}(l_{i_r}) \, , \quad \text{where} \quad w(s_{i_r}^K(a_r)) = l_{i_r} \neq 0 \, .$$

Since $s_{i_r}^K$ is injective, it preserves the isotropy groups and thus we have

$$\gamma^G_{s_{i_r}^K(a_r)}(l_{i_r}) = (s_{i_r}^K)_*^G(\gamma^G_{a_r}(l_{i_r})) \, .$$

Consequently, $w \in D_n$.

Notice that when $n = 0$, $F^G(K_0, M)$ consists only of nondegenerate elements, so that $D_0 = 0$. Hence $w \in D_0$ if and only if $w(b) = 0$ for all $b \in \nu_0(K) = K_0$.

Let now $\iota : \nu_n(K)^+ \hookrightarrow K_n$ be the extension of the inclusion such that $\iota(+)$ is the base point of K_n. Define φ as the composite

$$\varphi : F^G(\nu_n(K)^+, M) \xrightarrow{i_*^G} F^G(K_n, M) \xrightarrow{\pi} F^G(K_n, M)/D_n \, .$$

To see that φ is injective, take $u = \sum_{b \in \nu_n(K)} \gamma_b^G(l_b) \in F^G(\nu_n(K)^+, M)$ such that $\varphi(u) = 0$. Hence $\iota_*^G(u) \in D_n$, but since ι is injective, $\iota_*^G(\sum_{b \in \nu_n(K)} \gamma_b^G(l_b)) =$

$\sum_{b\in\nu_n(K)}\gamma_b^G(l_b)\in D_n$. Therefore, by the claim above, $\iota_*^G(u)$ is zero at all elements of $\nu_n(K)$. Thus $u=0$.

To see that φ is surjective, take $\pi(w)\in F^G(K_n,M)/D_n$. Let us write

$$w=\sum_{b\in\nu_n(K)}\gamma_b^G(l_b)+\sum_{b'\in K_n-\nu_n(K)}\gamma_{b'}^G(l_{b'})\,.$$

Take $u=\sum_{b\in\nu_n(K)}\gamma_b^G(l_b)\in F^G(\nu_n(K)^+,M)$. Then, by the claim above, $\varphi(u)=\pi(w)$.

We must now show that φ is an isomorphism of chain complexes. This follows from the fact that δ_n^G corresponds to the operator $\overline{\partial}_n$ under the isomorphism φ, as one easily verifies. ■

We now prove the next.

Proposition 1.4 *There is an isomorphism of chain complexes*

$$\psi':\{F(\nu_n(K)^+,\mathbb{Z}),\delta_n\}\longrightarrow\{H_n(|K|^n,|K|^{n-1}),\partial_n\}\,.$$

Proof: It is well known that the group $H_n(|K|^n,|K|^{n-1})$ is the free abelian group generated by the n-cells of $|K|$. The n-cells of $|K|$ are in a one-to-one correspondence with the elements of $\nu_n(K)$ (see [11]). The isomorphism is given as follows. Take a generator a of $F(\nu_n(K)^+,\mathbb{Z})$, namely $a\in\nu_n(K)$. Then we associate to a the relative cycle $\psi_a:(\Delta^n,\dot{\Delta}^n)\longrightarrow(|K|^n,|K|^{n-1})$ defined by $\psi_a(t)=[a,t]$, and we define the isomorphism ψ' by $\psi'(a)=[\psi_a]$.

We now prove that the isomorphism ψ' commutes with the boundary operators. We have, on the one hand

$$\partial_n\psi'(a)=\partial_n[\psi_a]=\left[\sum_{i=0}^n(-1)^i\psi_a d_{i\#}\right]$$

while on the other hand

$$\psi'\delta_n(a)=\psi'\left(\sum_{i=0}^n(-1)^i d_i^K(a)\right)\,.$$

But $\psi'(d_i^K(a))=[\psi_{d_i^K(a)}]$, and $\psi_{d_i^K(a)}(s)=[d_i^K(a),s]=[a,d_{i\#}(s)]=\psi_a d_{i\#}(s)$.

When $n=0$, one has an isomorphism $H_0(|K|^0,*)\cong F(\nu_0(K),\mathbb{Z})$, where $*=[a_0,1]$ and $a_0\in\nu_0(K)=K_0$ is the base point. Therefore the result follows. ■

Definition 1.5 Let T be a pointed G-set. There is a contravariant functor $\underline{F}(T,\mathbb{Z})$ from the category of canonical orbits G/H, which we denote by $\mathcal{O}(G)$, to the category of abelian groups given on objects by

$$\underline{F}(T,\mathbb{Z})(G/H) = F(T^H,\mathbb{Z})$$

and on morphisms, as follows. If $H \subset K$, that is, for $q : G/H \twoheadrightarrow G/K$, one has an inclusion $T^K \hookrightarrow T^H$ which induces

$$q^* : \underline{F}(T,\mathbb{Z})(G/K) \longrightarrow \underline{F}(T,\mathbb{Z})(G/H)\,.$$

Furthermore, if we take $R_{g^{-1}} : G/H \longrightarrow G/gHg^{-1}$, then the bijection $L_{g^{-1}} : T^{gHg^{-1}} \longrightarrow T^H$ given by $y \mapsto g^{-1}y$ induces

$$R^*_{g^{-1}} : \underline{F}(T,\mathbb{Z})(G/gHg^{-1}) \longrightarrow \underline{F}(T,\mathbb{Z})(G/H)\,.$$

We shall consider the categorical tensor product (or "coend" –see [12]) of $\underline{F}(T,\mathbb{Z})$ and M defined by

$$\underline{F}(T,\mathbb{Z}) \otimes_{\mathcal{O}(G)} M = \oplus_{H\subset G} F(T^H,\mathbb{Z}) \otimes M(G/H)/\!\sim$$

where one takes the quotient by the subgroup generated by the differences $f^*(a) \otimes l - a \otimes M_*(f)(l)$, where $f : G/H \longrightarrow G/K$ is any G-function.

Proposition 1.6 *Let T be a pointed G-set. Then there is an isomorphism*

$$F^G(T,M) \cong \underline{F}(T,\mathbb{Z}) \otimes_{\mathcal{O}(G)} M\,.$$

Proof: Take $x \in T$ and $l \in M(G/G_x)$. Then $1x \otimes l \in F(T^{G_x},\mathbb{Z}) \otimes M(G/G_x)$. Define $\varphi_x : M(G/G_x) \longrightarrow \underline{F}(T,\mathbb{Z}) \otimes_{\mathcal{O}(G)} M$ by

$$\varphi_x(l) = [1x \otimes l] \in \underline{F}(T,\mathbb{Z}) \otimes_{\mathcal{O}(G)} M\,.$$

We have

$$\begin{aligned}\varphi_{gx} M_*(R_{g^{-1}})(l) &= [gx \otimes M_*(R_{g^{-1}})(l)] \\ &= [R^*_{g^{-1}}(gx) \otimes l]\,.\end{aligned}$$

But, by definition, $R^*_{g^{-1}}(gx) = g^{-1}gx = x$, thus $\varphi_{gx} \circ M_*(R_{g^{-1}}) = \varphi_x$. Moreover, if $x_0 \in T$ is the base point, then $\varphi_{x_0}(l) = [1x_0 \otimes l] = 0$, since $1x_0 = 0$. Therefore, we have proved that the family φ_x satisfies the conditions of the universal property of $F^G(T,M)$ [2, 1.6], and thus it determines a homomorphism

$$\varphi : F^G(T,M) \longrightarrow \underline{F}(T,\mathbb{Z}) \otimes_{\mathcal{O}(G)} M\,,$$

such that $\varphi(\gamma^G_x(l)) = \varphi_x(l) = [1x \otimes l]$.

We now construct the inverse

$$\psi : \underline{F}(T, \mathbb{Z}) \otimes_{\mathcal{O}(G)} M \longrightarrow F^G(T, M)\,,$$

as follows. Define first

$$\psi_H : F(T^H, \mathbb{Z}) \otimes M(G/H)$$

by

$$\psi_H(u \otimes l) = \sum_{x \in T^H} u(x)\gamma_x^G M_*(p_x)(l)\,,$$

where $p_x : G/H \longrightarrow G/G_x$ is the canonical projection. Clearly ψ_H is well defined. We now prove that the homomorphisms ψ_H are compatible with the equivalence relation and therefore they determine ψ.

We analyze two cases. First consider $f = R_{g^{-1}} : G/H \longrightarrow G/gHg^{-1}$. Take $x \in T^{gHg^{-1}}$ and $l \in M(G/H)$. Observe that $p_{g^{-1}x} : G/H \longrightarrow G/G_{g^{-1}x}$ is the composite

$$p_{g^{-1}x} : G/H \xrightarrow{R_{g^{-1}}} G/gHg^{-1} \xrightarrow{p_x} G/G_x \xrightarrow{R_g} G/g^{-1}G_x g$$

and that $\gamma_x^G = \gamma_{g^{-1}x}^G \circ M_*(R_g)$. Then

$$\begin{aligned}
\psi_{gHg^{-1}}(x \otimes M_*(R_{g^{-1}})(l)) &= \gamma_x^G M_*(p_x) M_*(R_{g^{-1}})(l) \\
&= \gamma_{g^{-1}x}^G M_*(R_g) M_*(p_x) M_*(R_{g^{-1}})(l) \\
&= \gamma_{g^{-1}x}^G M_*(p_{g^{-1}x})(l) = \psi_H(R_{g^{-1}}^*(x) \otimes l)
\end{aligned} \tag{1.7}$$

Now consider $f = q : G/H \longrightarrow G/K$. Take $x \in T^K$ and $l \in M(G/H)$. Then $x \otimes M_*(q)(l) \sim q^*(x) \otimes l$. Let $p'_x : G/K \longrightarrow G/G_x$ be the projection; then clearly $p'_x \circ q = p_x$. We have

$$\begin{aligned}
\psi_K(x \otimes M_*(q)(l)) &= \gamma_x^G M_*(p'_x) M_*(q)(l) \\
&= \gamma_x^G M_*(p_x)(l) \\
&= \psi_H(p_x^*(x) \otimes l)\,.
\end{aligned} \tag{1.8}$$

Therefore, by (1.7) and (1.8), the homomorphisms ψ_H define ψ.

We now prove that φ is the inverse of ψ. Consider a generator $\gamma_x^G(l) \in F^G(T, M)$, thus $x \in T^{G_x}$ and $l \in M(G/G_x)$. Then

$$\psi\varphi(\gamma_x^G(l)) = \psi(\varphi_x(l)) = \psi([1x \otimes l]) = \psi_{G_x}(1x \otimes l) = \gamma_x^G(l)$$

and thus $\psi \circ \varphi = \mathrm{id}$. On the other hand, take $x \in T^H$ and $l \in M(G/H)$. Then $[x \otimes l]$, where $1x := x$ is a generator and

$$\begin{aligned}
\varphi\psi([x \otimes l]) &= \varphi(\psi_H(x \otimes l)) = \varphi(\gamma_x^G M_*(p_x)(l)) = \varphi_x M_*(p_x)(l) \\
&= [x \otimes M_*(p_x)(l)] = [p_x^*(x) \otimes l]\,.
\end{aligned}$$

Since $p_x^* : F(T^{G_x}, \mathbb{Z}) \longrightarrow F(T^H, \mathbb{Z})$ is the inclusion, $[p_x^*(x) \otimes l] = [x \otimes l]$. Therefore, $\varphi \circ \psi = \mathrm{id}$. ∎

Lemma 1.9 *Let K be a simplicial pointed G-set and denote by K^n, resp. $|K|^n$, the n-skeleton. Then*

(a) $\nu_n(K^H) = \nu_n(K)^H$;

(b) $L \subset K \Rightarrow \nu_n(L) \subset \nu_n(K)$;

(c) $(K^n)^H = (K^H)^n$,

F $|K^n| = |K|^n$;

(e) $|K^H| \approx |K|^H$;

(f) $(|K|^n)^H = |K^H|^n$.

Proof: (a) The inclusion $\nu_n(K)^H \subset \nu_n(K^H)$ is clear since the degenerate operator $s_i^{K^H}$ is the restriction of s_i^K. Conversely, take $a \in \nu_n(K^H)$. If a were degenerate in K, then $a = s_i^K(b)$. Hence

$$a = ha = hs_i^K(b) = s_i^K(hb) \ \forall\ h \in H\,.$$

Therefore, $s_i^K(b) = s_i^K(hb)$, and since s_i^K is injective, $b \in K^H$ and so $a = s_i^{K^H}(b)$, which is a contradiction. Thus $a \in \nu_n(K)^H$.

(b) Take $a \in \nu_n(L)$. If a were degenerate in K, then $a = s_i^K(b)$, $b \in K_{n-1}$. Then $d_i^K(a) = d_i^K s_i^K(b)$. But $d_i^K(a) \in L_{n-1}$ and $d_i^K s_i^K(b) = b$. Hence $b \in L_{n-1}$ and a would be degenerate in L, which is a contradiction.

(c) We have

$$\begin{aligned}
&(K^H)^n = \{c \in K^H \mid c \in K_r^H \text{ or } c = s^{K^H}(d)\,,\ d \in K_r^H\,,\ r \le n\}\,, \text{ and} \\
&(K^n)^H = \{a \in K \mid a \in K_r \text{ or } a = s^K(b)\,,\ b \in K_r\,,\ r \le n\,,\ ha = a\ \forall\ h \in H\}\,.
\end{aligned}$$

Clearly $(K^H)^n \subset (K^n)^H$. On the other hand, if $a \in (K^n)^H$, then $a \in K_r^H$ or $a = ha = hs^K(b) = s^K(hb) = s^K(b)$, since s^K is equivariant. But s^K is also injective, thus $hb = b$ and so $b \in K_r^H$. Therefore $(K^n)^H \subset (K^H)^n$.

(d) This follows from the fact that the cells of $|K|^n$ are in a one-to-one correspondence with the nondegenerate elements in K_m $(m \le n)$.

(e) By [9, 4.3.8], the inclusion $i : K^H \hookrightarrow K$ induces an embedding $|i| : |K^H| \hookrightarrow |K|$. It is clear that the image of $|i|$ is contained in $|K|^H$. Conversely, take $[a,t] \in |K|^H$, where (a,t) is a nondegenerate representative. By [2, 2.4], $G_{[a,t]} = G_a$. Hence $a \in K^H$.

(f) Consider $(|K|^n)^H$. Then, using the statements (d), (e), (c), and (d), we have the following equalities:

$$(|K|^n)^H = |K^n|^H = |(K^n)^H| = |(K^H)^n| = |K^H|^n\,,$$

thus we have the result. ■

In what follows we shall consider the chain complex $\{\underline{C}_n(|K|)\otimes_{\mathcal{O}(G)} M, \partial_n \otimes \mathrm{id}_M\}$, where

$$\underline{C}_n(|K|)(G/H) = H_n((|K|^n)^H, (|K|^{n-1})^H)$$

(see [12, Ch. I, §4]).

Proposition 1.10 *Let K be a simplicial pointed G-set. Then there is an isomorphism between the chain complex $\{F^G(\nu_n(K)^+, M), \delta_n^G\}$ and the chain complex $\{\underline{C}_n(|K|)\otimes_{\mathcal{O}(G)} M, \partial_n \otimes \mathrm{id}_M\}$.*

Proof: Applying Proposition 1.4 to the simplicial set K^H, we get an isomorphism of chain complexes

$$\{F(\nu_n(K^H)^+, \mathbb{Z}), \delta_n\} \cong \{H_n(|K^H|^n, |K^H|^{n-1}), \partial_n\}\,.$$

By Lemma 1.9 (a),

$$F(\nu_n(K^H)^+, \mathbb{Z}) \cong F(\nu_n(K)^{H+}, \mathbb{Z})\,,$$

and by (f),

$$H_n(|K^H|^n, |K^H|^{n-1}) = H_n(|K^n|^H, |K^{n-1}|^H)\,.$$

Therefore, there is an isomorphism of chain complexes

$$\{F(\nu_n(K)^{H+}, \mathbb{Z}), \delta_n\} \cong \{\underline{C}_n(|K|)(G/H), \partial_n\}\,.$$

Hence there is an isomorphism of chain complexes

$$\{\underline{F}(\nu_n(K)^+, \mathbb{Z})\otimes_{\mathcal{O}(G)} M, \delta_n \otimes \mathrm{id}_M\} \cong \{\underline{C}_n(|K|)\otimes_{\mathcal{O}(G)} M, \partial_n \otimes \mathrm{id}_M\}\,.$$

By Proposition 1.6, there is an isomorphism of groups

$$\varphi : F^G(\nu_n(K)^+, M) \longrightarrow \underline{F}(\nu_n(K)^+, \mathbb{Z})\otimes_{\mathcal{O}(G)} M$$

for each n. So we only have to check that φ is a chain morphism. To do this, take a generator $\gamma_a^G(l) \in F^G(\nu(K)^+, M)$. We have on the one hand,

$$\begin{aligned}\varphi\delta_n^G(\gamma_a^G(l)) &= \varphi\left(\sum_{i=0}^{n}(-1)^i \gamma^G_{d_i^K(a)} M_*(\widehat{d_i^K{}_a})(l)\right)\\ &= \left[\sum_{i=0}^{n}(-1)^i d_i^K(a)\otimes M_*(\widehat{d_i^K{}_a})(l)\right]\,,\end{aligned}$$

while on the other hand, since $\varphi(\gamma_a^G(l)) = [a \otimes l]$ we have

$$(\oplus\delta_n \otimes \mathrm{id})\varphi(\gamma_a^G(l)) = \left[\left(\sum_{i=0}^{n}(-1)^i d_i^K(a)\right) \otimes l\right]$$
$$= \left[\sum_{i=0}^{n}(-1)^i d_i^K(a) \otimes l\right].$$

Each summand inside the first brackets lies in $F(\nu_n(K^{G_{d_i^K(a)}}), \mathbb{Z}) \otimes M(G/G_{d_i^K(a)})$ and each summand inside the second brackets lies $F(\nu_{n-1}(K^{G_a}), \mathbb{Z}) \otimes M(G/G_a)$. Now we shall see that these summands are equivalent. Notice first that by (b) in the lemma above, since $K^{G_{d_i^K(a)}} \subset K^{G_a}$, then $\nu_{n-1}(K^{G_{d_i^K(a)}}) \subset \nu_{n-1}(K^{G_a})$. Let ι be the latter inclusion. Then we have

$$d_i^K(a) \otimes M_*(\widehat{d_i^K}_a)(l) \sim \iota^* d_i^K(a) \otimes l = d_i^K(a) \otimes l\,.$$

Thus the two brackets coincide, as we wanted to prove. ■

The next is the main result of this section.

Theorem 1.11 *Let K be a simplicial pointed G-set, and let M be a coefficient system for G. Then there is a natural isomorphism*

$$\pi_q(|F^G(K,M)|) \cong \widetilde{H}_q^G(|K|;M)\,.$$

Proof: By [12, Ch. I, §4], $\widetilde{H}_n^G(|K|;M) = H_n(\underline{C}_*(|K|) \otimes_{\mathcal{O}(G)} M, \partial_* \otimes \mathrm{id})$. Therefore, applying Proposition 1.10, we obtain

$$\widetilde{H}_q^G(|K|;M) \cong H_q(F^G(\nu_*(K)^+, M), \delta_*^G)\,. \tag{1.12}$$

By Proposition 1.3,

$$H_q(F^G(\nu_*(K)^+, M), \delta_*^G) \cong H_q(F^G(K_n, M)/D_n, \overline{\partial}_*)\,. \tag{1.13}$$

By [11], the projection

$$(F^G(K,M), \partial_*) \twoheadrightarrow (F^G(K_n, M)/D_n, \overline{\partial}_*) \tag{1.14}$$

is a chain homotopy equivalence, thus

$$H_q(F^G(K_n, M)/D_n, \overline{\partial}_*) \cong H_q(F^G(K,M), \partial_*)\,. \tag{1.15}$$

Finally, by [11, 22.1,16.6,16.1], we have the following isomorphisms:

$$\begin{tikzcd} H_q(F^G(K,M),\partial_*) \arrow[rrd, dashed, "\cong"'] & \pi_q(F^G(K,M)) \arrow[l, "i_*"', "\cong"] \arrow[r, "\Psi", "\cong"'] & \pi_q(\mathcal{S}|F^G(K,M)|) \arrow[d, "\phi", "\cong"'] \\ & & \pi_q(|F^G(K,M)|)\,. \end{tikzcd} \tag{1.16}$$

Now the result follows combining the isomorphisms (1.12), (1.13), (1.15), and (1.16). ■

In [5, 4.13] we show that any regular G-CW-complex X is G-homeomorphic to the geometric realization of an ordered G-simplicial complex $T(X)$, therefore X is also homeomorphic to the geometric realization of the associated simplicial G-set $K(X)$ (see the paragraph just before Proposition 4.4). Hence the previous result implies the following.

Corollary 1.17 *Let X be a regular G-CW-complex and M a coefficient system. Then there is an isomorphism*

$$\pi_q(|F^G(K(X), M)|) \cong \widetilde{H}_q^G(X; M)\,,$$

where $K(X)$ is the simplicial G-set associated to X. ■

As an application of the homotopical approach to the Bredon-Illman homology, we have the following. We first recall the definition of $F^G(X, M)$, where X is any pointed G-space and M is an arbitrary (covariant) coefficient system.

Consider the simplicial group $F^G(\mathcal{S}(X), M)$, where $\mathcal{S}(X)$ is the singular simplicial G-set of X. We denote by $F^G(X, M)$ the topological group $|F^G(\mathcal{S}(X), M)|$. Then we have an isomorphism $\pi_q(F^G(X, M)) \cong \widetilde{H}_q^G(X; M)$ for all q (see [2, 3]), where $\widetilde{H}_*^G(X; M)$ is the Bredon-Illman homology of the pointed G-space X.

REMARK 1.18 Notice that if X has the G-homotopy type of a G-CW-complex, then by [2, 4.7] one has a G-homotopy equivalence $\rho_X : |\mathcal{S}(X)| \longrightarrow X$. This induces an isomorphism $\widetilde{H}_*^G(|\mathcal{S}(X)|; M) \cong \widetilde{H}_*^G(X; M)$. By Theorem 1.11,

$$\widetilde{H}_q^G(|\mathcal{S}(X)|; M) \cong \pi_q(|F^G(\mathcal{S}(X), M)|) := \pi_q(F^G(X, M))\,.$$

Therefore, $\pi_q(F^G(X, M)) \cong \widetilde{H}_q^G(X; M)$.

However, the result in [2, 3] is valid for any G-space X, and it is this more general result which we shall use in what follows.

Theorem 1.19 *Let $f : X \longrightarrow Y$ be an equivariant weak homotopy equivalence between arbitrary G-spaces, and let M be a coefficient system for G. Then $f_* : H_*^G(X; M) \longrightarrow H_*^G(Y; M)$ is an isomorphism.*

Proof: Recall that, by definition, $f : X \longrightarrow Y$ is an equivariant weak homotopy equivalence if $f^H : X^H \longrightarrow Y^H$ is an ordinary weak homotopy equivalence for all $H \subset G$. Consider the simplicial map $\mathcal{S}(f^H) : \mathcal{S}(X^H) \longrightarrow \mathcal{S}(Y^H)$. Since

$\mathcal{S}(Z^H) = \mathcal{S}(Z)^H$ for every Z, $\mathcal{S}(f^H) = \mathcal{S}(f)^H : \mathcal{S}(X)^H \longrightarrow \mathcal{S}(Y)^H$. By [11, 16.1], there is a natural isomorphism $\varphi_Z : \pi_q(\mathcal{S}(Z)) \longrightarrow \pi_q(Z)$ for every space Z. Thus we have a commutative diagram

$$\begin{array}{ccc}
\pi_q(\mathcal{S}(X^H)) & \xrightarrow{\mathcal{S}(f^H)_\#} & \pi_q(\mathcal{S}(Y^H)) \\
{\scriptstyle \varphi_{X^H}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \varphi_{Y^H}} \\
\pi_q(X^H) & \xrightarrow[f^H_\#]{} & \pi_q(Y^H)\,.
\end{array}$$

Hence $\mathcal{S}(f) : \mathcal{S}(X) \longrightarrow \mathcal{S}(Y)$ is a weak homotopy equivalence in the category of simplicial G-sets (which is the same as the category of G-simplicial sets). Since $\mathcal{S}(Z)^H = \mathcal{S}(Z^H)$, $\mathcal{S}(Z)^H$ is a Kan simplicial set for all H, then $\mathcal{S}(f)$ is a G-morphism between fibrant simplicial G-sets. Hence by [8, 2.3], $\mathcal{S}(f)$ is a G-homotopy equivalence, and thus $\mathcal{S}(f)^+$ is also a G-homotopy equivalence. But $\mathcal{S}(Z)^+ \cong \mathcal{S}(Z^+)$ for all Z. Therefore, by [3, (4.18)(b)], one has a homotopy equivalence

$$|(\mathcal{S}(f^+))^G_*| : |F^G(\mathcal{S}(X^+), M)| \longrightarrow |F^G(\mathcal{S}(Y^+), M)|\,,$$

which induces an isomorphism of homotopy groups. By [2, 3], there is an isomorphism $\pi_q(|F^G(\mathcal{S}(Z^+), M)|) \cong H^G_q(Z; M)$, so the result follows. ■

2 G-Functions with multiplicity

In this section we recall the main facts about the transfer at the level of G-sets (for details, see [5]).

Definition 2.1 By an *n-fold G-function with multiplicity* we understand a G-function $p : E \longrightarrow S$ between G-sets with finite fibers, together with a G-invariant function $\mu : E \longrightarrow \mathbb{N}$, called *multiplicity function*, such that for each $x \in S$,

$$\sum_{a \in p^{-1}(x)} \mu(a) = n\,.$$

We say that the n-fold G-function with multiplicity $p : E \longrightarrow S$ is *pointed* if the spaces E and S have base points, which are fixed under the G-action, and p is a pointed function. Associated to $p : E \longrightarrow S$ one has the G-function

$$\varphi_p : S \longrightarrow \mathrm{SP}^n E$$

given by

$$\varphi_p(x) = \langle \underbrace{a_1, \ldots, a_1}_{\mu(a_1)}, \ldots, \underbrace{a_r, \ldots, a_r}_{\mu(a_r)} \rangle\,.$$

Definition 2.2 Let $p : E \longrightarrow S$ and $p' : E' \longrightarrow S'$ be n-fold G-functions with multiplicity functions μ and μ', respectively. A *morphism* from p to p' is a pair of G-functions $(\widetilde{f}, f)$ such that

(a) the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ S \arrow[r, "f"'] & S' , \end{tikzcd}$$

(b) for each $x \in S$, the restriction $\widetilde{f}|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$ is surjective,

(c) for each $x \in S$ and $a' \in p'^{-1}(f(x))$, one has the equality

$$\mu'(a') = \sum_{p(a)=x\,,\ \widetilde{f}(a)=a'} \mu(a)\,, \quad \text{and} \tag{2.3}$$

(d) for each $a \in E$ one has the formula

$$G_a = G_{p(a)} \cap G_{\widetilde{f}(a)} \tag{2.4}$$

for the isotropy groups.

There is a category whose objects are G-functions with multiplicity and its morphisms are as just defined.

We have the next useful characterization of a morphism of G-functions with multiplicity [5, Prop. 2.4].

Proposition 2.5 *Let $p : E \longrightarrow S$ and $p' : E' \longrightarrow S'$ be n-fold G-functions with multiplicity, and let $f : S \longrightarrow S$ and $\widetilde{f} : E \longrightarrow E'$ be G-functions such that $\widetilde{f} \circ p' = p \circ f$ and for $a \in E$, $G_a = G_{p(a)} \cap G_{\widetilde{f}(a)}$. Then $(\widetilde{f}, f)$ is a morphism from p to p' if and only if*

$$\varphi_{p'} \circ f = \mathrm{SP}^n \widetilde{f} \circ \varphi_p : S \longrightarrow \mathrm{SP}^n E'\,,$$

where the φs are as given in Definition 2.1. ■

Examples 2.6 We have the following examples.

(a) Let $p : E \longrightarrow S$ be a n-fold G-function with multiplicity μ, and let $f : T \longrightarrow S$ be a G-function. Consider the pullback diagram

(2.7)
$$\begin{tikzcd} f^*E \arrow[r, "\widetilde{f}"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ T \arrow[r, "f"'] & S , \end{tikzcd}$$

where $f^*E = T \times_S E = \{(y,a) \mid f(y) = p(a)\}$, and q and $\widetilde{f}$ are the projections. Clearly, q is also an n-fold G-function with multiplicity μ' given by $\mu'(y,a) = \mu(a)$, since $\mu'(g(y,a)) = \mu'(gy,ga) = \mu(ga) = \mu(a) = \mu'(y,a)$. Consider the restriction of $\widetilde{f}$ from the fiber $q^{-1}(y)$ to the fiber $p^{-1}(f(y))$. This bijective function induces a surjective function

$$q^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)} .$$

Clearly, conditions (a), (b), and (c) in the previous definition hold. Moreover, clearly $G_{(y,a)} = G_y \cap G_a$, thus condition (d) also holds. Hence $(\widetilde{f}, f)$ is a morphism from q to p.

(b) Let T be a G-set and consider the G-function $\pi : T^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n T$ given by $\pi\langle x_1, \ldots, x_n; i\rangle = \langle x_1, \ldots, x_n\rangle$, where G acts on both sets diagonally and trivially on the set $\overline{n} = \{1, 2, \ldots, n\}$. Define $\mu : T^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{N}$ by

$$\mu\langle x; i\rangle = \#x^{-1}(x(i)) ,$$

where one regards the ordered n-tuple $(x_1, \ldots, x_n)$ as a function $x : \overline{n} \longrightarrow T$. Then p is an n-fold G-function with multiplicity, since the sets $x^{-1}x(i)$ form a partition of the set $\overline{n}$. Furthermore, μ is clearly G-invariant. The function $\varphi_\pi : \mathrm{SP}^n T \longrightarrow \mathrm{SP}^n(T^n \times_{\Sigma_n} \overline{n})$ is given in this case by

$$\varphi_\pi\langle x_1, \ldots, x_n\rangle = \langle\langle x_1, \ldots, x_n; 1\rangle, \ldots, \langle x_1, \ldots, x_n; n\rangle\rangle .$$

Definition 2.8 Let $p : E \longrightarrow S$ be a n-fold G-function with multiplicity μ, and let M be a Mackey functor. Define a homomorphism

$$t_p : F(S, M) \longrightarrow F(E, M) ,$$

by

$$t_p(u)(a) = \mu(a) M^*(\widehat{p}_a) u(p(a)) ,$$

where $u \in F(S,M)$ and $a \in E$. If we assume that $u \in F^G(S,M)$, i.e., that $u(gx) = g \cdot u(x)$, then

$$\begin{aligned} t_p(u)(ga) &= \mu(ga)M^*(\widehat{p}_{ga})(u(p(ga))) \\ &= \mu(a)M^*(\widehat{p}_{ga})(g \cdot u(p(a))) \\ &= \mu(a)M^*(\widehat{p}_{ga})M_*(R_{g^{-1}})(u(p(a))) \\ &= \mu(a)M_*(R_{g^{-1}})M^*(\widehat{p}_a)(u(p(a))) \\ &= g \cdot (t_p(u)(a))\,, \end{aligned}$$

where the next to the last equality follows from the pullback property of the Mackey functor. Thus $t_p(u) \in F^G(E,M)$. Therefore, the homomorphism t_p restricts to a *transfer* homomorphism

$$t_p^G : F^G(S,M) \longrightarrow F^G(E,M)\,.$$

Remark 2.9 Let $p : E \longrightarrow S$ be a n-fold G-function with multiplicity μ. The isotropy group G_x acts on $p^{-1}(x)$ and the inclusion $p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$. Let $\{a_\iota\} \subset p^{-1}(x)$ be a set of representatives one for each G_x-orbit. Let $\gamma_x^G(l)$ be a generator of $F^G(S,M)$. Since the value of this function is zero on points which do not belong to the orbit Gx, and $\gamma_x^G(l)(x) = l$. One can give the transfer t_p^G on the generators $\gamma_x^G(l)$ by the formula

$$t_p^G(\gamma_x^G(l)) = \sum_{[a_\iota] \in p^{-1}(x)/G_x} \mu(a_\iota)\gamma_{a_\iota}^G(M^*(\widehat{p}_{a_\iota})(l))\,. \tag{2.11}$$

The next follows from what was done in [5, Section 2].

Proposition 2.12 *The transfer has the following properties: Naturality (with respect to morphisms of G-functions with multiplicity), Pullback, Normalization, Additivity, Quasiadditivity, Functoriality, Invariance under change of coefficients, and if M is homological, then the composite*

$$p_*^G \circ t_p^G : F^G(S,M) \longrightarrow F^G(S,M)$$

is multiplication by n. ∎

3 Ramified covering maps in the category of simplicial G-sets

In this section, we shall define G-equivariant simplicial ramified covering maps. Our definition is based on the concept of a weighted map given by Friedlander and Mazur [7]. We show that these simplicial ramified covering G-maps have properties analogous to those proved by Smith [13] and Dold [6] for topological ramified covering maps.

Definition 3.1 Let $p : K \longrightarrow Q$ be a pointed simplicial function between pointed simplicial sets. We say that p is an *n-fold G-equivariant simplicial ramified covering map* if the following conditions hold:

1. For each m, $p_m : K_m \longrightarrow Q_m$ has finite fibers.
2. The function $d_i^K|_{p_m^{-1}(x)} : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i^Q(x))$ is surjective for all i.
3. There is a family of G-invariant *multiplicity functions* $\mu_m : K_m \longrightarrow \mathbb{N}$, such that:
   (a) For all $x \in Q_m$, one has $\sum_{a\in p_m^{-1}(x)} \mu_m(a) = n$.
   (b) $\mu_{m+1} \circ s_i^K = \mu_m : K_m \longrightarrow \mathbb{N}$.
   (c) For all $x \in Q_m$ and $a \in p_m^{-1}(x)$ one has
   $$\mu_{m-1}(d_i^K(a)) = \sum_{\alpha=1}^{r} \mu_m(a_\alpha)\,,$$
   where $\{a_1, \ldots, a_r\} = (d_i^K)^{-1}(d_i^K(a)) \cap p_m^{-1}(x)$.

We call p *special* if, furthermore, the following condition holds.

4. For each $a \in K_m$, one has $G_a = G_{p_m(a)} \cap G_{d_i^K(a)}$ for all i.

Remark 3.2 Properties 3 (a) and (b) imply that the restrictions $s_i^K| : p_m^{-1}(x) \longrightarrow p_{m+1}^{-1}(s_i^Q(x))$ are bijective.

Remark 3.3 The corresponding definition in the nonequivariant setting is given in [4].

The concept of a simplicial ramified covering map is particularly well behaved. We have the following result, which is already valid in the nonequivariant case. We define $R_m = \{x \in Q_m \mid \exists a \in p_m^{-1}(x) \text{ with } \mu_m(a) > 1\}$. We call the elements of R_m the *ramification points* of p_m.

Proposition 3.4 *Let $p : K \longrightarrow Q$ be an n-fold simplicial ramified covering map. Then the sets of ramification points R_m form a simplicial subset of Q.*

Proof: Take $x \in R_m$ and take $d_i^Q : Q_m \longrightarrow Q_{m-1}$. If $d_i^Q(x) \notin R_{m-1}$, then the cardinality $|p_{m-1}^{-1}(d_i^Q(x))|$ would be n, while $|p_m^{-1}(x)|$ is less than n. This contradicts the fact that $d_i^K : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i^Q(x))$ is surjective. Hence d_i^Q restricts to $d_i^R : R_m \longrightarrow R_{m-1}$.

On the other hand, if again $x \in R_m$ and we take $s_i^Q : Q_m \longrightarrow Q_{m+1}$, we have $a \in p_m^{-1}(x)$ with $\mu_m(a) > 1$. But since $\mu_{m+1}(s_i^K(a)) = \mu_m(a) > 1$, $s_i^Q(x) \in R_{m+1}$ and s_i^Q restricts to $s_i^R : R_m \longrightarrow R_{m+1}$. ■

Corollary 3.5 *Let $p : K \longrightarrow Q$ be an n-fold simplicial ramified covering G-map. Then $|R| \subset |Q|$ is a G-invariant subcomplex.*

Proof: By [9, 4.3.8], $|R|$ is a subcomplex of $|K|$, and it is G-invariant since μ_m is G-invariant for all m. ■

The following result provides a sufficient condition for a simplicial ramified covering G-map to be special.

Proposition 3.6 *Let $p : K \longrightarrow Q$ be an n-fold simplicial ramified covering G-map, such that G acts freely on the ramification points $R \subset Q$. Then p is special.*

Proof: We have to prove the equality $G_a = G_{p_m(a)} \cap G_{d_i^K(a)}$ for all $a \in K_m$, all i, and all m. For simplicity, call $x = p_m(a)$. There are two cases:

Case I: $x \in R_m$. Then $G_x = \{e\}$ and since $G_a \subset G_x$, the equality follows.

Case II: $x \notin R_m$. We have two subcases:

(a) $d_i^Q(x) \notin R_{m-1}$. In this case, if $b \in p_{m-1}^{-1}(d_i^Q(x))$, then $\mu_{m-1}(b) = 1$. Hence $\mu_m(a') = 1$ for all $a' \in p_m^{-1}(x)$ and therefore, the restriction $d_i^K|_{p_m^{-1}(x)} : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i^Q(x))$ is bijective. Take $g \in G_x \cap G_{d_i^K(a)}$, then $p_m(a) = p_m(ga) = x$ and $d_i^K(a) = d_i^K(ga)$. Hence $a = ga$ so that $g \in G_a$, and the equality follows.

(b) $d_i^Q(x) \in R_{m-1}$. Since $G_a \subset G_{d_i^K(a)} \subset G_{d_i^Q(x)} = \{e\}$ the equality holds.

■

Proposition 3.7 *Let $p : K \longrightarrow Q$ be a map of simplicial G-sets. Then p is a special n-fold G-equivariant simplicial ramified covering map if and only if p_m is an n-fold G-function with multiplicity μ_m and the pairs (d_i^K, d_i^Q), (s_i^K, s_i^Q) are morphisms of G-functions with multiplicity for all face and degeneracy operators.*

Proof: The conditions on the face operators d_i in Definition 3.1 are exactly the same as the conditions for the pair (d_i^K, d_i^Q) to be a morphism of G-functions with multiplicity. Moreover, since the degeneracy operators s_i are always injective, then condition 3(b) is equivalent to the condition of the pair (s_i^K, s_i^Q) being a morphism. ■

Proposition 3.8 *Let $p : K \longrightarrow Q$ be an n-fold G-equivariant simplicial ramified covering map. If p_m is isovariant for all m, then p is special.*

Proof: If p_m is isovariant, then $G_a = G_{p_m(a)}$ for all a. Since d_i^K is G-equivariant, $G_a \subset G_{d_i^K(a)}$. Thus condition 4 holds. ■

Remark 3.9 In 4.6 we give an example of a ramified covering G-map which is special, but not isovariant.

Corresponding to [4, Thm. 3.5], we have the following.

Proposition 3.10 *Let $p : K \longrightarrow Q$ be an n-fold G-equivariant (special) simplicial ramified covering map, and let $f : Q' \longrightarrow Q$ be a simplicial map. Then the pullback of p over f, $p' : K' = Q' \times_Q K \longrightarrow Q'$, is an n-fold (special) simplicial G-equivariant ramified covering map.*

Proof: By [4, Prop. 1.4], we have that p' is an n-fold simplicial ramified covering map. Since the map p' is clearly G-equivariant, we only have to prove that if p is special, then also p' is special. To see this, take $a' = (b', a) \in Q'_m \times_{Q_m} K_m$. Since p is special, we have $G_a = G_{p_m(a)} \cap G_{d_i^K(a)}$. Notice that since $f_m(b') = p_m(a)$, one has $G_{b'} \subset G_{p_m(a)}$. Hence

$$G_{a'} = G_{b'} \cap G_a = G_{b'} \cap G_{p_m(a)} \cap G_{d_i^K(a)} = G_{b'} \cap G_{d_i^K(a)} \,.$$

On the other hand,

$$G_{p'_m(a')} \cap G_{d_i^{K'}(a')} = G_{b'} \cap G_{d_i^{Q'}(b')} \cap G_{d_i^K(a)} = G_{b'} \cap G_{d_i^K(a)} \,.$$

Hence p' is special. ■

The following is the equivariant version of [4, Prop. 1.6].

Proposition 3.11 *Let T be a simplicial G-set. Then the simplicial function $\pi : T^n \times_{\Sigma_n} \overline{n} \longrightarrow T^n/\Sigma_n$, where $\overline{n} = \{1, 2, \ldots, n\}$, is a G-equivariant simplicial n-fold ramified covering map.*

Proof: T is a contravariant functor $\Delta \longrightarrow G$-$\mathcal{S}$et, where Δ is the category whose objects are the sets $\mathbf{n}$, $n \geq 0$, and whose morphisms are order-preserving functions (see [10] or [11]). Consider the functors

$$E_n : G\text{-}\mathcal{S}\text{et} \longrightarrow G\text{-}\mathcal{S}\text{et} \quad \text{and} \quad B_n : G\text{-}\mathcal{S}\text{et} \longrightarrow G\text{-}\mathcal{S}\text{et}$$

given by $S \longmapsto S^n \times_{\Sigma_n} \overline{n}$ and $S \longmapsto S^n/\Sigma_n$, respectively, where G acts diagonally on S^n. Then $T^n \times_{\Sigma_n} \overline{n} = E_n \circ T$ and $T^n/\Sigma_n = B_n \circ T$. The natural transformation $E_n \longrightarrow B_n$ that maps $\langle s_1, \ldots, s_n; j\rangle$ to $\langle s_1, \ldots, s_n\rangle$ determines the function of simplicial sets

$$\pi : T^n \times_{\Sigma_n} \overline{n} \longrightarrow T^n/\Sigma_n\,.$$

■

Proposition 3.12 *The functions $\varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n K_m$ defined in 2.1 determine a map $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ of simplicial G-sets.*

Proof: Since the multiplicity functions μ_m are G-invariant, the functions φ_{p_m} are clearly G-equivariant. By [4, Prop. 1.8 and Thm. 1.9], these functions define a map of simplicial sets. ■

Proposition 3.13 *Let $p : K \longrightarrow Q$ be a simplicial G-equivariant ramified covering map and take $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$. Then p is the pullback over φ_p of the simplicial G-equivariant ramified covering map $\pi : K^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n K$.*

Proof: By [4, Thm. 1.9], it follows that p is the pullback of π over φ. Moreover, since the face operators of K are isovariant, by 3.11, π is special, and by 3.10, p must be special. ■

The next is the equivariant version of [4, Thm. 3.1].

Theorem 3.14 *Let $p : K \longrightarrow Q$ be a map of simplicial G-sets. Then p is an n-fold simplicial G-equivariant ramified covering map with multiplicity functions μ_m if and only if there is a map of simplicial G-sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ such that for each m the following hold:*

1. *If $a \in K_m$, then $a \in \varphi_{p_m}(p_m(a))$.*
2. *The composition $\mathrm{SP}^n p_m \circ \varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n Q_m$ is the diagonal map.*

Proof: Assume first that $p : K \longrightarrow Q$ is an n-fold G-equivariant simplicial ramified covering map with multiplicity functions μ_m. Define $\varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n K_m$ by

$$\varphi_{p_m}(x) = \langle \underbrace{a_1, \ldots, a_1}_{\mu_m(a_1)}, \ldots, \underbrace{a_r, \ldots, a_r}_{\mu_m(a_r)} \rangle\,,$$

where $p_m^{-1}(x) = \{a_1, \dots, a_r\}$. By [4, Prop. 1.8], the functions φ_{p_m} determine a map φ_p of simplicial sets. Since the functions μ_m are G-invariant, φ_p is a map of simplicial G-sets. By [4, Prop. 3.1], the functions φ_{p_m} satisfy conditions 1 and 2.

Conversely, suppose that there is a map of simplicial G-sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ which satisfies conditions 1 and 2. Take $a \in K_m$ and consider $\varphi_{p_m}(p_m(a)) \in \mathrm{SP}^n K_m$. If $(a_1, \dots, a_n) \in K_m^n$ is a representative of $\varphi_{p_m}(p_m(a))$. Then define $\mu_m : K_m \longrightarrow \mathbb{N}$ by

$$\mu_m(a) = \#\{i \in \overline{n} \mid a_i = a\}\,.$$

Again by [4, Prop. 3.1], $p : K \longrightarrow Q$ together with the family $\{\mu_m\}$ is an n-fold simplicial ramified covering map. Since φ_{p_m} is G-equivariant, the functions μ_m are G-invariant. ■

We have the following.

Theorem 3.15 *Let $p : K \longrightarrow Q$ be an n-fold G-equivariant simplicial ramified covering map. Then there exists a simplicial G-set W with a simplicial action of the symmetric group Σ_n such that there are simplicial equivariant isomorphisms $\alpha : W/\Sigma_{n-1} \longrightarrow K$ and $\beta : W/\Sigma_n \longrightarrow Q$ such that the following diagram commutes:*

$$\begin{tikzcd} W/\Sigma_{n-1} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & K \arrow[d, "p"] \\ W/\Sigma_n \arrow[r, "\beta"'] & Q\,, \end{tikzcd}$$

where $\pi : W/\Sigma_{n-1} \longrightarrow W/\Sigma_n$ is the canonical projection.

Proof: Define the simplicial set W as follows. Take

$$W_m = \{(x; a_1, \dots, a_n) \in Q_m \times K_m^n \mid \varphi_{p_m}(x) = \langle a_1, \dots, a_n\rangle\}\,.$$

If $f : \mathbf{k} \longrightarrow \mathbf{m}$ is a morphism in Δ, define $f^W : W_m \longrightarrow W_k$ by

$$f^W(x; a_1, \dots, a_n) = (f^Q(x); f^K(a_1), \dots, f^K(a_n))\,.$$

This is well defined, since φ_p is a map of simplicial sets.

We define a right action of Σ_n on W_m by

$$(x; a_1, \dots, a_n)\sigma = (x; a_{\sigma(1)}, \dots, a_{\sigma(n)})\,.$$

We consider Σ_{n-1} as the subgroup of those permutations that leave the first coordinate fixed. Let $\alpha_m : W_m/\Sigma_{n-1} \longrightarrow K_m$ and $\beta_m : W_m/\Sigma_n \longrightarrow Q_m$ be given by

$$\alpha_m([x; a_1, \dots, a_n]_{\Sigma_{n-1}}) = a_1 \quad \text{and} \quad \beta_m([x; a_1, \dots, a_n]_{\Sigma_n}) = x\,.$$

Let $\pi_m : W_m/\Sigma_{n-1} \longrightarrow W_m/\Sigma_n$ be the canonical surjection. Clearly $p_m \circ \alpha_m = \beta_m \circ \pi_m$. One can easily check that both α_m and β_m are bijective and determine maps α and β of simplicial sets. Observe that the Σ_n-action on W is simplicial.

Now we define a left action of G on each W_m by

$$g(x; a_1, \dots, a_n) = (gx; ga_1, \dots, ga_n)\,.$$

Since φ_{p_m} is G-equivariant, then this action is well defined. Notice that the left G-action and the right Σ_n-action satisfy the following associativity condition:

$$g((x; a_1, \dots, a_n)\sigma) = (g(x; a_1, \dots, a_n))\sigma\,.$$

This guarantees that the G-action passes to the quotients. One easily verifies that the isomorphisms α and β are G-equivariant. Observe that the G-action on W is simplicial. ■

Conversely, we have the following.

Proposition 3.16 *Let Γ be a finite group and $\Lambda \subset \Gamma$ be a subgroup of index n, and let W be a simplicial (left) G-set. Assume that Γ acts simplicially on the right on W, in such a way that*

$$g(w\gamma) = (gw)\gamma\,,$$

for all $w \in W$, $g \in G$, and $\gamma \in \Gamma$. Then the orbit map of simplicial sets

$$\pi : W/\Lambda \longrightarrow W/\Gamma$$

is an n-fold G-equivariant simplicial ramified covering map.

Proof: We shall prove that p satisfies conditions 1 and 2 of Theorem 3.14. Let $\varphi_\pi : W/\Gamma \longrightarrow \mathrm{SP}^n W/\Lambda$ be given for $[w]_\Gamma \in W_m/\Gamma$ by

$$\varphi_{\pi_m}([w]_\Gamma) = \langle [w\gamma_1]_\Lambda, \dots, [w\gamma_n]_\Lambda \rangle \in \mathrm{SP}^n W_m/\Lambda\,,$$

where $\Gamma/\Lambda = \{[\gamma_1], \dots, [\gamma_n]\}$ $(\gamma_1 = e \in \Gamma)$.

Since the action of Γ on W is simplicial, one easily verifies that φ_π is a map of simplicial sets. To see condition 1, take $a = [w]_\Lambda \in W_m/\Lambda$; since $\gamma_1 = e$, $a = [w\gamma_1]_\Lambda \in \varphi_{\pi_m}\pi_m(a) = \varphi_{\pi_m}([w]_\Gamma)$. To see condition 2, take $x = [w]_\Gamma \in W_m/\Gamma$. Then

$$\begin{aligned}
\mathrm{SP}^n \pi_m \varphi_{\pi_m}(x) &= \mathrm{SP}^n \pi_m(\langle [w\gamma_1]_\Lambda, \dots, [w\gamma_1]_\Lambda \rangle) \\
&= \langle [w\gamma_1]_\Gamma, \dots, [w\gamma_1]_\Gamma \rangle \\
&= \langle [w]_\Gamma, \dots, [w]_\Gamma \rangle \\
&= \langle x, \dots, x \rangle
\end{aligned}$$

■

Corollary 3.17 *Under the previous hypotheses, assume that the following condition holds for every $w \in W_m$ and all m:*

- *If $g \notin G_w$ and $gw = w\gamma$, then $\gamma \in \Lambda$.*

Then the n-fold G-equivariant simplicial ramified covering map $\pi : W/\Lambda \longrightarrow W/\Gamma$ is such that π_m is isovariant for all m; thus π is special. In particular, if for each $w \in W_m$ and $g \notin G_w$ we have that $gw \neq w\gamma$, $\gamma \in \Gamma$, then $\pi : W \longrightarrow W/\Gamma$ is special.

Proof: Take $w \in W_m$ and assume that $e \neq g \in G_{[w]_\Gamma}$, that is $g[w]_\Gamma = [w]_\Gamma$. Then $gw = w\gamma$, with $\gamma \in \Gamma$ and so, by the condition, $\gamma \in \Lambda$, and thus $g[w]_\Lambda = [w]_\Lambda$. Hence $G_{[w]_\Gamma} \subset G_{[w]_\Lambda}$ and so π_m is isovariant. The second part follows taking Λ to be the trivial subgroup of Γ. ∎

Similarly to [4, Thm. 4.2], we have the next.

Theorem 3.18 *Let $p : K \longrightarrow Q$ be an n-fold simplicial G-equivariant ramified covering map. Then $|p| : |K| \longrightarrow |Q|$ is a topological n-fold G-equivariant ramified covering map.* ∎

Definition 3.19 Let $p : K \longrightarrow Q$ be a simplicial G-equivariant ramified covering map. We can define $t^G_{p_m} : F^G(Q_m, M) \longrightarrow F^G(K_m, M)$ on generators, as before, by

$$t^G_{p_m}(\gamma^G_x(l)) = \sum_{\{[a]\} = p_m^{-1}(x)/G_x} \mu_m(a)\gamma^G_a M^*(\widehat{p_{m}}_a)(l)\,.$$

Theorem 3.20 *If $p : K \longrightarrow Q$ is a special simplicial G-equivariant ramified covering map, then the set of maps $\{t^G_{p_m} \mid m \in \mathbb{N}\}$ determines a morphism $t^G_p : F^G(Q, M) \longrightarrow F^G(K, M)$ of simplicial groups. Hence there is a continuous transfer $|t^G_p| : |F^G(Q, M)| \longrightarrow |F^G(K, M)|$.*

Proof: By [5, 2.8], t^G_p commutes with morphisms of G-functions with multiplicity. Since by 3.7 the pairs (d^K_i, d^Q_i), (s^K_i, s^Q_i) are morphisms of G-functions with multiplicity, the result follows. ∎

Corollary 3.21 *Let $p : K \longrightarrow Q$ be a special simplicial G-equivariant ramified covering map. Then there is a transfer in Bredon homology*

$$\tau_{|p|} : \widetilde{H}^G_*(|Q|; M) \longrightarrow \widetilde{H}^G_*(|K|; M)\,.$$

Proof: By the previous theorem we have the morphism t_p^G of simplicial abelian groups. Thus we have a continuous homomorphism

$$|t_p^G| : |F^G(Q, M)| \longrightarrow |F^G(K, M)|\,,$$

which induces a homomorphism

$$|t_p^G|_* : \pi_*(|F^G(Q, M)) \longrightarrow \pi_*(|F^G(K, M)|)\,.$$

The result follows by Theorem 1.11. ■

In [5, Thm. 4.17], given a ramified covering G-map $p : E \longrightarrow X$ such that X and E are strong ρ-spaces, and given a homological Mackey functor M for G, we proved the existence of a transfer $t_p^G : \mathbb{F}^G(X, M) \longrightarrow \mathbb{F}^G(E, M)$, where the topological groups $\mathbb{F}^G(X, M)$ and $\mathbb{F}^G(E, M)$ are such that their homotopy groups are isomorphic to the Bredon-Illman G-equivariant homology with coefficients in M. In particular, the geometric realization of any simplicial G-set is a strong ρ-space. Thus we have the following compatibility result of this transfer with the one defined in this paper, as follows.

Proposition 3.22 *Let $p : K \longrightarrow Q$ be a special simplicial G-equivariant ramified covering map and let M be a homological Mackey functor for G. Then for each simplicial pointed G-set S there is an isomorphism of topological abelian groups $\psi_M^G : |F^G(S, M)| \longrightarrow \mathbb{F}^G(|S|, M)$ and the following diagram commutes:*

$$\begin{tikzcd}
|F^G(Q, M)| \arrow[r, "|t_p^G|"] \arrow[d, "\psi_M^G"'] & |F^G(K, M)| \arrow[d, "\psi_M^G"] \\
\mathbb{F}^G(|Q|, M) \arrow[r, "t_{|p|}^G"'] & \mathbb{F}^G(|K|, M)\,.
\end{tikzcd}$$

Proof: The proof that under the assumptions ψ_M^G is an isomorphism of topological groups for the case $K = \mathcal{S}(X)$ was given in [3, Prop. (5.17)]. One can easily check that this proof is valid for any simplicial G-set K. Thus we only need to prove the commutativity of the diagram.

The elements of the form $[\gamma_x^G(l), t] \in |F^G(Q, M)|$ generate the group. Consider the pair $(x, t) \in Q_m \times \Delta^m$. We can write $t = d_\#(t')$, where t' is an interior point of some Δ^k and d is an order-preserving inclusion in Δ. On the other hand, $d^K(x) = s^K(y)$, where y is nondegenerate and s is an iteration of degeneracy operators. Thus we have

$$(x, t) = (x, d_\#(t')) \sim (d^K(x), t') = (s^K(y), t') \sim (y, s_\#(t'))\,,$$

where $s_{\#}(t')$ is also an interior point. Using this in the simplicial group $F^G(Q,M)$, we have

$$(\gamma_x^G(l),t) \sim (\gamma_y^G M_*(\widehat{d^K{}_x})(l), s_{\#}(t'))\,.$$

Hence we can assume that the generator $[\gamma_x^G(l),t] \in |F^G(Q,M)|$ is such that (x,t) is nondegenerate.

We thus have, on the one hand,

$$\begin{aligned}
\psi_M^G |t_p^G|([\gamma_x^G(l),t]) &= \psi_M^G\left(\left[\sum_{[a]\in p_m^{-1}(x)/G_x} \mu_m(a)\gamma_a^G M^*(\widehat{p_m{}_a})(l), t\right]\right)\\
&= \sum_{[a]\in p_m^{-1}(x)/G_x} \mu_m(a)\gamma_{[a,t]}^G M^*(\widehat{p_m{}_a})(l)\,;
\end{aligned}$$

while, on the other hand,

$$\begin{aligned}
t_{|p|}^G \psi_M^G([\gamma_x^G(l),t]) &= t_{|p|}^G\left(\gamma_{[x,t]}^G(l)\right)\\
&= \sum_{[[a,t]]\in|p|^{-1}([x,t])/G_{[x,t]}} \mu\left([a,t]\right)\gamma_{[a,t]}^G M^*(\widehat{|p|_{[a,t]}})(l)\,.
\end{aligned}$$

By definition, $\mu([a,t]) = \mu_m(a)$, since (a,t) is also nondegenerate. By [4, Lemma 4.1], there is a bijection $\beta : p_m^{-1}(x) \cong |p|^{-1}([x,t])$, and by [2, Prop. 2.4], $G_{[x,t]} = G_x$ and $G_{[a,t]} = G_a$, thus we also have $\widehat{p_m{}_a} = \widehat{|p|_{[a,t]}}$. Hence both composites are equal. ■

4 Ramified covering maps in the category of simplicial G-complexes

Recall ([14]) that a *simplicial complex* C is a family of nonempty finite subsets of a set V_C, whose elements are the *vertices* of C and which have the following two properties:

(i) For each $v \in V_C$, the set $\{v\} \in C$.

(ii) Given $\sigma \in C$ and $\sigma' \subset \sigma$, then $\sigma' \in C$.

A map $f : C \longrightarrow D$ of simplicial complexes is given by a function $f : V_C \longrightarrow V_D$ such that if $\{v_0,\ldots,v_q\} \in C$, then $\{f(v_0),\ldots,f(v_q)\} \in D$.

In what follows, we shall assume that any simplicial complex C is *ordered*, that is, the vertices of C have a partial order such that each simplex is totally ordered. Moreover, we can also assume that any simplicial map $p : C \longrightarrow D$ preserves the order. This can always be achieved by considering the barycentric subdivision of each of the simplicial complexes, $\mathrm{sd}(C)$, $\mathrm{sd}(D)$, with the order given by inclusion. We denote by $\sigma^{(i)}$ the ith face of any ordered m-simplex $\sigma = (v_0 < \cdots < v_m)$ in a simplicial complex, which is defined by $\sigma^{(i)} = (v_0 < \cdots < \widehat{v_i} < \cdots < v_m)$, where we omit the ith vertex.

Definition 4.1 Let $p : C \longrightarrow D$ be a simplicial map between simplicial complexes. We say that p is an *n-fold ramified covering map of simplicial complexes* if there exists a *multiplicity function* $\mu : C \longrightarrow \mathbb{N}$ such that the following conditions hold:

1. For each vertex w of D, the fiber $p^{-1}(w)$ is a finite nonempty set and if $\sigma \in p^{-1}(\tau)$, then σ and τ have the same dimension.
2. For each simplex $\tau \in D$ and each simplex $\widetilde{\sigma} \in C$, such that $p(\widetilde{\sigma}) = \tau^{(i)}$, there is a simplex $\sigma \in D$ such that $p(\sigma) = \tau$ and $\sigma^{(i)} = \widetilde{\sigma}$.
3. For each simplex τ in D,
$$\sum_{p(\sigma)=\tau} \mu(\sigma) = n\,.$$
4. For each simplex $\sigma \in C$,
$$\mu(\sigma^{(i)}) = \sum_{\substack{p(\sigma)=p(\sigma') \\ \sigma^{(i)}=\sigma'^{(i)}}} \mu(\sigma')\,.$$

Proposition 4.2 *Let $p : C \longrightarrow D$ be a simplicial map between simplicial complexes, and let $\mathrm{sd}(p) : \mathrm{sd}(C) \longrightarrow \mathrm{sd}(D)$ be the induced map between the barycentric subdivisions. If p is a ramified covering map of simplicial complexes with multiplicity function μ, then $\mathrm{sd}(p)$ is a ramified covering map of simplicial complexes with multiplicity function $\mathrm{sd}(\mu)$, where*
$$\mathrm{sd}(\mu)(\sigma_0 \subsetneq \cdots \subsetneq \sigma_m) = \mu(\sigma_m)\,.$$

Proof: Notice first that there is a bijection $\mathrm{sd}(p)^{-1}(\tau_0 \subsetneq \cdots \subsetneq \tau_m) \approx p^{-1}(\tau_m)$, since clearly $\sigma_m \in p^{-1}(\tau_m)$ determines σ_i such that $p(\sigma_i) = \tau_i$. Therefore, 1 holds for $\mathrm{sd}(p)$.

To see 2, let
$$\begin{aligned}\mathrm{sd}(p)(\sigma_0 \subsetneq \cdots \subsetneq \sigma_{i-1} \subsetneq \sigma_{i+1} \subsetneq \cdots \subsetneq \sigma_m) &= (\tau_0 \subsetneq \cdots \subsetneq \tau_{i-1} \subsetneq \tau_{i+1} \subsetneq \cdots \subsetneq \tau_m) \\ &= (\tau_0 \subsetneq \cdots \subsetneq \tau_m)^{(i)}\end{aligned}$$

Since $\tau_i \in D$ is a certain face of $\tau_{i+1} \in D$, one should take the corresponding face of σ_{i+1} and call it σ_i. Then, since p preserves faces, clearly

$$\mathrm{sd}(p)(\sigma_0 \subsetneq \cdots \subsetneq \sigma_m) = (\tau_0 \subsetneq \cdots \subsetneq \tau_m)$$

and

$$(\sigma_0 \subsetneq \cdots \subsetneq \sigma_m)^{(i)} = (\sigma_0 \subsetneq \cdots \subsetneq \sigma_{i-1} \subsetneq \sigma_{i+1} \subsetneq \cdots \subsetneq \sigma_m)\,.$$

Condition 3 follows immediately from the definition of $\mathrm{sd}(\mu)$ and the remark at the beginning of this proof.

Finally, to show condition 4, we have two cases:

Case 1. If $i < m$, then we have

$$\mathrm{sd}(\mu)((\tau_0 \subsetneq \cdots \subsetneq \tau_m)^{(i)}) = \mu(\tau_m)$$

and the condition follows immediately

Case 2. If $i = m$, then we have

$$\mathrm{sd}(\mu)((\tau_0 \subsetneq \cdots \subsetneq \tau_m)^{(i)}) = \mu(\tau_{m-1})\,.$$

τ_{m-1} is an iterated face of τ_m. We assume first that $\tau_{m-1} = \tau_m^{(j)}$. Then the condition follows immediately from condition 4 for p. In the general case, one can proceed inductively. ■

We understand by an *ordered simplicial G-complex* an ordered simplicial complex C together with an order-preserving action of G on the set of vertices V_C, such that each $g \in G$ induces a simplicial map. We can always assume that a simplical G-complex is ordered by passing to the barycentric subdivision, if necessary. A *simplicial G-map* $p : C \longrightarrow D$ of simplicial G-complexes is an order-preserving simplicial map which is G-equivariant. In what follows, we shall always consider ordered simplicial G-complexes.

Definition 4.3 We say that $p : C \longrightarrow D$ is an *n-fold G-equivariant ramified covering map of simplicial complexes* with multiplicity function μ, if p is an n-fold ramified covering map such that C and D are G-complexes, p is a G-map, and μ is G-invariant. We say that p is *special* if the following condition holds:

5. For each m-simplex $\sigma \in C$, one has $G_\sigma = G_{p(\sigma)} \cap G_{\sigma^{(i)}}$, for all $i = 0, \ldots, m$.

Proposition 4.2 allows us to assume that the G-actions in a G-equivariant ramified covering map of simplicial complexes preserve the orderings. From now on we shall assume that this is the case.

Recall ([4, Def. 5.4]) that given a simplicial complex C, one has a simplicial set $K(C)$ such that

$$K(C)_m = \{(v_0, \dots, v_m) \mid \{v_0, \dots, v_m\} \in C,\ v_0 \leq \cdots \leq v_m\}\,,$$

$d_i^{K(C)} : K(C)_m \longrightarrow K(C)_{m-1}$ is given by

$$d_i^{K(C)}(v_0, \dots, v_m) = (v_0, \dots, \widehat{v_i}, \dots, v_m)\,,$$

and $s_i^{K(C)} : K(C)_m \longrightarrow K(C)_{m+1}$ is given by

$$s_i^{K(C)}(v_0, \dots, v_m) = (v_0, \dots, v_i, v_i, \dots, v_m)\,.$$

If $p : C \longrightarrow D$ is an n-fold ramified covering map of simplicial complexes, call $K(p)_m : K(C)_m \longrightarrow K(D)_m$ the induced map of simplicial sets, given by $K(p)_m(v_0, \dots, v_m) = (p(v_0), \dots, p(v_m))$. Define $\mu_m : K(C)_m \longrightarrow \mathbb{N}$ by $\mu_m(\sigma) = \mu(\sigma')$, where $\sigma' \in K(C)_l$, $l \leq m$, is the unique nondegenerate simplex such that $s^{K(C)}(\sigma') = \sigma$.

Proposition 4.4 *Let $p : C \longrightarrow D$ be an n-fold G-equivariant ramified covering map of simplicial complexes. Then $K(p) : K(C) \longrightarrow K(D)$ is an n-fold simplicial G-equivariant ramified covering map. Furthermore, if p is special, then so is $K(p)$.*

Proof: By [4, Thm. 5.6], $K(p)$ is an n-fold simplicial ramified covering map and is clearly G-equivariant with the obvious actions. One easily verifies that μ_m is G-invariant for every m.

Now assume that p is special and take $a = (v_0 \leq \cdots \leq v_m) \in K(C)_m$. Let

$$v'_0 = v_0 = \cdots = v_{j_1-1} < v'_1 = v_{j_1} = \cdots = v_{j_2-1} < \cdots < v'_{m'} = v_{j_{m'}} = \cdots = v_m\,,$$

and take i such that $0 \leq i \leq m$. We have that $(v'_0 < \cdots < v'_{m'})$ is the unique nondegenerate simplex associated to $(v_0 \leq \cdots \leq v_m)$. There are two cases:

Case 1. The vertex v_i appears more than once in $a = (v_0 \leq \cdots \leq v_m)$. In this case, a and $a^{(i)}$ have the same associated nondegenerate simplex, and condition 4 in Definition 3.1 follows trivially.

Case 2. The vertex v_i appears once in $a = (v_0 \leq \cdots \leq v_m)$. In this case, the associated nondegenerate simplex of $a^{(i)}$ is $(v'_0 < \cdots < v'_{m'})^{(j)}$, where $v_i = v'_j$. Condition 4 in Definition 3.1 follows from condition 5 in Definition 4.3 applied to the associated nondegenerate simplexes. ∎

Remark 4.5 Let us recall that given a simplicial complex C, its *geometric realization* is given by

$$|C| = \{\alpha : V_C \longrightarrow I \mid \alpha^{-1}(0,1] \in C \text{ and } \sum_{v \in V_C} \alpha(v) = 1\}\,.$$

It has the coherent topology with respect to the family of realizations $|\sigma|$ that are homeomorphic to Δ^n for some n. If $\sigma = \{v_0 < \cdots < v_m\}$, then the homeomorphism $|\sigma| \longrightarrow \Delta^n$ is given by $\alpha \mapsto (\alpha(v_0), \ldots, \alpha(v_m))$.

Given an n-fold ramified covering map of simplicial complexes $p : C \longrightarrow D$, by Property 1 in Definition 4.1, $p(\sigma) = \{p(v_0) < \cdots < p(v_m)\}$. Therefore we have the commutative square

$$\begin{tikzcd} |\sigma| \arrow[r, "|p|"] \arrow[d, "\approx"'] & |p(\sigma)| \arrow[d, "\approx"] \\ \Delta^m \arrow[r, "\mathrm{id}"'] & \Delta^m\,. \end{tikzcd}$$

Hence $|p|$ maps the realization of every simplex of C homeomorphically onto the realization of a simplex of D.

Example 4.6 Figure 1 shows a 3-fold $\mathbb{Z}_2$-equivariant ramified covering map of simplicial complexes $p : C \longrightarrow D$. The vertices have their natural order and $\mathbb{Z}_2$ acts on C exchanging 1 and 2, as well as 3 and 4, and leaving 0 fixed. On D it exchanges $3'$ and $4'$ and leaves $0'$ fixed. The projection is given by $p(0) = p(1) = p(2) = 0'$, $p(3) = 3'$, and $p(4) = 4'$. The multiplicity map is given by $\mu(0) = \mu(1) = \mu(2) = 1$, and $\mu(3) = \mu(4) = 3$. Moreover μ evaluated at any 1-simplex is 1. Thus the ramification points are $3'$ and $4'$, where $\mathbb{Z}_2$ acts freely. Notice that the actions on the total space and on the base space are not free, since the vertices 0 and $0'$ are fixed points. Take $K(p) : K(C) \longrightarrow K(D)$. By Proposition 3.6, $K(p)$ is special, but it is not isovariant.

Proposition 4.7 *Let $p : C \longrightarrow D$ be a special n-fold G-equivariant ramified covering map of simplicial complexes, and let M be a Mackey functor for G. Then there is a continuous transfer $t_p^G : |F^G(K(D), M)| \longrightarrow |F^G(K(C), M)|$.*

Proof: By Proposition 4.4, we have a special simplicial n-fold ramified covering G-map $K(p) : K(C) \longrightarrow K(D)$. By Theorem 3.20, we have a transfer

$$t_{K(p)}^G : F^G(K(D), M) \longrightarrow F^G(K(C), M)\,.$$

The result follows by defining $t_p^G = |t_{K(p)}^G|$. ∎

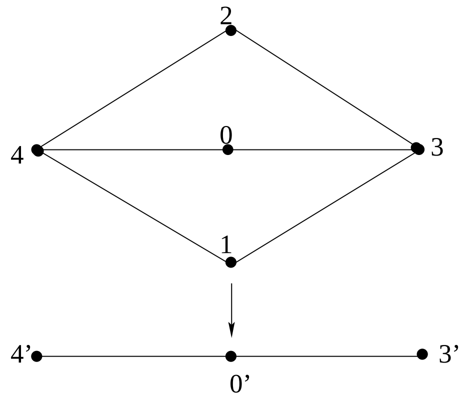

Figure 1: A 3-fold ramified covering $\mathbb{Z}_2$-map

REMARK 4.8 In the previous result, notice that since $|p| : |C| \longrightarrow |D|$ is a ramified covering G-map in the category of strong ρ-spaces, if the Mackey functor is homological, then by [5, Thm. 4.12] it need not be special in order for $t^G_{|p|} : \mathbb{F}^G(|D|, M) \longrightarrow \mathbb{F}^G(|C|, M)$ to be continuous.

In this case, as shown in Proposition 3.22, the topological groups are regular CW-complexes, since they are isomorphic to geometric realizations of simplicial groups. However, the transfer $t^G_{|p|}$ need not be a regular map (i.e. a cellular map that sends open cells onto open cells), unless the simplicial ramified covering G-map p is special.

To finish this section we define the transfer in Bredon homology by just applying the homotopy-group functors π_q to the transfer between topological groups. Using Theorem 1.11 and Propositions 4.7 and 2.12, we obtain the following.

Theorem 4.9 *Let $p : C \longrightarrow D$ be a special n-fold G-equivariant ramified covering map of simplicial complexes, and let M be a Mackey functor for G. Then there exists a transfer*

$$\tau_{|p|} : \widetilde{H}^G_*(|D|; M) \longrightarrow \widetilde{H}^G_*(|C|; M)$$

with the following properties: Pullback, Normalization, Additivity, Quasiadditivity, Functoriality, Invariance under change of coefficients, and if M is homological, then the composite

$$|p|_* \circ \tau_{|p|} : \widetilde{H}^G(|D|; M) \longrightarrow \widetilde{H}^G(|D|; M)$$

is multiplication by n.

Proof: Assume that $p : C \longrightarrow D$ is a special n-fold G-equivariant ramified covering map of simplicial complexes and that $f : D' \longrightarrow D$ is a G-equivariant pointed simplicial map. The pullback property follows from the fact that there are canonical homeomorphisms

$$|D'| \times_{|D|} |C| \approx |K(D')| \times_{|K(D)|} |K(C)| \approx |K(D') \times_{K(D)} K(C)|\,.$$

The first one follows from the homeomorphism mentioned in the proof of 4.7. To see the second one, notice that there is a natural homeomorphism $|Q' \times K| \approx |Q'| \times |K|$ for arbitrary simplicial sets Q' and K (see [11]), which restricts to a homeomorphism $|Q' \times_Q K| \approx |Q'| \times_{|Q|} |K|$ for any maps $K \xrightarrow{p} Q \xleftarrow{f} Q'$. Furthermore, under these homeomorphisms, the pullback diagram

$$\begin{tikzcd}
|D'| \times_{|D|} |C| \arrow[r, "\widetilde{|f|}"] \arrow[d, "|p|'"'] & |C| \arrow[d, "|p|"] \\
|D'| \arrow[r, "|f|"'] & |D|
\end{tikzcd}$$

corresponds to the diagram

$$\begin{tikzcd}
|K(D') \times_{K(D)} K(C)| \arrow[r, "\widetilde{K(f)}"] \arrow[d, "|K(p)'|"'] & |K(C)| \arrow[d, "|K(p)|"] \\
|K(D')| \arrow[r, "|K(f)|"'] & |K(D)|\,.
\end{tikzcd}$$

Therefore, since the pullback property of the transfer holds in the category of simplicial sets, it holds also in this case.

In order to prove the additivity property, assume first that for each $\alpha = 1, 2, \ldots, k$, $p_\alpha : C_\alpha \longrightarrow D$ is an n_α-fold G-equivariant ramified covering map of simplicial complexes. One can take the wedge sum $C = C_1 \vee C_2 \vee \cdots \vee C_k$. If the set of vertices of each C_α is partially ordered, so that every simplex is totally ordered, then the partial orders define a partial order in the set of vertices of C such that it induces the given order in the set of vertices of each C_α and each simplex in C, which is a simplex in some C_α, is totally ordered. Then one has a homeomorphism of topological spaces

$$|C_1 \vee C_2 \vee \cdots \vee C_k| \approx |C_1| \vee |C_2| \vee \cdots \vee |C_k|\,.$$

By [4, Thm. 4.2], each $|p_\alpha| : |C_\alpha| \longrightarrow |D|$ is a (topological) n_α-fold ramified covering G-map. Hence, from [5, 3.2(a)], $\pi : |C_1| \vee |C_2| \vee \cdots \vee |C_k| \longrightarrow |D|$, given by $\pi|_{|C_\alpha|} = |p_\alpha|$, is an $(n_1 + n_2 + \cdots + n_k)$-fold ramified covering G-map.

By the additivity property of the transfer for G-functions with multiplicity [5, 2.19], the desired additivity property follows, namely, for all $\xi \in H_*^G(|D|; M)$,

$$\tau_\pi(\xi) = i_{1*}\tau_{|p_1|}(\xi) + i_{2*}\tau_{|p_2|}(\xi) + \cdots + i_{k*}\tau_{|p_k|}(\xi) \in H_*^G(|C_1| \vee |C_2| \vee \cdots \vee |C_k|; M)\,,$$

where i_α is the inclusion. Notice that using [4, Thm. 3.1], one can easily show that $p : K(C_1) \vee K(C_2) \vee \cdots \vee K(C_k) \longrightarrow K(D)$ given by $p|_{K(C_\alpha)} = K(p_\alpha)$ is a G-equivariant ramified covering map of simplicial sets, and it has the property that its realization corresponds to π. Thus the transfer of π corresponds to the realization of the transfer of p defined on simplicial sets.

The functoriality follows from the fact that if $p : C \longrightarrow D$ and $q : D \longrightarrow E$ are G-equivariant ramified covering maps of simplicial complexes, then by [1, 4.20] the composite $|q| \circ |p|$ is a (topological) ramified covering G-map and the corresponding property of G-functions with multiplicity [5, 2.21]. Notice that the composite $q \circ p$ is a G-equivariant ramified covering map of simplicial complexes such that $|q \circ p| = |q| \circ |p|$ and $t^G_{|q|\circ|p|} = t^G_{|q\circ p|}$. ∎

References

[1] M. Aguilar, C. Prieto, Transfers for ramified covering maps in homology and cohomology, *Int. J. Math. and Math. Sci.* vol. **2006** (2006) 1–28.

[2] M. Aguilar, C. Prieto, Equivariant homotopical homology with coefficients in a Mackey functor, *Topology and its Applications* **154** (2007) 2826–2848 (with a correction, ibid. **156** (2009), 1609–1613.

[3] M. Aguilar, C. Prieto, Topological groups and Mackey functors, to appear in *Bol. Soc. Mat. Mex.* **14** (2008).

[4] M. Aguilar, C. Prieto, Simplicial ramified covering maps, *Topology and its Applications* **156** (2008) 205–216.

[5] M. Aguilar, C. Prieto, The transfer for ramified covering G-maps, to appear in Forum Math (2009).

[6] A. Dold, Ramified coverings, orbit projections and symmetric powers, *Math. Proc. Camb. Phil. Soc.* **99** (1986), 65–72.

[7] E. M. Friedlander, B. Mazur, Filtrations on the homology of algebraic varieties. With an appendix by Daniel Quillen, *Mem. Amer. Math. Soc.* **110** (1994) no. 529, x+110

[8] R. Fritsch, M. Golasiński, Simplicial and categorical diagrams, and their equivariant applications, *Theory Appl. Categ.* **4** (1998), 73–81.

[9] R. Fritsch, R. A. Piccinini, *Cellular Structures in Topology*, Cambridge University Press, Cambridge, UK, 1990.

[10] P. G. Goerss, J. F. Jardine, *Simplicial Homotopy Theory*, Birkhäuser, Basel Boston Berlin, 1999

[11] J. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, Chicago London 1992.

[12] J. P. May et al., *Equivariant Homotopy and Cohomology Theory*, CBMS 91 Amer. Math. Soc. Providence 1996

[13] L. Smith, Transfer and ramified coverings, *Math. Proc. Camb. Phil. Soc.* **93** (1983), 485–493.

[14] E. H. Spanier, *Algebraic Topology*, Springer, New York Heidelberg Berlin (McGraw-Hill) 1966

April 3, 2009